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PHYSICS:

MECHANICS:

Laboratory works

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as a study aid for the foreign students
for the specialties 134 Aviation, rocket and space machinery;
173 Avionics; 152 Metrology and information-measurement engineering
of the Faculty of Aerospace Systems

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ANNOTATION
For the educational publication
“Physics: Mechanics: Laboratory works”

Methodical recommendations for laboratory works in Physics for students who study section "Mechanics" in physics and are under the Bachelor’s degree study program the for the specialties 134 Aviation, rocket and space machinery; 173 Avionics; 152 Metrology and information-measurement engineering of the Faculty of Aerospace Systems. It also could be used for other students’ specialties at the National Technical University of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”.

Methodical recommendations for laboratory works in Physics are designed for the foreign students and written in English, they are understandable and at the same time they correspond to the "Physics" course curriculum for the Faculty of Aerospace Systems by the level of material presentation. Methodical recommendations are a practical guide for performing laboratory works in the laboratories of the Faculty of Physics and Mathematics. They provide students with an opportunity to get acquainted with fundamental laws of physics and to verify directly the implementation of these laws in experiments, to form a sufficient level of competence for carrying out physical experiments, processing data and estimating results.

There are laboratory works from the section "Mechanics" in the present publication, namely, such topics as "Study of the experimental data processing methods: the study of the mathematical pendulum oscillations", "Study of the rigid body dynamics on the example of the physical pendulum", "The study of the fundamental law of the dynamics of rotational motion on the Oberbeck pendulum", "Determination of gravitational acceleration by means of a reversible pendulum".

The text of the protocol of each laboratory work is accompanied by necessary explanations, illustrations, tables, description of the experimental setup, the procedure order and processing of the experimental results, control questions.

Laboratory work 1-1

Study of the experimental data processing methods: example of the mathematical pendulum oscillations

Objective: to acquire skills of making a histogram, to study methods for processing of experimental measurements.

Equipment: mathematical pendulum, stopwatch.

1.1. Theoretical information

A rigid body is considered to be a pendulum if it can oscillate about a fixed axis under the action of gravity. There are mathematical and physical pendulums.

Mathematical pendulum (simple gravity pendulum) is an idealized system consisting of a particle-like bob of mass m suspended by a massless, inextensible string of length l and exhibiting oscillatory motion about a point O (Fig. 1.1). The length of the pendulum l must be much larger than the size of the bob so it can be considered as a point mass.

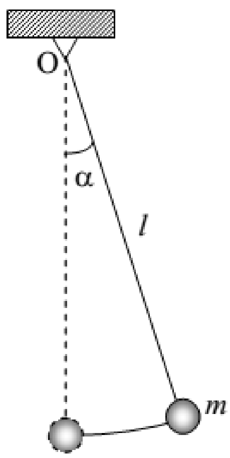

Figure 1.1.

If angular displacement of the pendulum is small and friction in the system can be neglected, then the mathematical pendulum performs harmonic oscillations, the period of which is determined by the length of the pendulum *l* and the acceleration due to gravity *g*:

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

The period of oscillations of the pendulum can be calculated by the formula or measured experimentally using the stopwatch. Measurement of the period of the mathematical pendulum with a stopwatch is a direct measurement. During the experiment the results are influenced by a variety of factors, so they are only a certain approximation to the true value. The best approximation can be obtained by measuring the period several times and determining the mean value $\langle T \rangle$. The degree of approximation is estimated by calculating errors.

See the "Theory of errors” appendix for successful completion of this work.

1.2. Experimental setup and method of measurement

The experimental setup is a heavy bob suspended by an inextensible wire, whose length is much larger than the size of the bob. The time is measured with an electronic stopwatch with 0.001 sec accuracy.

By measuring the time Δt_i of five full oscillations, the value of the oscillation period is obtained by the formula:

$$T_i = \frac{\Delta t_i}{5} \qquad (1.1)$$

The oscillations can be considered as harmonic (subject to the law of cosine or sinus) if the pendulum is displaced by small angles (about 4^0) from the equilibrium position.

1.3. Procedure

1. Prepare tables, similar to Table 1.1 and Table 1.2, which are shown below. (If possible, the tables should be prepared in the MS Exel program, and you can make calculating formulas according to the formulas given below).

2. Bring the pendulum into oscillatory motion. Measure the time of five oscillations using a stopwatch, tabulate results with accuracy of 0.01 s (Table 1.1).

Perform 50 measurements.

Rewrite the same results into the first 50 lines of Table 1.3.

3. Perform another series of 50 measurements; tabulate the data in Table 1.3, starting from the line 51.

4. Record the stopwatch accuracy: σ_t = ...

1.4. Processing of measurements

1. Using the formula (1.1) calculate the oscillation period for each of the 50 measurements with accuracy of 0.001 s. Tabulate the obtained results.

2. Calculate the average value of the period $<T>=\frac{T_1+T_2+...+T_n}{n}=\frac{\sum_{i=1}^{50}T_i}{50}$.

3. Calculate deviation from the mean value for each value of the period $\Delta T_i = T_i - <T>$ and $\left(T_i - <T>\right)^2$. Tabulate results.

4. Perform the actions from items 1 - 3 for the data in Table 1.3.

All calculations should be carried out with the accuracy of 0.001 s.

5. According to the tables 1.1 and 1.3, count the number Δn_i of deviations ΔT_i which lie within intervals of equal width 0.01 s, from -0.10 s to +0.10 s. Tabulate the obtained numbers: Table 1.2 (50 measurements) and Table 1.4 (100 measurements).

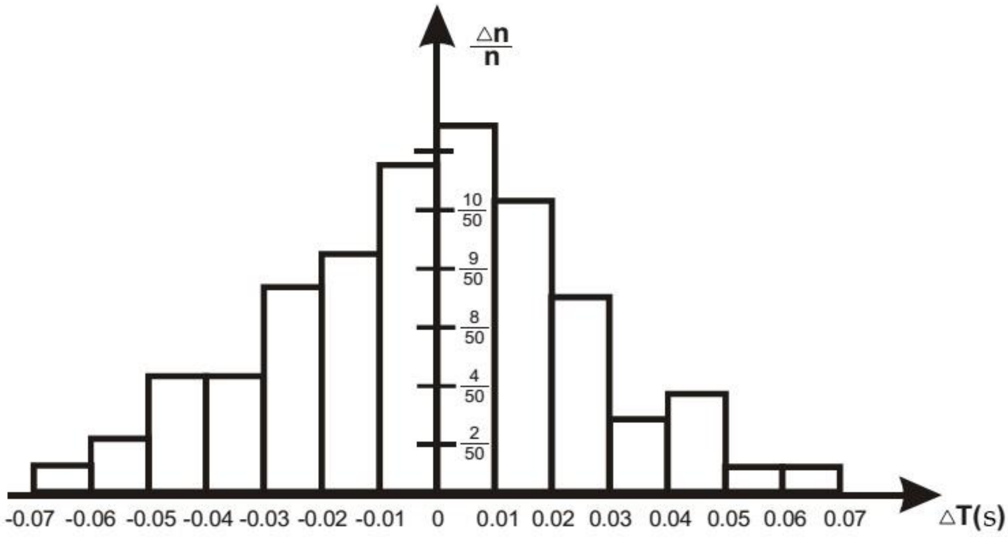

Figure 1.2.

6. Calculate the relative number of values $\Delta n_i / n$ that lie within each of the intervals (1, 2, 3, ... 20) by dividing Δn_i by n = 50 in the table 1.2, and by n = 100 in the table 1.4. Write the obtained results in the table 1.5.

7. Make histograms for the series of 50 and 100 measurements. The vertical axis of the histogram corresponds to the values of $\Delta n_i / n$, while the horizontal axis corresponds to the values of random deviation ΔT_i. An example of the histogram for n = 50 is shown in Fig. 1.2.

8. Calculate the sample standard error $S_{<T>}$ for n = 50 and n = 100 using the formula:

$$S_{<T>} = \sqrt{\frac{\sum_{i=1}^{n}\left(T_i - <T>\right)^2}{n(n-1)}} = \sqrt{\frac{\sum_{i=1}^{n}\Delta T_i^2}{n(n-1)}} .$$

9. Calculate the total standard error due to systematic errors using the formula

$$\sigma_{<T>\Sigma} = \frac{\sigma_{<\Delta T>}}{m} = \frac{\delta}{m\sqrt{12}} ,$$

where δ is the stopwatch accuracy, m = 5 (the number of oscillations).

11. Check the three sigma rule and write the final result in accordance with the instructions from the section “Theory of errors”.

1.5. Control quiz

1. What is the mathematical pendulum?

2. What oscillations are called harmonic?

3. Name the types of measurements and give classification of the measurement errors.

4. How to make a histogram?

5. What is the sample mean of the direct measurements results?

6. Derive the formulas for $\sigma_{<g>\Sigma}$ and $S_{<g>}$.

7. Formulate the law of universal gravitation. How to define the gravitational acceleration g in the present work?

Table 1.1 (example)

n	Time of 5 oscillations Δt_i, s	Period of oscillations $T_i = \dfrac{\Delta t_i}{5}$, s	$\Delta T_i = T_i - \langle T \rangle$, s	ΔT_i^2, s^2
1				
2				
…				
50				
		$\sum_{i=1}^{50} T_i =$		$\sum_{i=1}^{50} \Delta T_i^2 =$

$$\langle T \rangle = \frac{\sum_{i=1}^{50} T_i}{50} =$$

Table 1.2.

$-0.10<\Delta T<-0.09$	$-0.09<\Delta T<-0.08$	$-0.08<\Delta T<-0.07$	$-0.07<\Delta T<-0.06$	$-0.06<\Delta T<-0.05$
1	2	3	4	5
$-0.05<\Delta T<-0.04$	$-0.04<\Delta T<-0.03$	$-0.03<\Delta T<-0.02$	$-0.02<\Delta T<-0.01$	$-0.01<\Delta T<-0.00$
6	7	8	9	10
$0.00<\Delta T<0.01$	$0.01<\Delta T<0.02$	$0.02<\Delta T<0.03$	$0.03<\Delta T<0.04$	$0.04<\Delta T<0.05$
11	12	13	14	15
$0.05<\Delta T<0.06$	$0.06<\Delta T<0.07$	$0.07<\Delta T<0.08$	$0.08<\Delta T<0.09$	$0.09<\Delta T<0.10$
16	17	18	19	20

Table 1.3 (example)

n	Time of 5 oscillations Δt_i, s	Period of oscillations $T_i = \dfrac{\Delta t_i}{5}$, s	$\Delta T_i = T_i - \langle T \rangle$, s	ΔT_i^2, s^2
1				
2				
…				
100				
		$\sum_{i=1}^{100} T_i =$		$\sum_{i=1}^{100} \Delta T_i^2 =$

$$\langle T \rangle = \frac{\sum_{i=1}^{100} T_i}{100} =$$

Table 1.4.

$-0.10<\Delta T<-0.09$	$-0.09<\Delta T<-0.08$	$-0.08<\Delta T<-0.07$	$-0.07<\Delta T<-0.06$	$-0.06<\Delta T<-0.05$
1	2	3	4	5
$-0.05<\Delta T<-0.04$	$-0.04<\Delta T<-0.03$	$-0.03<\Delta T<-0.02$	$-0.02<\Delta T<-0.01$	$-0.01<\Delta T<-0.00$
6	7	8	9	10
$0.00<\Delta T<0.01$	$0.01<\Delta T<0.02$	$0.02<\Delta T<0.03$	$0.03<\Delta T<0.04$	$0.04<\Delta T<0.05$
11	12	13	14	15
$0.05<\Delta T<0.06$	$0.06<\Delta T<0.07$	$0.07<\Delta T<0.08$	$0.08<\Delta T<0.09$	$0.09<\Delta T<0.10$
16	17	18	19	20

Table 1.5.

Intervals of deviations	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Delta n_i / n$ for $n = 50$																				
$\Delta n_i / n$ for $n = 100$																				

Errors:

$S_{<T>} =$ $\sigma_{<T>\Sigma} =$

Final result:

$$T = \_\_\_\_\_\_\_\_\_\_ \pm \_\_\_\_\_\_\_\_\_\_ \text{ s.}$$

Laboratory work 1-2

Study of the rigid body dynamics on the example of physical pendulum

Objective: to study the laws of oscillatory motion on the example of physical pendulum, to determine the gravitational acceleration

Equipment: physical pendulum (homogenous steel rod), ruler, stopwatch.

2.1. Theoretical information

Physical pendulum (or gravity pendulum) is a rigid body, which can freely oscillate by gravity about a fixed horizontal axis. In the present work, the physical pendulum is a homogeneous steel rod of length L. The rod has a measuring scale on it and there is a fixing prism which can be shifted along the rod. The sharp edge of the prism makes the axis of oscillation. By shifting the fixing prism one can vary the distance a between point O representing the rotation axis and point C representing the center of mass of the pendulum (Fig.2.1).

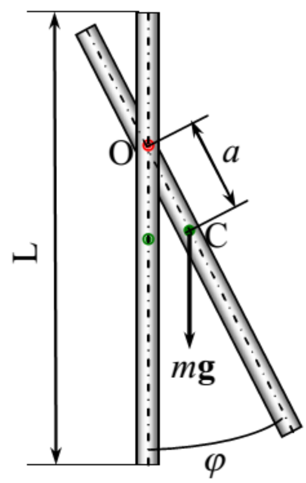

Figure 2.1.

Here we neglect forces of friction and resistance. Then the pendulum motion is determined only by the gravity torque, and its magnitude is:

$$M = mga \cdot \sin\varphi,$$

where a is the distance OC between the rotation axis and the center of mass, φ is the angular displacement of the pendulum. When the pendulum moves away from the equilibrium position (moves to the left in Fig. 2.1), the torque and the angular acceleration vectors have opposite directions, so their projections on the rotation axis have opposite signs. Therefore, the fundamental equation of rotational motion dynamics ($M = I\beta$, where M is the torque, I is the moment of inertia, $\beta = \frac{d^2\varphi}{dt^2} \equiv \varphi''$ is the angular acceleration) can be written as:

$$I\varphi'' = -mga \cdot \sin\varphi \qquad (2.1)$$

In the case of small-angle approximation, when $\sin\varphi \simeq \varphi$, the equation (2.1) can be written as:

$$\varphi'' + \omega_0^2\varphi = 0, \qquad (2.2)$$

where $\omega_0^2 = mga/I$. The solution of this differential equation is the equation of harmonic oscillation

$$\varphi = \varphi_0\cos(\omega_0 t + \alpha), \qquad (2.3)$$

where $\omega_0 = \sqrt{mga/I}$ is the angular frequency, φ_0 is the amplitude, α is the initial phase. It can be easily verified by substituting the function (2.3) into the equation (2.2). The amplitude φ_0 and the initial phase α depend on the way how the pendulum oscillation is started, i.e. are determined by so-called problem's initial conditions – by the initial angular displacement $\varphi_0 = \varphi\ (t = 0)$ and initial angular velocity $\frac{d\varphi}{dt}(t = 0)$.

The period of oscillation $T = 2\pi/\omega_0$ is determined by the pendulum's parameters and by the gravitational acceleration g:

$$T = 2\pi\sqrt{\frac{I}{mga}}\,. \qquad (2.4)$$

Let's denote as I_0 the pendulum's moment of inertia about the axis that passes through the center of mass C and is parallel to the axis of oscillation. According to the Steiner theorem (parallel axis theorem):

$$I = I_0 + ma^2, \qquad (2.5)$$

where I_0 is the moment of inertia about the axis through the center of mass, m is the mass of the body, a is the perpendicular distance between the axes. So, we have

$$T = 2\pi\sqrt{\frac{I_0}{mga} + \frac{a}{g}} \qquad (2.6)$$

Equation (2.6) establishes dependence of the oscillation period T on the distance a between the point of suspension and the center of mass of the pendulum.

Let us explore behavior of the function $T(a)$ at very high ($a \to\infty$) and very small ($a \to 0$) values of a. Obviously, when $a \to\infty$, $T \simeq \sqrt{a/g}$ (the ratio $I_0/mga \to 0$), i.e. $T(a) \sim a^{1/2}$. When $a \to 0$ then $a/g \to 0$ and $T \simeq \sqrt{I_0/mga}$, i.e. $T(a) \sim a^{-1/2}$. So, at $a \to\infty$ the period $T(a) \to\infty$ as $a^{1/2}$, while at $a \to 0$ the period $T(a)$ also tends to infinity, but this time as $a^{-1/2}$. Function (2.6) is continuous on the interval $(0, \infty)$ and approaches infinity at the interval boundaries. Accordingly, it must reach a certain minimum value $a_0(0, \infty)$. Having the function (2.6) analyzed to find its extremum one can easily see that its minimum is $a_0 = \sqrt{I_0/m}$.

Moreover, the formula (2.6) describes dependence $T(a)$ both for "direct" and "reverse" pendulum. All these considerations allow to make a graph $T(a)$, as shown in Fig. 2.2. The graph allows to find the oscillation periods T_1 and T_2 for suspension points O_1 and O_2, for example. The figure shows a thin rod pendulum, but, of course, all the obtained results are applicable to any physical pendulum.

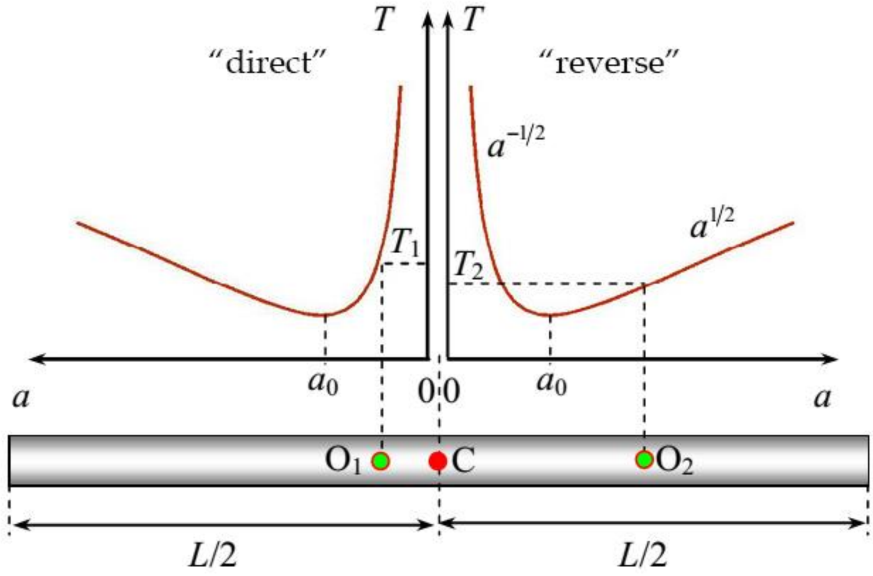

Figure 2.2. Dependence of physical pendulum oscillation period T on the distance a between the point of suspension and the center of mass.

For the homogenous rod $I_0 = (1/12)mL^2$, and the formula (2.6) can be rewritten as follows:

$$T^2a = (4\pi^2/g)a^2 + \pi^2L^2/3g. \qquad (2.7)$$

This allows to simplify the experimental verification of theoretical dependence $T(a)$, reducing it to a simple linear function of variables T^2a and a^2. Graph of the function T^2a vs a^2 is a straight line with a slope

$$k = 4\pi^2/g. \qquad (2.8)$$

The line has axial displacement along the T^2a axis

$$b = \pi^2L^2/3g, \qquad (2.9)$$

as shown in Fig.2.3.

If, having the experimental errors taken into account, the obtained data points fit on a straight line, this is the evidence of fairness of the theoretical dependence (2.6). In this case, one can draw "the best" straight line closest to all the experimental data points (T^2a; a^2) and calculate the slope of this line

$$k = \frac{\Delta(T^2a)}{\Delta a^2},$$

and then the gravitational acceleration g by the formula (2.8).

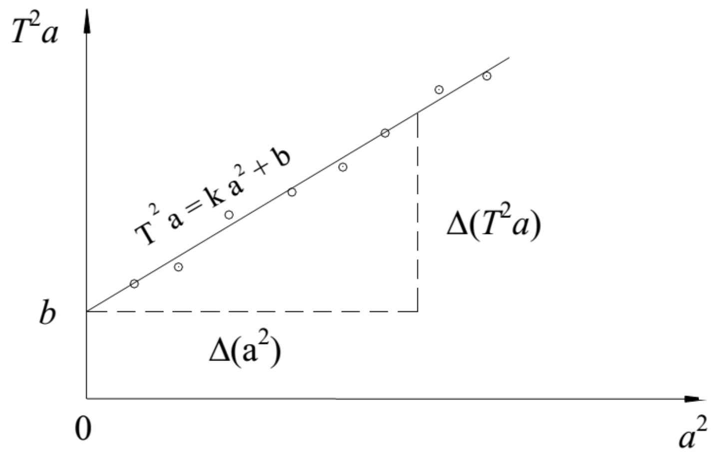

Figure 2.3. Experimental verification of the theoretical dependence $T(a)$.

The points at Fig.2.3 denote values of T^2a and a^2 calculated using the experimentally measured T and a.

The values T and a are received by shifting the suspension point O of the pendulum and measuring corresponding parameters.

2.2. Procedure

1. Review the design of the physical pendulum. Determine the position of the pendulum's center of mass by balancing it on a convenient surface.

2. Fix the prism on the first scale point of the pendulum, i.e. on the maximum distance from the center of mass. Measure the distance a using a ruler.

Bring the pendulum into oscillatory motion with the amplitude not exceeding 10^0 ($\sin\varphi \simeq \varphi$).

Measure the time of 10 complete oscillations at least three times and use the obtained data to calculate the average value of the oscillation period $\langle T \rangle$. Tabulate the experimental data (Table 2.1).

3. Shifting the fixing prism by 2-3 scale points, determine for each value of a the average value of the oscillation period $\langle T \rangle$, as described above in the item 2.

Repeat these measurements for 14-15 positions of the fixing prism.

4. According to the obtained values <T> calculate the values of T^2a and a^2 (taking $T \equiv \langle T \rangle$). Tabulate the calculated results (Table 2.1).

5. Graph the dependence $T(a)$ on a sheet of plotting paper. Define T_{min} and the corresponding value $a = a_0$, which should be compared with theoretical value for the rod pendulum:

$$a_0 = \sqrt{\frac{I_0}{m}} = \frac{L}{\sqrt{12}} = 0.29L\,.$$

6. Plot the experimental points T^2a vs a^2 on the sheet of plotting paper. Draw a straight line closest to all the points. Make a conclusion regarding the fairness of the theoretical dependence $T(a)$.

7. Define the slope k and parameter b of the obtained straight line.

8. According to the formula (2.8) calculate the gravitational acceleration g and compare it with the tabular value.

Using the value of parameter b define the pendulum length L_{ex} and compare it with result of direct measurement of the pendulum length L_{mes} (Table 2.2).

2.3. Control quiz

1. Derive the equation of the physical pendulum motion. What is its solution for the small-angle approximation? (harmonic oscillation)

2. Formulate the fundamental equation of rotational motion dynamics.

3. Show by direct substitution that the function (2.3) is solution of the differential equation (2.2).

4. Which factors determine the amplitude and the initial phase of the oscillation?

5. Formulate and prove the parallel axis theorem.

6. Derive the dependence of the oscillation period T on the distance a between the point of suspension and the center of mass. Consider behavior of $T(a)$ at $a \to \infty$ and $a \to 0$. Show that $T_{\min}$ corresponds to the $a_0 = \sqrt{\frac{I_0}{m}}$.

7. How to verify experimentally the theoretical dependence $T(a)$?

8. How to define the gravitational acceleration in the present work?

Table 2.1

	a, m	T (s) = $t/10$	$<T>$ (s)	a^2 (m^2)	$<T>^2 a$ (s^2·m)
1					
2					
3					
4					
5					
6					
7					
8					
9					

10					
11					
12					
13					
14					
15					

Table 2.2

T_{min} (s) =	k (s^2/m) =
a_0 (m) =	b ($m \cdot s^2$) =
$g_{tab} = 9.8\ m/s^2$; g_{ex} =	L_{ex} (m) = ; L_{mes} (m) =
Error $\varepsilon = (\lvert g_{ex} - g_{tab}\rvert / g_{tab}) \cdot 100\%$ =	

Laboratory work 1-3

The study of the fundamental law of dynamics of rotational motion on the Oberbeck pendulum

Objective: experimental verification of the fundamental equation of dynamics of rotational motion of a rigid body and determination of the moment of inertia

Equipment: Oberbeck pendulum, set of loads, ruler, stopwatch, Vernier caliper.

3.1. Theoretical information

As follows from the fundamental laws of classical mechanics, the fundamental equation of dynamics of rotational motion of a rigid body about a fixed axis is:

$$M_{\Sigma} = I\beta , \qquad (3.1)$$

where I is the moment of inertia about the axis of rotation, β is the angular acceleration, M_{Σ} is the net external torque projected onto the axis of rotation.

Experimental verification of (3.1) is the verification of the fundamental postulates of classical mechanics.

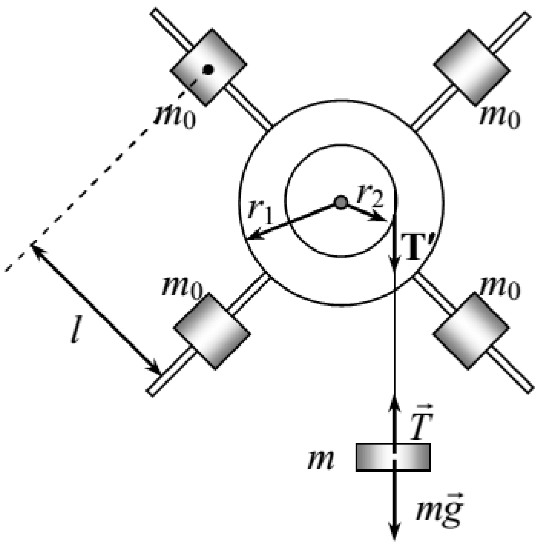

Fig. 3.1 Schematic diagram of the experimental setup

Figure 3.1 shows the scheme of the experimental setup (the Oberbeck cross). It consists of four rods attached to the sleeve at right angles with respect to each other. Two pulleys of different radii r_1 and r_2 are attached to the same sleeve. Four identical weights of the same mass m_0 can be moved along the rods and fixed at different distances l, which enables you to customize the moment of inertia of the system. The whole construction is free to rotate about a horizontal axis passing through the center of the sleeve. Various masses m can be hung on a string wound around the pulley of either of two radii to bring the system into rotation. Position of the weight m is read from a vertical scale.

The weight is under the action of the gravitational force $m\vec{g}$ and the tension force $\vec{T}$, as shown in Fig. 3.1. According to the Newton's second law,

$$m\vec{a} = m\vec{g} + \vec{T}\,, \qquad (3.2)$$

where $\vec{a}$ is acceleration of the weight.

Projection of (3.2) on the direction of the weight’s acceleration (downwards) gives as the equations of motion in scalar form:

$$ma = mg - T\,. \qquad (3.2)$$

According to the Newton's third law, the oppositely directed force $\vec{T}'$ is exerted on the pulley, and its magnitude $T' = T$. The force $\vec{T}'$ provides an external torque acting on the pulley. The magnitude of this torque about the axis of rotation is

$$M = T \cdot r. \qquad (3.4)$$

The equations of motion of the pendulum can be greatly simplified if the pendulum is previously balanced, that is, if it is in the state of neutral equilibrium when no external forces are acting. In this case, the center of mass of the pendulum coincides with the point O located on the axis of rotation, and the gravitational torque about that axis is zero. Then, the pendulum motion is determined by the torque M due to the tension force and the torque M_f due to friction, which allows us to write the fundamental equation of rotational motion (3.1) in the following way:

$$I\beta = M - M_f \,. \tag{3.5}$$

Solving the system of equation (3.3), (3.4), (3.5) and using the known relationship between the angular and linear acceleration

$$\beta = a/r, \tag{3.6}$$

we obtain:

$$a = \frac{mgr - M_f}{I}\,. \tag{3.7}$$

The friction torque can be considered as constant during motion. In this case, expression (3.7) means that motion of the weight is uniformly accelerated (a = const).

By measuring time t, during which the weight moves from rest at zero height to the distance h, it is possible to experimentally determine angular acceleration of the pendulum. As $h = at^2/2$, we obtain:

$$\beta = \frac{2h}{rt^2}\,. \tag{3.8}$$

From equations (3.3) and (3.4) we obtain the tension torque about the axis of rotation:

$$M = m(g - a)r\,. \tag{3.9}$$

(Note that β and M can be determined using equations (3.8) and (3.9) which are obtained independently from the main equation (3.5)).

Let's rewrite equation (3.5) in a more convenient form:

$$M = M_f + I\beta\,. \tag{3.10}$$

This expression means that the dependence $M(\beta)$ is a straight line with an angular coefficient k, which is numerically equal to the moment of inertia of the system:

$$I = k = \frac{\Delta M}{\Delta\beta}, \tag{3.11}$$

while the intersection point of this line and the M-axis on the graph corresponds to the magnitude of the friction torque M_f, see Figure 3.2.

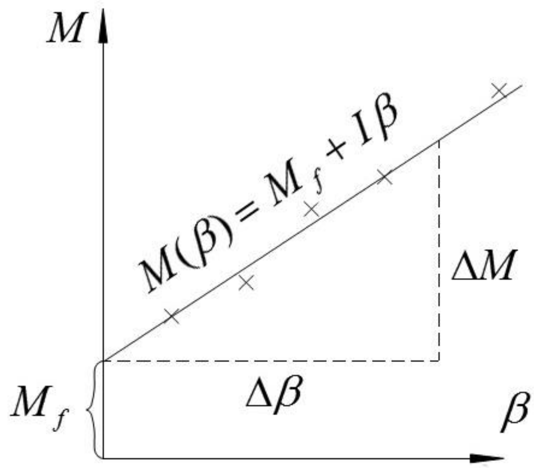

Figure 3.2. Dependence of the torque on the angular acceleration. Deviations of the experimental points (M_i, β_i) from the line $M(\beta) = M_f + I\beta$ lay within the experimental error.

Having the set of weights of different masses, you can widely vary the mass m, and, hence, the values of M and β, i.e. you can measure the experimental dependence $M(\beta)$. If the obtained experimental points (M_i, β_i) (with the experimental error taken into account) fit a straight line, then it is the confirmation of fairness of the expression (3.10), and therefore, of the fundamental equation of dynamics of rotational motion (3.1). In this case, a straight line closest to all the (M_i, β_i) can be plotted through the experimental points. It allows to determine the friction torque and to calculated the moment of inertia of the system using the formula (3.11) (Fig. 3.2).

3.2. Procedure

1. Review design of the Oberbeck pendulum. (Before the experiment, verify if the pendulum rotates about the axis sufficiently freely. Make sure that the screw that fixates the sleeve does not tighten during the pendulum rotation. Otherwise, you will not obtain agreement with the theory, because the pendulum motion will be affected by additional forces and torques, and the equation of motion will be complicated)

2. Position the weights m_0 at some distance L from the axis of rotation (preferably, in the first experiment take the maximum distance $L = L_{max}$). Check if the pendulum is

balanced. To do that, turn it by an angle ≈ 45^0. The pendulum must remain at rest in every new position.

3. Wind the string with the weight m_1 hanging from it around the pulley of larger radius ($r = r_1$).

4. Release the weight and measure the time it takes to fall over a distance $h = 1$ m. Tabulate the values of mass of the weight, radius of the pulley, time (Table 3.1). Repeat three times measurements of time and determine the average value.

5. Repeat such experiment for 5 – 6 different values of the mass m, each time adding another weight or combining them. Tabulate all the data measured.

6. Fulfil actions of the items 4 – 6 for the smaller radius of the pulley ($r = r_2$) with the value of $L = L_{max}$ remaining the same. Tabulate the obtained results (Table 3.1).

7. Change the moment of inertia I of the system by positioning the weights m_0 at the minimum distance from the axis of rotation ($L = L_{min}$). Repeat the experiments described in the items 3 – 6. Tabulate the obtained results (Table 3.2).

3.3. Processing of measurements

1. For each of the experiments calculate the angular acceleration β (formula (3.8)) substituting the average value of time $<t>$. Tabulate the obtained results.

2. For each of the experiments with different weights m calculate the tension torque. As $a << g$, you can use the approximate formula instead of the formula (3.9):

$$M \approx mgr . \tag{3.12}$$

Tabulate the obtained results.

3. For each of the values of the moment of inertia graph the dependence $M(\beta)$ on a sheet of plotting paper. From these graphs define the values of the friction torque M_f and the moment of inertia I (formula 3.11, see Fig.3.2). Compare the obtained results. Find the average value of M_f and average values of I_{max} and I_{min}.

4. Estimate the experimental errors for β and one of the values of the tension torque M. To do that, use the formulas given by the "Theory of errors"

$$\left(\frac{\sigma_\beta}{\beta}\right)^2 = \left(\frac{\sigma_h}{h}\right)^2 + \left(\frac{\sigma_r}{r}\right)^2 + 4\left(\frac{\sigma_t}{t}\right)^2; \tag{3.13}$$

$$\frac{S_{<\beta>}}{\beta} = 2\frac{S_{<t>}}{t}; \tag{3.14}$$

$$\left(\frac{\sigma_M}{M}\right)^2 = \left(\frac{\sigma_m}{m}\right)^2 + \left(\frac{\sigma_g}{g}\right)^2 + \left(\frac{\sigma_r}{r}\right)^2, \tag{3.15}$$

where $S_{<\beta>}$ and $S_{<t>}$ are standard errors of corresponding average values; σ_β, σ_h, …, σ_r are systematic errors of the β, h, … , r.

5. On one of the experimental graphs plot the values $\sigma_{<\beta>} = \sqrt{S^2_{<\beta>} + \sigma^2_{<\beta>}}$ and σ_M, which characterize the experimental error as shown in Fig.3.2. Make a conclusion regarding the fairness of (3.10) within the experimental error. Tabulate the data for error calculations (Table 3.3).

6. Calculate the average value of the moment of inertia $<I_{max}>$ using the two values from the Table 3.1, $<I_{min}>$ using the two values from the Table 3.2, and in the same way the average values of the friction torque:

$<I_{max}> =$ $<I_{min}> =$

$< M_f > =$

Table 3.1.

	$L = L_{max}$									
i	$r = r_1 =$ (m)					$r = r_2 =$ (m)				
	m (kg) $\cdot 10^{-3}$	*M*, N·m	*t*, s	<*t*>, s	*β*, rad/s	*m* (kg) $\cdot 10^{-3}$	*M*, N·m	*t*, s	<*t*>, s	*β*, rad/s
1										
2										
3										
4										
5										
6										
	$M_f =$ (N·m) $I_{max} =$ (kg·m^2)					$M_f =$ (N·m) $I_{max} =$ (kg·m^2)				

Table 3.2.

	$L = L_{min}$									
i	$r = r_1 =$ (m)					$r = r_2 =$ (m)				
	m (kg) $\cdot 10^{-3}$	M, N·m	t, s	$\langle t \rangle$, s	β, rad/s	m (kg) $\cdot 10^{-3}$	M, N·m	t, s	$\langle t \rangle$, s	β, rad/s
1										
2										
3										
4										
5										
6										
	$M_f =$ (N·m) $I_{min} =$ (kg·m^2)					$M_f =$ (N·m) $I_{min} =$ (kg·m^2)				

Table 3.3.

$\sigma_t =$	$\sigma_m =$	$\sigma_g =$	$\sigma_\beta =$
$\sigma_h =$	$\sigma_r =$	$\sigma_M =$	$\sigma_{\langle\beta\rangle} =$
$S_{\langle t\rangle} =$	$S_{\langle\beta\rangle} =$	$\sigma_M / M =$	$\sigma_\beta / \beta =$

3.4. Control quiz

1. Torque and angular momentum of a system of material points about a point *O*. Relationship between them: fundamental equation of dynamics of rotational motion for the system of material points.
2. The law of conservation of angular momentum for a system of material points.
3. Angular momentum and torque about an axis. Fundamental equation of dynamics of rotational motion about a fixed axis.
4. Moment of inertia of a rigid body about a fixed axis. Parallel axes theorem.
5. What method is used for definition of the moment of inertia of the pendulum in this work? What does the moment of inertia of the pendulum depend on?
6. How to define the friction torque from the $M(\beta)$ plot?
7. How to estimate the experimental error?

Laboratory work 1-4

Determination of gravitational acceleration by means of reversible pendulum

Objective: to study the reversible pendulum, to determine the gravitational acceleration

Equipment: reversible pendulum, electronic stopwatch, ruler.

4.1. Theoretical information

Physical pendulum (or gravity pendulum) is a rigid body, which can oscillate by gravity about a fixed horizontal axis (Fig.4.1). Point *O*, where the horizontal axis and the vertical plane passing through the pendulum's center of mass *C* intersect, is called the point of suspension. The pendulum's displacement from the equilibrium position is characterized by the displacement angle φ.

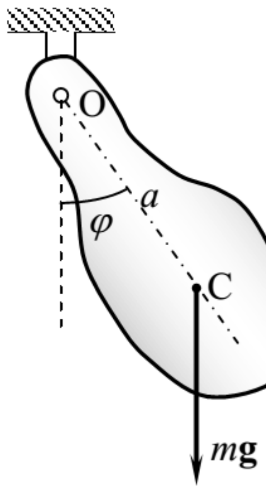

Figure 4.1

We assume that torques of friction and resistance can be neglected. Then the pendulum motion is determined only by gravity torque, and its magnitude is:

$M = mga \cdot \sin\varphi,$

where a is the distance OC between the point of suspension and the center of mass.

Applying the fundamental equation of rotational motion dynamics, we can write:

$$I\varphi'' = -mga \cdot \sin\varphi, \tag{4.1}$$

where I is the moment of inertia about the axis O, $\varphi'' = \dfrac{d^2\varphi}{dt^2}$ is the angular acceleration.

In the case of small-angle approximation, when angular displacement φ is small and $\sin\varphi \simeq \varphi$, the equation (4.1) can be written as:

$$\varphi'' + \omega_0^2\varphi = 0, \tag{4.2}$$

where $\omega_0^2 = mga/I$. The solution of this differential equation is harmonic oscillation with angular frequency $\omega_0 = \sqrt{mga/I}$, and equation of motion is given by the formula

$$\varphi = \varphi_0\cos(\omega_0 t + \alpha), \tag{4.3}$$

where φ is the angular displacement, φ_0 is the amplitude, α is the initial phase.

The period of oscillation:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I}{mga}}. \tag{4.4}$$

If the pendulum's moment of inertia about the axis through the center of mass C and parallel to the axis of oscillation O equals I_0 then according to the Steiner theorem (parallel axis theorem):

$$I = I_0 + ma^2, \tag{4.5}$$

Let's substitute this expression into the formula (4.4):

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I_0 + ma^2}{mga}} = 2\pi\sqrt{\frac{I_0}{mga} + \frac{a}{g}}. \tag{4.6}$$

Equation (4.6) allows to study behavior of the function $T(a)$ at very high ($a \to \infty$) and very small ($a \to 0$) values of a. Obviously, when $a \to \infty$ $T \simeq \sqrt{a/g}$ (ratio

$I_0/mga \to 0$), i.e. $T(a) \sim a^{1/2}$. When $a \to 0$ then $a/g \to 0$ and $T \simeq \sqrt{I_0/mga}$, i.e. $T(a) \sim a^{-1/2}$. So, at $a \to \infty$ the period $T(a) \to \infty$ as $a^{1/2}$, while at $a \to 0$ the period $T(a)$ also approaches infinity, but this time as $a^{-1/2}$.

Function (4.6) is continuous on the interval $(0, \infty)$ and approaches infinity at the interval boundaries. Accordingly, it must attain a certain minimum value T_{min} at a_0 $(0, \infty)$. Moreover, the formula (4.6) describes dependence $T(a)$ both for "direct" and "reverse" pendulum. All these considerations allow to make a graph $T(a)$, as shown in Fig. 4.2.

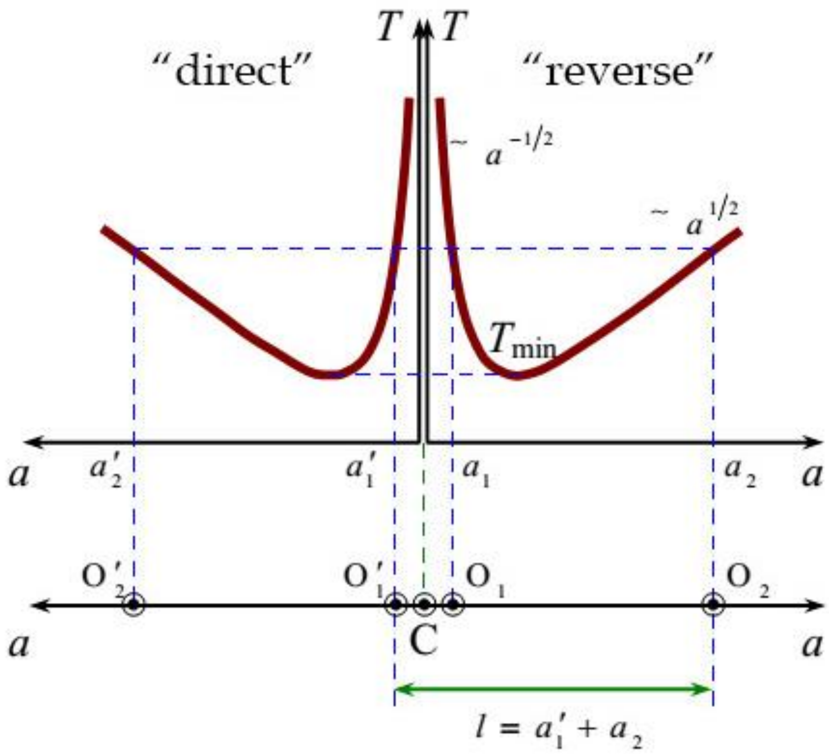

Figure 4.2. Dependence of the physical pendulum oscillation period T on the distance a between the point of suspension and the center of mass.

Equation (4.6) allows to determine experimentally the gravitational acceleration g. Indeed, having the pendulum suspended at different distances a_1 and a_2 from the center of mass, one can measure corresponding periods of oscillation T_1 and T_2. Using the formula (4.6), we obtain the system of equations:

$$T_1^2 = 4\pi^2\left[(I_0 + ma_1^2)/mga_1\right];$$

$$T_2^2 = 4\pi^2\left[(I_0 + ma_2^2)/mga_2\right].$$

Excluding the I_0, we obtain:

$$g = 4\pi^2(a_1^2 - a_2^2)/(a_1T_1^2 - a_2T_2^2)\,. \tag{4.7}$$

Axes T should be considered as coincident: $a_1 = a'_1$; $a_2 = a'_2$ (see Fig. 4.2). So, equal values of the period T (for $T > T_{min}$) are attained for the pendulum suspended in points O_1, O_2, O'_1, O'_2.

However, formula (4.7) can be simplified significantly. Suppose that we've managed to find positions of points O_2 and O'_1 located on opposite sides from the center of mass (see. Fig.4.2). In this case, $T_1 = T_2 = T$, and the formula (4.7) takes a simpler form:

$$g = 4\pi^2 l/T^2, \tag{4.8}$$

where $l = a'_1 + a_2$.

All the values included in the formula (4.8) can be easily measured with great accuracy. The biggest difficulty is to determine the point of suspension, in which periods of "direct" and "reverse" pendulums coincide (that's why it's called reversible).

There are many different designs of the reversible pendulum, one of them is shown in Fig.4.3. There are two fixing prisms (Π_1 and Π_2) fastened on a steel rod and two bobs (Γ_1 and Γ_2). By moving the bobs one can vary the period of oscillation in a wide range. The rod has scale allowing to evaluate position of the movable elements of the construction. Fig.4.4 illustrates their impact on the periods T_1 and T_2. Shifting of the prism Π_2 has greater influence on the period T_2 than shifting of the prism Π_1 has on the period T_1. Position of the center of mass is almost unchanged since the prisms are quite lightweight. However, a slight shift of the bob Γ_2 in the direction of the arrow (see Fig. 4.4) results in a significant shift of the center of mass C. This means that distance a_2 increases while distance a_1 decreases by the same value. The both periods decrease, however T_2 decreases much faster, so the values of T_1 and T_2 can be equalized. The difference of periods T_1 and T_2 will lie within a random spread of results of repeated measurements. This allows to consider a set of values T_1 and T_2 as a single data sample of T and to calculate corresponding mean value $\langle T \rangle$ and sample standard error $S_{<T>}$. In fact, we consider $T_1 - T_2$ as random (stochastic) error.

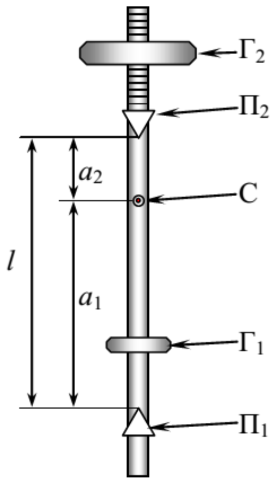

Figure 4.3. Construction of the reversible pendulum

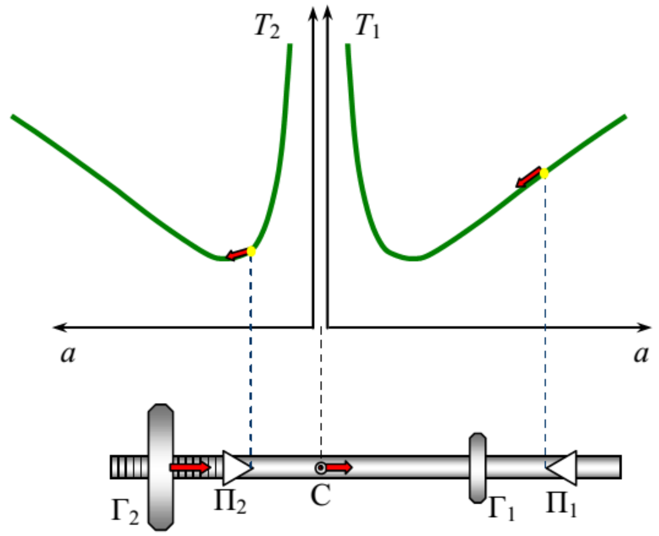

Figure 4.4. The influence of various elements translocation on the values of periods T_1 and T_2. C denotes the center of mass position (here Π_1, Π_2 are movable fixing prisms, Γ_1, Γ_2 are movable bobs)

Let's see how random error affects the error for g. For that purpose, we should use formula (4.7) instead of (4.8), which does not consider differences in periods. Corresponding calculations give following result:

$$S_{<g>} = \sqrt{\left(\frac{\partial g}{\partial T_1} S_{<T>}\right)^2 + \left(\frac{\partial g}{\partial T_2} S_{<T>}\right)^2} = \frac{8\pi^2 l\sqrt{a_1^2 + a_2^2}}{|a_1 - a_2| T^3} S_{<T>},$$

where $S_{<g>}$ is sample standard error for g. The expression for relative error is quite simple:

$$\frac{S_{<g>}}{g} = \frac{2\sqrt{a_1^2 + a_2^2}}{|a_1 - a_2|} \frac{S_{<T>}}{T}. \qquad (4.9)$$

Similarly, the systematic error is calculated:

$$\frac{\sigma_g}{g} = \sqrt{\left(\frac{\sigma_l}{l}\right)^2 + \frac{4(a_1^2 + a_2^2)}{(a_1 - a_2)^2}\left(\frac{\sigma_T}{T}\right)^2 + 4\left(\frac{\sigma_\pi}{\pi}\right)^2}, \qquad (4.10)$$

where σ_l, σ_T, σ_π are systematic errors for l, T, π. These expressions show that the relative error for g increases indefinitely if difference $a_1 - a_2$ tends to zero, i.e. if $T \to T_{\min}$ (Fig. 4. 2). Therefore, the experiment should be planned in a way when a_1 and a_2 differ significantly from each other. However, it is easy to show that large difference $a_1 - a_2$ leads to faster oscillation damping, resulting decrease in accuracy of the period measurement. Satisfactory results can be obtained if we choose

$$3 > a_1/a_2 > 1.5.$$

4.2. Procedure

1. Review design of the reversible pendulum. Place the bob Γ_2 at the closest point to the prism Π_2.

2. Suspend the pendulum on one of the fixing prisms and bring it into oscillatory motion. The amplitude must not exceed 10^0. Measure time of 10 oscillations t_1. Repeat this measurement two more times. Calculate the mean value of time $<t_1>$. Tabulate results of the measurements (Table. 4.1).

3. Reverse the pendulum, suspend it on the other prism and measure time of 10 oscillations t_2 three times. Calculate the mean value of time $<t_2>$. Tabulate results of the measurements (Table. 4.1).

4. Shift the bob Γ_2 along the scale on the rod by 1 or 2 points and measure times t_1 and t_2 as described above in the items 2, 3. Calculate the mean values of time $<t_1>$ and $<t_2>$ for different positions of Γ_2.

5. Graph the dependences $<t_1>$ vs n and $<t_2>$ vs n on the same diagram, where n is the number of scale points. The intersection of these two curves determines the optimal position of the bob Γ_2, for which the values of period $T_1 = <t_1>/10$ and $T_2 =<t_2>/10$ are the closest. Mark the intersection point as n_0.

6. Suspend the pendulum on the prism Π_2 and fix the bob Γ_2 in the position n_0. Bring the pendulum into oscillatory motion with angular displacement within 10° and measure time t of 50 oscillations. Repeat this measurement two more times.

7. Suspend the pendulum on the prism $П_1$ without changing position of the bob $Г_2$. Repeat measurements of time t of 50 oscillations (three times) (see. item 6). Tabulate these measurements (table 4.2.).

8. For each of the six series of measurements determine the oscillation period T. Calculate the mean value of the period $<T>$.

9. Measure parameter l (distance between the prisms $П_1$ and $П_2$).

10. According to the formula (4.8) determine the gravitational acceleration $<g>$ by substituting the mean value $<T>$ of the period.

11. Estimate the experimental error using expressions (4.9) and (4.10).

Experimental parameters:

l (m) = ; a_1 (m) = ; a_2 (m) = ;

σ_l = ; σ_T = ; σ_π =

Table 4.1

n	For the prism $П_1$		For the prism $П_2$	
	t_1, s	$<t_1>$, s	t_2, s	$<t_2>$, s

Table 4.2

	Time of 50 oscillations, s	Period T, s	T_i - <T>, s	(T_i - <T>)^2, s^2
	The prism Π_1			
1				
2				
3				
	The prism Π_2			
1				
2				
3				
$<T>=\frac{\sum_{i=1}^{6} T_i}{6}=$				$\sum_{i=1}^{6}(T_i-<T>)^2=$

Errors:

$$S_{<T>} = \sqrt{\frac{\sum_{i=1}^{6}(T_i - <T>)^2}{6 \cdot 5}} =$$

$$\frac{S_{<g>}}{g} \cdot 100\% = \qquad ; \frac{\sigma_g}{g} \cdot 100\% =$$

Final results:

$<g> =$; $S_{<g>} =$; $\sigma_{<g>} =$

4.3. Control quiz

1. Torque and angular momentum of a system of material points about a point O. Relationship between them: fundamental equation of dynamics of rotational motion for the system of material points.
2. The law of conservation of angular momentum for a system of material points.
3. Angular momentum and torque about an axis. Fundamental equation of dynamics of rotational motion about a fixed axis.
4. Moment of inertia of a rigid body about a fixed axis. Parallel axes theorem.
5. Equations of motion of the physical pendulum. Its solution for small angular displacements: harmonic oscillations.
6. Dependence of the oscillation period T on the distance a between the center of mass and the point of suspension.
7. The method for measuring the gravitational acceleration g by means of the reversible pendulum.
8. How should the experiment be planned in order to ensure the minimum error for g?

Appendix: Theory of errors

References:

- Sante R. Scuro, Visual Physics Laboratory, Texas A&M University, College Station, TX 77843
- J.R. Taylor, An Introduction to Error Analysis (University Science Books, Mill Valley, California, 1982)

The measurement of a physical quantity can never be made with perfect accuracy, there will always be some error or uncertainty present. For any measurement there are an infinite number of factors that can cause a value obtained experimentally to deviate from the true (theoretical) value. Therefore, when experimental results are reported, they are accompanied by an estimate of the experimental error, called the uncertainty. This uncertainty indicates how reliable the experimenter believes the results to be.

A1. Types of Errors

There are many different types of errors that can occur in an experiment, but they will generally fall into one of two categories: random errors or systematic errors.

A1.1. Random errors

Random errors usually result from human and from accidental errors. Accidental errors are brought about by changing experimental conditions that are beyond the control of the experimenter; examples are vibrations in the equipment, changes in the humidity, fluctuating temperatures, etc. By their very nature, random errors cannot be quantified exactly since the magnitude of the random errors and their effect on the experimental values is different for every repetition of the experiment. For example, while using a stopwatch to measure time intervals, the major uncertainty is usually not from reading the dial, but from our reaction time in starting and stopping the watch. Most likely you'll obtain somewhat different values for every one of your measurements.

So, statistical methods are usually used to obtain an estimate of the random errors in the experiment.

A1.2. Systematic errors

A systematic error is an error that will occur consistently in only one direction each time the experiment is performed, i.e., the value of the measurement will always be greater (or lesser) than the real value. So, systematic errors always bias results in one specific direction.

Uncertainty in a measurement can arise from different origins: the measuring device, the procedure of how you measure, and the observed quantity itself. Usually the largest of these will determine the uncertainty in your data. Proper calibration and adjustment of the equipment will help reduce the systematic errors.

However, almost all direct measurements involve reading a scale (ruler, caliper, stopwatch, analog voltmeter, etc.) or a digital display. One of the unavoidable sources of errors is a reading error which refers to the uncertainties caused by the limitations of our measuring equipment and/or our own limitations at the time of measurement.

The uncertainty associated with the reading of the scale and the need to interpolate between scale markings is relatively easy to estimate. For example, consider the millimeter (mm) markings on a ruler scale. For a person with a normal vision it is reasonable to say that the length could be read to the nearest millimeter at best. Therefore, a reasonable estimate of the uncertainty in this case would be $\Delta l = \pm 0.5$ mm. A rule for evaluating the reading error on analogue readout is to use half of the smallest division.

For instruments that have a digital readout, you may assume that the reading error is ± 1/2 of the last digit displayed; e.g. if the reading of the timer in the free fall experiment is 682.6 ms, the error can be assumed to be ±0.05 ms, and you should quote (682.60±0.05) ms.

A2. Estimating uncertainties in repeating measurements

Suppose that we want to measure the value of a quantity, x. Let us imagine that we do this by making a total number of n measurements of x, the values of each measurement being denoted by x_i. The "correct" measured value lies somewhere between the lowest value and the biggest value. We assume that the best estimate of the measured value x_{best} is the **<u>average</u>** or **<u>mean value</u>** of x:

$$< x >= \frac{x_1 + x_2 + ... + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i \,. \qquad \text{(a1)}$$

The **<u>residual</u>** (or the **<u>absolute error</u>**) of an observed value is the difference between the observed value and the best estimated value of the quantity of interest:

$$\Delta x_i = x_i - < x > . \qquad \text{(a2)}$$

The absolute error indicates the reliability of the measurement, but the quality of the measurement also depends on the value of x_{best}. For example, an uncertainty of 1 cm in a distance of 1 km would indicate an unusually precise measurement, whereas the same uncertainty of 1 cm in a distance of 10 cm would result in a crude estimate. **<u>Fractional (relative) error</u>** gives us an indication how reliable our experiment is. Fractional uncertainty is defined as

$$\varepsilon = \left|\frac{\Delta x}{x}\right| , \qquad \text{(a3)}$$

where Δx is the absolute uncertainty, x is the best estimated value. If the quantity measured has a true (theoretical) value, then x is the true value. Fractional uncertainty can be also represented in percentile form $\varepsilon = \left|\frac{\Delta x}{x_{best}}\right| \cdot 100\%$.

Note that the fractional uncertainty is a dimensionless quantity. Fractional uncertainties of about 10% or so are usually characteristic of rather rough measurements. Fractional uncertainties of 1 or 2% indicate fairly accurate measurements.

The data $x_1, x_2, ..., x_n$ are dispersed around the mean value. Usually it is also required the measure of the spread in the values of x_i. A measure of this dispersion is called the **standard deviation** and is given by

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\Delta x_i^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - <x>)^2}\ . \qquad \text{(a4)}$$

The smaller the standard deviation is, the more closely the data is grouped about the mean.

S is associated with the error in each individual measurement x_i. However, what we really need to know is the error in our best estimate of x, which is the mean value. Clearly, this error is less than S, the error in any individual measurement. A standard deviation S of individual measurements divided by the square root of the total number of measurements is called the **standard deviation of the mean** (**standard error**) $S_{<x>}$:

$$S_{<x>} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}\Delta x_i^2} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - <x>)^2}\ . \qquad \text{(a5)}$$

Note that we could reduce the standard error by taking more measurements. However there is no point in taking more than the number required to reduce the standard error to below the reading error. So, typically the number of measurements n is from 3 to 10. The uncertainty in any measured quantity has the same dimensions as the measured quantity.

In general, the result of any measurement of physical quantity must include both the value itself (the best estimate $<x>$ of what we believe is a true value of the physical quantity) and its error (experimental uncertainty Δx_{exp}), given by the standard error.

That is, the true value of x probably lies between ($<x> - \Delta x_{exp}$) and ($<x> + \Delta x_{exp}$). This interval gives a probable range. It is certainly possible that the correct value lies slightly outside this range. That's why this interval is called the **confidence interval**, and the desired level of confidence is set by the researcher establishing the probability of that we are reasonably confident that the actual quantity lies within this interval. The half-width of confidence interval

$$\Delta x_{\exp} = t_{\alpha,n} \cdot S_{<x>}, \qquad \text{(a6)}$$

where $S_{<x>}$ is the standard error, $t_{\alpha,n}$ is the so-called **<u>Student's coefficient</u>** depending on the size of the sample *n* and the confidence level *α* (see Table a1). Most commonly, the 95% confidence level is used.

Table a1

Confidence level *α*	Number of measurements *n*								
	3	5	7	8	10	20	40	60	100
0.8	1.89	1.53	1.44	1.42	1.38	1.33	1.30	1.30	1.29
0.9	2.92	2.13	1.94	1.89	1.73	1.73	1.68	1.67	1.66
0.95	4.3	2.78	2.45	2.36	2.26	2.09	2.02	2.00	1.98

So, the final result of measurement of the physical quantity must be written in the form

$$< x > - t_{\alpha,n} \cdot S_{<x>} \le \; x \; \le \; < x > - t_{\alpha,n} \cdot S_{<x>}, \text{ or}$$

$$x = < x > \pm t_{\alpha,n} \cdot S_{<x>}, \qquad \text{(a7)}$$

which means that the true value of *x* lies within this interval with the probability (confidence level) *α*.

A3. Propagation of errors

In the majority of experiments the quantity of interest is not measured directly, but must be calculated from other quantities. Such measurements are called indirect. Since all measurements have uncertainties associated with them, clearly any calculated quantity will have an uncertainty that is related to the uncertainties of the direct measurements. The procedure used to estimate the error for the calculated quantities is called the propagation of errors.

Let us consider how to calculate errors in case of indirect measurements.

Suppose the variables *A, B, C,...* represent independent measurable quantities that will be used to obtain a value for some calculated quantity *U*. Since *U* is a function of *A, B, C,...*, it can be written as *U*=f(*A, B, C,...*). The measurements of the quantities *A, B, C,...* yield estimates of the expectation values (the mean values) <*A*>, <*B*>, <*C*>, … and the associated standard errors $S_{<A>}, S_{<B>}, S_{<C>}, ...$ for each variable. To find the expectation value, or best estimate, for the quantity *U*, the expectation value of each measured variable is substituted into the equation for *U*:

$$<U>=f(<A>, <B>, <C>,...). \quad \text{(a8)}$$

If the errors for *A, B, C,...* are independent, random, and sufficiently small, it can be shown from the differential calculus that the uncertainty for *U* is given by

$$S_{<U>} = \sqrt{\left(\frac{\partial U}{\partial A}\right)^2 S_{<A>}{}^2 + \left(\frac{\partial U}{\partial B}\right)^2 S_{<B>}{}^2 + \left(\frac{\partial U}{\partial C}\right)^2 S_{<C>}{}^2 + \dots}, \quad \text{(a9)}$$

where the partial derivatives are evaluated using the expectation values <*A*>, <*B*>, <*C*>, … as the values for the independent variables. The correct notation to express the final estimate for the calculated quantity *U* is given by

$$U = <U> \pm t_{\alpha,n} \cdot S_{<U>} \text{ with confidence level } \alpha. \quad \text{(a10)}$$

The total systematic error in the case of indirect measurements can be estimated by the similar formula:

$$\sigma_{<U>\Sigma} = \sqrt{\left(\frac{\partial U}{\partial A}\right)^2 \sigma_{<A>\Sigma}{}^2 + \left(\frac{\partial U}{\partial B}\right)^2 \sigma_{<B>\Sigma}{}^2 + \left(\frac{\partial U}{\partial C}\right)^2 \sigma_{<C>\Sigma}{}^2 + \dots}, \quad \text{(a11)}$$

where $\sigma_{<A>\Sigma}, \sigma_{<B>\Sigma}, \sigma_{<C>\Sigma}$, … are systematic errors for the corresponding values. Usually, they are determined by the reading error:

$$\sigma_{<A>\Sigma} = \frac{\delta}{\sqrt{12}}, \ldots \quad \text{(a12)}$$

where δ is the reading error for A.

A4. Comparing systematic and random errors

Note that depending on the type of the experiment the prevailing error could be random or reading error. Which one is the actual error of precision in the quantity? For practical purposes, the "3σ" criterion is used.

– If $S_{<U>} > 3\sigma_{\Sigma}$, then the reading error may be neglected, and the standard error should be used as the error.

– If $\sigma_{\Sigma} > 3S_{<U>}$, then the random error may be neglected, and the systematic error should be used as the error:

$$U = <U> \pm \gamma_{\alpha} \cdot \sigma_{\Sigma}, \quad \text{(a13)}$$

where γ_{α} is the Chebyshev coefficient depending on the confidence level α.

γ_{α} = 1.8 when α = 0.7;

γ_{α} = 2.2 when α = 0.8;

γ_{α} = 3.2 when α = 0.9.

In case the reading error and random error are comparable in value $\sigma_{\Sigma} \approx S_{<U>}$, both should be taken into account and treated as two independent errors.

Be aware that if the dominant source of error is the reading error, taking multiple measurements will not improve the precision.

A5. Practical hints

Let us denote as Δy and various Δx either standard deviations, standard errors or systematic errors, depending on the circumstances.

Rule 1: If two mutually independent quantities are being added or subtracted

$y = x_1 + x_2$ or $y = x_1 - x_2$, then $\frac{\partial y}{\partial x_1} = 1$, $\frac{\partial y}{\partial x_2} = 1$, and

$$\Delta y = \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}\ . \tag{a14}$$

Rule 2: If two mutually independent quantities are being multiplied or divided

$y = x_1 \cdot x_2$ or $y = x_1 / x_2$, then

$$\frac{\Delta y}{y} = \sqrt{\left(\frac{\Delta x_1}{x_1}\right)^2 + \left(\frac{\Delta x_2}{x_2}\right)^2}\ . \tag{a15}$$

Rule 3: If a quantity is raised to a power

$y = x^n$ and if Δx is small, then $\Delta y \approx n x^{n-1} \Delta x$, therefore

$$\frac{\Delta y}{y} \approx n \frac{\Delta x}{x}\ . \tag{a16}$$

Example:

Consider $y = \frac{A \cdot B^2}{C}$. According to the Rule 2,

$$\frac{\Delta y}{y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta\left(B^2\right)}{B^2}\right)^2 + \left(\frac{\Delta C}{C}\right)^2}\ .$$

Using the Rule 3, $\Delta\left(B^2\right) = 2B\Delta B$, and we obtain:

$$\frac{\Delta y}{y} = \sqrt{\left(\frac{\Delta A}{A}\right)^2 + 4\left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta C}{C}\right)^2}\ .$$

Calculating the error, keep in mind that the error, by its nature, denotes the uncertainty in the last one or two significant digits of the main number and therefore any additional digits obtained from multiplication or division should be rounded off at the meaningful position. It is a mistake to write: *x*= (56.7 ± 0.914606) cm, instead, write: *x*= (56.7 ± 0.9) cm.

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