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r-HYPERGEOMETRIC FUNCTION AND ITS APPLICATION

In this paper with the help of the (τ,β)-generalized confluent hypergeometric function the *r*-hypergeometric function is considered. The aim of it is to study the main properties of the *r*-hypergeometric function, in particular, to study the relation of Erdelyi' types, the Mellin transform, the composite relation with integral operator of Erdelyi–Kober' type. In the study used common methods of the theory of special functions, the theory of integral transforms and operators of fractional integration. We also obtained the representation of the *r*-hypergeometric function by the fractional integral. Some applications of the *r*-hypergeometric functions to the solving of integral Volterra' equations in closed form are given. The results can be used for further development of the theory of special functions and their applications in different sciences.

Introduction

Further studying of the special functions is prospective and very useful for different branches of science. This is due to the needs of the development of effective analytical methods for solving a wide class of boundary value problems of mathematical physics, the theory of differential and integral equations, the theory of modeling and automatic control, many problems of applied mathematics and so on.

The continuous development of the mechanics of solid medium, aerodynamics, quantum mechanics, probability theory, astronomy and other has led to the generalization and the creation of new classes of the special functions. With the theory of generalized special functions associated large number of different mathematical problems. Thus, the special functions are widely used in the construction of various integral transforms (Saigo, Kober, Saxena et al.), the theory of which (with kernels as special functions) in recent years has developed in the works of A.A. Kilbas, S.L. Kalla, A.M. Mathai, M. Saigo, R.K. Saxena and others [1–5], and which allow to obtain solutions in analytical form of many important classes of differential and integral equations.

In this article we consider the *r*-generalized hypergeometric function, its properties and some applications for solving integral equations in closed form.

Statement of the problem

The aim of the paper is to study the main properties of the *r*-hypergeometric function, in particular, to study the relation of Erdelyi' type, the Mellin' transform, the composite relation with integral operator of Erdelyi–Kober' type, to obtain the representation of the *r*-hypergeometric function by the fractional integral. Also to consider the application of the *r*-hypergeometric functions to solving integral Volterra' equations in closed form.

Main properties of *r*-hypergeometric function

Let us consider the *r*-hypergeometric function in the following form:

$$ {}_r\tilde{F}^{\tau,\beta}(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\times $$
$$ \times(1-zt)^{-a}\,{}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{t(1-t)}\right)dt, \qquad (1) $$

where $\{a,b,c\}\subset C$, $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, $r>0$; $r=0$, $|z|<1$, $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0$, $\{\tau,\beta\}\subset R$, $\tau>0$, $\tau-\beta<1$, $\mathrm{B}(b,c-b)$ is the classical beta-function [6], ${}_1\Phi_1^{\tau,\beta}(\ldots)$ is the (τ,β) – generalized confluent hypergeometric function [7]:

$$ {}_1\Phi_1^{\tau,\beta}(a;c;z) = \frac{1}{\mathrm{B}(b,c-b)}\times $$
$$ \times\int_0^1 t^{a-1}(1-t)^{c-a-1}\,{}_1\Psi_1\left[\begin{matrix}(c;\tau);\\(c;\beta);\end{matrix}zt^{\tau}\right]dt, \qquad (2) $$

where ${}_1\Psi_1(\ldots)$ is the Fox–Wright function [1]. As $\tau=\beta=0$, $r=0$ in (2) we have the classical confluent hypergeometric function ${}_1\Phi_1(a;c;z)$ [1]; as $r=0$ in (1) we have the Gauss' hypergeometric function ${}_2F_1^{\tau,\beta}(a,b;c;z)$ [6].

In the case of fulfilling of the conditions of existence of ${}_r\tilde{F}^{\tau,\beta}(a,b;c;z)$ following formulae are valid.

Lemma 1 (the relation of Erdelyi' type). As $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, $\{a,b,c\} \subset C$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $r > 0$; $r = 0$, $|z| < 1$, $\{\tau,\beta\} \subset R$, $\tau > 0$, $\tau - \beta < 1$, the following relation of Erdelyi' type is valid:

$$ {}_r\tilde{F}^{\tau,\beta}(a,b;c;z) = $$

$$ = (1-z)^{-a}\, {}_r\tilde{F}^{\tau,\beta}\left(a, c-b; c; \frac{z}{z-1}\right). \quad (3) $$

P r o o f. Let us use formula (1), substitution $t = 1-\omega$, expression $1 - z(1-\omega) = (1-z)\left(1 - \frac{z}{z-1}\omega\right)$, then we receive

$$ {}_r\tilde{F}^{\tau,\beta}(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 (1-\omega)^{b-1}\omega^{c-b-1} \times $$

$$ \times\left(1 - \frac{z}{z-1}\omega\right)^{-a} (1-z)^{-a}\, {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{\omega(1-\omega)}\right) d\omega = $$

$$ = \frac{(1-z)^{-a}}{\mathrm{B}(b,c-b)} \int_0^1 (1-\omega)^{b-1}\omega^{c-b-1} \times $$

$$ \times\left(1 - \frac{z}{z-1}\omega\right)^{-a} {}_1\Phi_1^{\tau,\beta}\left(\alpha;\gamma;-\frac{r}{\omega(1-\omega)}\right) d\omega = $$

$$ = (1-z)^{-a}\, {}_r\tilde{F}^{\tau,\beta}\left(a, c-b; c; \frac{z}{z-1}\right). $$

Lemma 2 (on the Mellin' transform for ${}_r\tilde{F}^{\tau}(a,b;c;z)$). As the conditions: $\tau \in R_+$, $\{a,b,c\} \subset \subset C$, $\mathrm{Re}(c-b) > s$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $r > 0$; $r = 0$, $|z| < 1$, $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > s\tau$, $\{r,z\} \subset C$, the following relation is valid:

$$ M\{{}_r\tilde{F}^{\tau}(a,b;c;z), s\} = A\, {}_2F_1(a, b+s; c+2s; z), \quad (4) $$

where

$$ A = \frac{\mathrm{B}(b+s, c-b+s)}{\mathrm{B}(b,c-b)} \frac{\Gamma(\gamma)\Gamma(s)\Gamma(\alpha - s\tau)}{\Gamma(\alpha)\Gamma(\gamma - s\tau)}. \quad (5) $$

***Corollary* 1.** The formula

$$ {}_r\tilde{F}^{\tau}(a,b;c;z) = $$

$$ = \frac{1}{2\pi i} \frac{A}{B(b,c-b)} \int_{x-i\infty}^{x+i\infty} {}_2F_1(a, b+s; c+2s; r) r^{-s} ds, \quad (6) $$

is obtained from (4) with help inversion formula of the Mellin' transform. The formula (6) gives interesting connection between r-generalized hypergeometric function ${}_r\tilde{F}^{\tau}(a,b;c;z)$ and the classical hypergeometric function ${}_2F_1(a,b;c;z)$.

Theorem 1 (the representation of the *r*-hypergeometric function by the fractional integral). As the conditions of existence of the function ${}_r\tilde{F}^{\tau,\beta}$ are fulfilled, and $a + n \geq 0$, $c \pm m > a$, the *r*-hypergeometric function has the following representation by the fractional Riemann—Liouvile integral:

$$ {}_r\tilde{F}^{\tau,\beta}\left(a,b;c;1-\frac{x}{p}\right) = \frac{\Gamma(c)}{\Gamma(cm-a)}(x-p)^{1-c} \times $$

$$ \times\, p^a \left(\frac{d}{dx}\right)^m I_{x-}^{\alpha} (x-p)^{c+m-a-1} p^{-a}, \quad (7) $$

where I_{x-}^{α} is the right-hand fractional integral of the order α (the fractional Riemann—Liouvile integral [8]).

On some applications of the *r*-hypergeometric functions

Let us introduce integral operator $N_{c-\lambda}^{a}$ in the form:

$$ (N_{c-\lambda}^{a} f)(x) \equiv $$

$$ \equiv x^{\lambda} \int_0^{\infty} (xt)^{a-1}\, {}_1^r\Phi_1^{\tau,\beta}(a;c;-xt) f(t)\, dt, \quad (8) $$

where $x > 0$, $\{\tau,\beta\} \subset R$, $\tau > 0$, $\tau - \beta < 1$, $\mathrm{Re}(c) > > \mathrm{Re}(a) > 0$, ${}_1^r\Phi_1^{\tau,\beta}$ is the following function:

$$ {}_1^r\Phi_1^{\tau,\beta}(a;c;x) = \frac{1}{\mathrm{B}(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{xt} $$

$$ {}_1\Phi_1^{\tau,\beta}\left(\delta;\gamma;-\frac{r}{t(1-t)}\right) dt, \quad (9) $$

here $\{a,c,\gamma,\delta\} \subset C$, $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $r > 0$, $\mathrm{Re}\,\gamma > \mathrm{Re}\,\delta > 0$, $\gamma > 0$, $\delta > 0$, $\{\tau,\beta\} \subset R$, $\tau > 0$, $\tau - \beta < 1$, ${}_1\Phi_1^{\tau,\beta}(\ldots)$ is the function of form (2).

The following theorem is valid.

Theorem 2 (on the composite relation). As $0 < \frac{1}{p} \leq 1$, $0 \leq \lambda + \frac{2}{p} - 1 < c - a$, $\frac{1}{p} < a$, $\frac{1}{q} = 1 - \frac{1}{p} - \lambda > 0$ for integral operator (8) the following composite relation is valid:

$$(N^a_{c-\lambda} f)(x) = \frac{\Gamma(c)}{\Gamma(a)} I^{c-a}_{\frac{2}{p}-1,\lambda+\frac{2}{p}-1} T^a f(x)\,, \quad (10)$$

where T^a is operator of the modified Laplace' integral transform:

$$(T^a f)(x) = x^{a-\frac{2}{p}} \int_0^\infty t^{a-1} e^{-xt} f(t)dt\,,\ (x>0)\,, \quad (11)$$

$I^{c-a}_{\frac{2}{p}-1,\,\lambda+\frac{2}{p}-1}$ is the integral operator of Erdelyi–Kober' type:

$$I^{c-a}_{\frac{2}{p}-1,\lambda+\frac{2}{p}-1} f(x) = \frac{x^{\lambda-c+a}}{\Gamma(c-a)} \int_0^x (x-t)^{c-a-1}\, t^{\frac{2}{p}-1} \times$$

$$\times\, {}_1\Phi_1^{\tau,\beta}\left(\delta;\gamma;-\frac{r}{\frac{t}{x}\left(1-\frac{t}{x}\right)}\right) f(t)dt\,, \quad (12)$$

P r o o f. Let us apply operator (12) to (10):

$$I^{c-a}_{\frac{2}{p}-1,\lambda+\frac{2}{p}-1} (N^a_{c-\lambda} f) = \frac{x^{\lambda-c+a}}{\Gamma(c-a)} \int_0^x (x-t)^{c-a-1}\, t^{\frac{2}{p}-1} \times$$

$$\times\, {}_1\Phi_1^{\tau,\beta}\left(\delta;\gamma;-\frac{r}{\frac{t}{x}\left(1-\frac{t}{x}\right)}\right) t^{c-a-\frac{2}{p}} dt \times$$

$$\times \int_0^\infty \xi^{c-a-1} e^{-t\xi} f(\xi)d\xi = \frac{x^{\lambda-c+a}}{\Gamma(c-a)} \int_0^\infty \xi^{c-a-1} f(\xi)d\xi \times$$

$$\times \int_0^x (x-t)^{c-a-1}\, t^{c-a-1} e^{-t\xi}\, {}_1\Phi_1^{\tau,\beta}\left(\delta;\gamma;-\frac{r}{\frac{t}{x}\left(1-\frac{t}{x}\right)}\right) dt =$$

$$= \frac{x^{\lambda-c+a}}{\Gamma(c-a)} \int_0^\infty \xi^{c-a-1} f(\xi)d\xi \int_0^1 (x-t)^{2c-2a-1} \times$$

$$\times (1-\omega)^{c-a-1} \omega^{c-a-1} e^{-\omega\xi x}\, {}_1\Phi_1^{\tau,\beta}\left(\delta;\gamma;-\frac{r}{\omega(1-\omega)}\right) d\omega\,.$$

Here the legality of interchanging of the order of integration and summation, and substitution $t = \omega x$ are used. Taking into account (9) we get (10).

a) Using (7) it easy to receive the solution of Volterra' integral equation:

$$\int_x^1 (\omega - x)^{c-1}\, {}_r\tilde{F}^{\tau,\beta}\left(a,b;c;1-\frac{x}{\omega}\right) f(\omega)\,d\omega = \psi(x)\,, \quad (13)$$

where $f(\omega) \in C^m[x,1]$, $c+m \ge a+n \ge 0$; $f^{(k)}(1) = 0$, $k = 0,1,\dots,m-1$.

We get the solution of equation (13) in form:

$$f(x) = I_{1-}^{c-a} x^a I_{1-}^{-a} \psi(x)\,, \quad (14)$$

or the solution in integral form:

$$f(x) = \frac{1}{\Gamma(a-c)} \int_x^1 (u-x)^{a-c-1} u^a \times$$

$$\times \left[\left(-\frac{d}{du}\right)^a \psi(u)\right] du\,. \quad (15)$$

b) Following to [9] we receive solution of the following integral equation:

$$\int_x^\beta \frac{(t^n - x^n)^{c-1}}{\Gamma(c)}\, {}_r\tilde{F}^{\tau,\beta}\left(a,b;c;1-\frac{x^n}{t^n}\right) \times$$

$$\times\, n t^{n-1} f(t)dt = g(x)\,, \quad (16)$$

where $\alpha \le x \le \beta$, $f \in L$.

Here we use the following integral operator:

$$(R(a,b;c;n)f)(x) = \int_x^\beta \frac{(t^n - x^n)^{c-1}}{\Gamma(c)}\, {}_r\tilde{F}^{\tau,\beta} \times$$

$$\times \left(a,b;c;1-\frac{x^n}{t^n}\right) n t^{n-1} f(t)dt \quad (17)$$

$$(x \in [\alpha,\beta], 0 < \alpha < \beta < \infty, \mathrm{Re}(c) > 0)\,.$$

Let us note the main properties of operator $(R(a,b;c;n)f)(x)$:

i) If a,b,c are complex numbers with $\mathrm{Re}(c) > 0$ and $n > 0$, then integral operator $R(a,b;c;n)$ for almost all x in $[\alpha,\beta]$ is bounded linear operator on L into itself;

ii) If $\mathrm{Re}(c) > 0$, $\mathrm{Re}(\mu) > \mathrm{Re}(c)$, $\mathrm{Re}(\mu) > 0$, then the following relation is valid:

$$J^\mu_{x^n} R(a,b;c;n) = R(a,b;c+\mu;n)\,, \quad (18)$$

where

$$(J^\mu_{x^n} f)(x) = \int_x^\beta \frac{(t^n - x^n)^{\mu-1}}{\Gamma(\mu)}\, n t^{n-1} f(t)dt. \quad (19)$$

iii) If $\mathrm{Re}(c) > 0$ and $f \in L$, then for almost all $x \in [\alpha,\beta]$

$$R(a,b;c;n)\ f(x) = J_{x^n}^{c-b} x^{-na}\ J_{x^n}^{b}\ x^{na} f(x) \quad (20)$$

for $\mathrm{Re}(c) > 0$;

$$R(a,b;c;n)\ f(x) =$$

$$x^{-n(a+b-c)} J_{x^n}^{b} x^{n(a-c)}\ J_{x^n}^{c-b}\ x^{nb} f(x), \quad (21)$$

for $\mathrm{Re}\, b < \mathrm{Re}\, c$.

Using (3), (20) we get the next result.

Theorem 3. If $\mathrm{Re}(c) > 0$, $\mathrm{Re}(c) > 0$, $g \in L_c$, then the integral equation (16) has solution f in L given by

$$f(x) = x^{-na} J_{x^n}^{-b} x^{na} J_{x^n}^{b-c}\ g(x), \quad (22)$$

where $L_\mu = \{f : f = J^\mu g,\ g \in L\}$ and $J^\mu g = J_x^\mu g$.

P r o o f. It is immediate from statement

$$J_{x^n}^{-c} R(a,b;c;n) f(x) = f(x) +$$

$$+ nab \int_x^\beta {}_r\tilde{F}\left(a+1, b+1; 2; 1 - \frac{x^n}{t^n}\right) \frac{f(t)}{t} dt,$$

and simple verifications.

Also hold such lemmas:

Lemma 3. If $\mathrm{Re}\,\mu > 0, n > 0$ and $f \in L$ then the integral

$$\int_x^\beta \frac{(t^n - x^n)^{c-1}}{\Gamma(c)}\ {}_r\tilde{F}^{\tau,\beta}\left(a,b;c;1 - \frac{x^n}{t^n}\right) n t^{n-1} f(t)\, dt$$

exists for almost all $x \in [\alpha, \beta]$ and defines a function in L.

Lemma 4. If a, b, c are complex numbers with $\mathrm{Re}(c) > 0$ and $n > 0$, then the integral operator $R(a,b;c;n)$ on L, defined by

$$(R(a,b;c;n)f)(x) = \int_x^\beta \frac{(t^n - x^n)^{c-1}}{\Gamma(c)}\ {}_r\tilde{F}^{\tau,\beta} \times$$

$$\times \left(a, b; c; 1 - \frac{x^n}{t^n}\right) n t^{n-1} f(t) dt$$

for almost all $x \in [\alpha, \beta]$, is bounded linear operator on L into itself.

Conclusions

With the help of the (τ, β)-generalized confluent hypergeometric function the r-hypergeometric function is considered. The main properties of the r-hypergeometric function and important relations, in particular, the composite relation with integral operator of Erdelyi–Kober' type, the relation of Erdelyi' type, the Mellin transform are obtained. Also it is given the representation of the r-hypergeometric function by the fractional integral. Some applications of the r-hypergeometric functions to the solving of integral Volterra' equations in closed form are given. These results can be used to further a deeper application of the r-hypergeometric function for solving new classes of integral equations.

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