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HIGHER MATHEMATICS

Integral Calculus
of a Function of One Variable

<u>Elements of Theory.</u>

Kyiv
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This textbook is designed for students of the first year of technical university. It covers one of the most important areas to be studied in the first semester: Integral Calculus of a Function of One Variable.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of Calculus are explained and illustrated by figures and examples.

CONTENTS

Introduction

This textbook is designed for students of the first year of technical university. It covers one of the most important areas to be studied in the first semester: Integral Calculus of a Function of One Variable.

The manual can be helpful for students who want to understand and be able to use standard integration techniques, apply integration for solving some tasks from geometry and physics and so on.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of Integral Calculus are explained and illustrated by figures and examples.

The first two parts deal with the concept of indefinite integrals, their properties and main techniques of integration: by substitution and by parts. We also considered the ways of integration of rational, trigonometric and irrational functions.

The third section is concerned with the bases of definite integral: Fundamental Theorem of Calculus and main integration techniques for definite integral.

Next part deals with improper integrals including using the comparison test for convergence of improper integrals.

In the fifth section we take a look at some applications of integrals: determining area of a region, the arc length of a curve, the surface area and the volume of a solid of revolution, the center of mass and moments of inertia of a region and curve.

There are also four appendices concerned with graphs of some elementary functions, the polar coordinates parametric representation of a function and some knowledge about derivatives.

1. The Indefinite Integral

1.1 The Indefinite Integral and its properties

I. *The Concept of an Antiderivative*

Previously we considered the following problem: given a function f, find the derivative f'. Now let us solve the reverse problem: given a function f, find a function F such that $F' = f$.

Such inverse operation is called *integration*, that is the process of finding the function $F(x)$ that has its derivative equal to the given function $f(x)$.

Definition. Differentiable function $F(x)$ is called *the primitive* (*antiderivative*) of the function $f(x)$, if $F'(x) = f(x)$ or $dF = f(x)dx$.

Example: Find the antiderivative for function $f(x) = 2x$.

It is well known that $(x^2)' = 2x$, hence $F(x) = x^2$. There are many other primitives of $2x$, such as $x^2 + 1$, $x^2 - 3{,}6$ and $x^2 + \ln 2$. In general, if C is any real number (arbitrary constant), then $x^2 + C$ is an antiderivative of $2x$, because $(x^2 + C)' = 2x$.

Theorem 1.1.

If functions $F_1(x)$ and $F_2(x)$ are two primitives of function $f(x)$ on the interval $[a,b]$, then the difference between them is a constant ($F_1(x) - F_2(x) = C$).

Proof.

Let us consider the function $\varphi(x) = F_1(x) - F_2(x)$.

According to definition of an antiderivative we have

$$F_1'(x) = f(x),$$
$$F_2'(x) = f(x)$$

for any value of x on the interval $[a,b]$.

Hence,

$$\varphi'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0, \quad \forall x \in [a,b].$$

From $\varphi'(x)=0$ it follows that $\varphi(x)$ is a constant.

Since $\varphi(x)$ is differentiable, $\varphi(x)$ is continuous, and we can apply the Mean Value Theorem to the function $\varphi(x)$ on the interval $[a,b]$:

$$\varphi(x)-\varphi(a)=(x-a)\varphi'(c),$$

where $a<c<x$.

Since $\varphi'(x)=0$,

$$\varphi(x)-\varphi(a)=0,$$

$$\varphi(x)=\varphi(a).$$

Thus, the function $\varphi(x)$ is a constant for any x of the interval $[a,b]$.

From this theorem it follows that the primitive $F(x)$ is unique up to an additive constant and all functions $F(x)+C$ (C is an arbitrary constant) are primitives of $f(x)$ too, as $(F(x)+C)'=f(x)$.

Definition. The set of primitives $F(x)+C$ (C is an arbitrary constant) is called *the indefinite integral* of the function $f(x)$ and denoted by

$$\int f(x)dx=F(x)+C,$$

where C is *the constant of integration*.

Function $f(x)$ is called *the integrand* and x is *the integration variable*.

Properties of Indefinite Integrals:

1. $\left(\int f(x)dx\right)'=\left(F(x)+C\right)'=f(x).$

This equation follows directly from the definition of indefinite integral.

2. $\int f'(x)dx=f(x)+C.$

The truth of this property can easily be checked by differentiation of both sides of the equation

$$\begin{array}{ccc}\left(\int f'(x)dx\right)' & = & \left(f(x)+C\right)' \\ \Downarrow & & \Downarrow \\ f'(x) & = & f'(x).\end{array}$$

3. $\forall K \in \mathbb{R}, \quad K \neq 0: \quad \int Kf(x)dx = K\int f(x)dx.$

Let us differentiate both sides of the equation

$$\left(\int Kf(x)dx\right)' = \left(K\int f(x)dx\right)'$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$Kf(x) \quad = \quad K\left(\int f(x)dx\right)' = Kf(x).$$

4. $\int\left(f_1(x) + f_2(x)\right)dx = \int f_1(x)dx + \int f_2(x)dx.$

Let us find the derivatives of both sides of the equation

$$\left(\int\left(f_1(x) + f_2(x)\right)dx\right)' = \left(\int f_1(x)dx + \int f_2(x)dx\right)'$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f_1(x) + f_2(x) \quad = \quad \left(\int f_1(x)dx\right)' + \left(\int f_2(x)dx\right)' = f_1(x) + f_2(x).$$

1.2 *Table of Integrals. Examples*

According to the definition of the indefinite integral, the table of derivatives is transformed into the table of common indefinite integrals.

$\int 0dx = C$	$\int dx = x + C$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$
$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$	$\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \frac{1}{\cos^2 x}dx = \tan x + C$	$\int \frac{1}{\sin^2 x}dx = -\cot x + C$
$\int \sinh x dx = \cosh x + C$	$\int \cosh x dx = \sin x + C$
$\int \frac{1}{\cosh^2 x}dx = \tanh x + C$	$\int \frac{1}{\sinh^2 x}dx = -\coth x + C$

$\int\frac{1}{x^2+1}dx=\begin{cases}\arctan x+C\\-\operatorname{arccot}x+C\end{cases}$	$\int\frac{1}{x^2+a^2}dx=\begin{cases}\frac{1}{a}\arctan\frac{x}{a}+C\\-\frac{1}{a}\operatorname{arccot}\frac{x}{a}+C\end{cases}$
$\int\frac{1}{\sqrt{1-x^2}}dx=\begin{cases}\arcsin x+C\\-\arccos x+C\end{cases}$	$\int\frac{1}{\sqrt{a^2-x^2}}dx=\begin{cases}\arcsin\frac{x}{a}+C\\-\arccos\frac{x}{a}+C\end{cases}$
$\int\frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|+C$	$\int\frac{1}{\sqrt{x^2\pm a^2}}dx=\ln\left\|x+\sqrt{x^2\pm a^2}\right\|+C$

The most of these formulas have a correspondence to the formulas from the table of derivatives (see Appendix 4.), but some of them does not have. The truth of these formulas can easily be checked by differentiation.

For example

$$\left(\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\right)'=\frac{1}{2a}\left(\ln|x-a|-\ln|x+a|\right)'=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)=\frac{1}{x^2-a^2};$$

$$\left(\ln\left|x+\sqrt{x^2+a^2}\right|+C\right)'=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)=$$

$$=\frac{1}{\sqrt{x^2+a^2}}.$$

Finding indefinite integrals is often more complicated than finding derivatives. For some elementary functions, it is impossible to find primitives in terms of other elementary functions.

Examples.

1. $$\int\left(x-3x^3+2\sqrt[5]{x^3}-6\right)dx=\int xdx-3\int x^3dx+2\int x^{\frac{3}{5}}dx-6\int dx=\frac{x^{1+1}}{1+1}-3\frac{x^{3+1}}{3+1}+$$

$$+2\frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1}-6x+C=\frac{x^2}{2}-\frac{3x^4}{4}+\frac{5x^{\frac{8}{5}}}{4}-6x+C=\frac{1}{2}x^2-\frac{3}{4}x^4+\frac{5}{4}\sqrt[5]{x^8}-6x+C;$$

2. $$\int\left(\frac{5}{x^4}+\frac{1}{\sqrt{x^3}}\right)dx=5\int x^{-4}dx+\int x^{-\frac{3}{2}}dx=5\frac{x^{-4+1}}{-4+1}+\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}+C=-\frac{5}{3x^3}-2\frac{1}{\sqrt{x}}+C;$$

3. $\int \frac{(x^2-2)^2}{x^2}dx = \int \frac{x^4-4x^2+4}{x^2}dx = \int\left(x^2-4+\frac{4}{x^2}\right)dx = \frac{x^3}{3}-4x-\frac{4}{x}+C$;

4. $\int \frac{2^{x+1}+5^{x-2}}{10^x}dx = \int \frac{2\cdot 2^x+5^{-2}\cdot 5^x}{2^x\cdot 5^x}dx = \int\left(2\cdot\left(\frac{1}{5}\right)^x+\frac{1}{25}\cdot\left(\frac{1}{2}\right)^x\right)dx =$

$= 2\cdot\left(\frac{1}{5}\right)^x\cdot\frac{1}{\ln\frac{1}{5}}+\frac{1}{25}\cdot\left(\frac{1}{2}\right)^x\cdot\frac{1}{\ln\frac{1}{2}}+C = -\frac{2}{5^x\ln 5}-\frac{1}{2^x 25\ln 2}+C$;

5. $\int \sin^2\frac{x}{2}dx = \left|\begin{array}{l}\text{transform the integrand}\\ \text{using the formula}\\ \sin^2\alpha=\frac{1}{2}(1-\cos 2\alpha)\end{array}\right| = \frac{1}{2}\int(1-\cos x)dx = \frac{1}{2}(x-\sin x)+C$;

6. $\int \tanh^2 x dx = \left|\begin{array}{l}\text{transform the integrand}\\ \tanh^2 x = 1-\frac{1}{\cosh^2 x}\end{array}\right| = \int dx - \int\frac{1}{\cosh^2 x}dx = x-\tanh x+C$;

7. $\int \frac{1}{4x^2+100}dx = \frac{1}{4}\int\frac{1}{x^2+5^2}dx = \frac{1}{4}\cdot\frac{1}{5}\arctan\frac{x}{5}+C = \frac{1}{20}\arctan\frac{x}{5}+C$;

8. $\int \frac{1}{7-x^2}dx = -\int\frac{1}{x^2-7}dx = -\frac{1}{2\sqrt{7}}\ln\left|\frac{x-\sqrt{7}}{x+\sqrt{7}}\right|+C = \frac{1}{2\sqrt{7}}\ln\left|\frac{x+\sqrt{7}}{x-\sqrt{7}}\right|+C$.

WARNINGS.

1. Integration variable

The dx tells us that we are integrating with respect to x (all other variables in the integrand are considered to be constants).

$$\int 3x^2dx = x^3+C; \qquad \int 3t^2dt = t^3+C; \qquad \int 3\sin^2 x d\sin x = \sin^3 x + C;$$

$$\int 3x^2dt = 3x^2t+C.$$

2. Do not drop the dx at the end of integral, because it shows where the integral ends and what is the variable of integration.

$$\int(3x^2+9)dx = x^3+9x+C; \qquad \int 3x^2dx+9 = x^3+C+9;$$

$$\int(3x^2+9)dz = (3x^2+9)z+C;$$

$$\int 3x^2+9 = \left|\begin{array}{l}\text{where is the end of integral?}\\ \text{what is the variable of integration?}\end{array}\right| \Rightarrow \left|\begin{array}{l}\text{it is impossible to solve}\\ \text{this problem}\end{array}\right|$$

2. Techniques of Integration

2.1 *Integration by Substitution*

I. For evaluating indefinite integrals it is convenient to use the following rule.

Let $\int f(x)dx = F(x) + C$. Then $\forall \ a \in \mathbb{R},\ a \neq 0,\ b \in \mathbb{R}$

$$\int f(ax+b)dt = \frac{1}{a}F(ax+b) + C. \tag{2.1}$$

For proving it is enough to differentiate the left and right sides of (2.1).

$$\left(\int f(ax+b)dt\right)' = \left(\frac{1}{a}F(ax+b) + C\right)'$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$f(ax+b) \ = \ \frac{1}{a}\left(F(ax+b)\right)_x' = \frac{1}{a}f(ax+b)\cdot(ax+b)' = \frac{1}{a}f(ax+b)\cdot a = f(ax+b)$$

The derivatives of the both sides are equal.

Examples:

1. $\int (5x-1)^3 dx = \frac{1}{5}\cdot\frac{(5x-1)^4}{4} + C = \frac{(5x-1)^4}{20} + C$;

2. $\int e^{3x+5}dx = \frac{1}{3}e^{3x+5} + C$;

3. $\int \cos^2 x dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$;

4. $\int \sin 4x \cos 3x dx = \frac{1}{2}\int (\sin(4x-3x) + \sin(4x+3x))dx = \frac{1}{2}\int (\sin x + \sin 7x)dx =$

$= \frac{1}{2}\left(-\cos x - \frac{1}{7}\cos 7x\right) + C$;

5. $\int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{(x+2)^2+1}} = \ln\left|x+2+\sqrt{x^2+4x+5}\right| + C$;

6. $\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \frac{1}{2}\cdot\frac{1}{3}\arctan\frac{2x}{3} + C = \frac{1}{6}\arctan\frac{2x}{3} + C$;

7. $\int \frac{dx}{4x^2-4x-3} = \int \frac{dx}{(2x-1)^2-2^2} = \frac{1}{2}\cdot\frac{1}{4}\ln\left|\frac{2x-1-2}{2x-1+2}\right| + C = \frac{1}{8}\ln\left|\frac{2x-3}{2x+1}\right| + C$.

II. Integration by Changing of Variable

Let $\int f(x)dx = F(x) + C$. Consider the differentiable function $u = u(t)$. Then

$$\int f(u(t))u'(t)dt = \int f(u(t))du(t) = F(u(t)) + C. \qquad (2.2)$$

or (another way of notation)

$$\int f(u(t))u'(t)dt = \left| \begin{matrix} x = u(t) \\ dx = u'(t)dt \end{matrix} \right| = \int f(x)dx = F(x) + C = F(u(t)) + C.$$

This formula is based on the chain rule for derivatives and used to transform one integral into another that is easier to be solved.

Example:

1. $\int 2xe^{x^2} dx = \left| \begin{matrix} u = x^2 \\ du = 2xdx \end{matrix} \right| = \int e^u du = e^u + C = e^{x^2} + C$

or $\int 2xe^{x^2} dx = \int e^{x^2} \underbrace{2xdx}_{dx^2} = \int e^{x^2} dx^2 = e^{x^2} + C$;

2. $\int \frac{\ln^5 x}{x} dx = \left| \begin{matrix} u = \ln x \\ du = \frac{1}{x} dx \end{matrix} \right| = \int u^5 du = \frac{u^6}{6} + C = \frac{\ln^6 x}{6} + C$

or $\int \frac{\ln^5 x}{x} dx = \int \ln^5 x \cdot \underbrace{\frac{1}{x} dx}_{d\ln x} = \int \ln^5 x d\ln x = \frac{\ln^6 x}{6} + C$;

3. $\int \frac{e^x + 1}{x + e^x} dx = \left| \begin{matrix} u = x + e^x \\ du = (1 + e^x)dx \end{matrix} \right| = \int \frac{1}{u} du = \ln|u| + C = \ln\left|x + e^x\right| + C$

or $\int \frac{e^x + 1}{x + e^x} dx = \int \frac{1}{x + e^x} \underbrace{(1 + e^x)dx}_{d(x+e^x)} = \int \frac{1}{x + e^x} d(x + e^x) == \ln\left|x + e^x\right| + C$;

4. $\int x\sqrt{x-2}dx = \left| \begin{matrix} u = \sqrt{x-2} \\ u^2 + 2 = x \\ 2udu = dx \end{matrix} \right| = \int u(u^2 + 2)2udu = 2\int u^2(u^2 + 2)du = 2\int (u^4 + 2u^2)du =$

$= 2\left(\frac{u^5}{5} + \frac{2u^3}{3} \right) + C = \frac{2\sqrt{(x-2)^5}}{5} + \frac{4\sqrt{(x-2)^3}}{3} + C$.

The method of substitution is one of the basic methods of integration. Often when we use another method, we resort to substitution in the intermediate stages of integration. The success of calculation depends on choosing the appropriate substitution (it should simplify the given integral).

2.2 *Integration by Parts*

Let functions $u(x)$ and $v(x)$ be differentiable, consider

$$\left(u(x)\cdot v(x)\right)' = u(x)'\cdot v(x) + u(x)\cdot v'(x).$$

Integrate both sides with respect to x

$$\int \left(u(x)\cdot v(x)\right)' dx = \int u(x)'\cdot v(x) + u(x)\cdot v'(x)dx.$$

Apply the definition of indefinite integral

$$u(x)\cdot v(x) = \int u(x)'\cdot v(x)dx + \int u(x)\cdot v'(x)dx.$$

Then we obtain *the formula of integration by parts*

$$\int u(x)\cdot v'(x)dx = u(x)\cdot v(x) - \int v(x)\cdot u'(x)dx.$$

or

$$\int udv = uv - \int vdu. \qquad (2.3)$$

This formula makes it possible to calculate the integral of the product of two functions.

On practice we should make the following steps:

1. Choose correctly u **and** dv**;**

2. Calculate the differential du**:** $du = u'(x)dx$**;**

3. Find $v(x)$**:** $v(x) = \int dv$**;**

4. Use the formula $\int udv = uv - \int vdu$ **;**

5. Simplify and calculate.

There are several rules for choosing correctly u and dv.

I. For integrals of the form

$$\int x^k e^{ax}dx \qquad \int x^k \sin axdx \qquad \int x^k \sinh axdx$$

$$\int x^k b^{ax}dx \qquad \int x^k \cos axdx \qquad \int x^k \cosh axdx$$

we choose $u(x) = x^k$.

Examples:

1. $\int xe^x dx = \begin{vmatrix} u = x & du = dx \\ dv = e^x dx & v = \int e^x dx = e^x \end{vmatrix} = xe^x - \int e^x dx = xe^x - e^x + C.$

2. It is possible to use formula (2.3) several times.

$$\int x^2 \sin 2x dx = \begin{vmatrix} u = x^2 & du = 2x dx \\ dv = \sin 2x dx & v = \int \sin 2x dx = -\dfrac{\cos 2x}{2} \end{vmatrix} = x^2 \cdot \frac{\cos 2x}{2} -$$

$$-\int \frac{-\cos 2x}{2} \cdot 2x dx = \frac{x^2 \cos 2x}{2} + \int x \cos 2x dx = \begin{vmatrix} u = x & du = dx \\ dv = \cos 2x dx & v = \int \cos 2x dx = \dfrac{\sin 2x}{2} \end{vmatrix} =$$

$$= \frac{x^2 \cos 2x}{2} + x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} dx = \frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + C.$$

II. For integrals of the form

$$\int x^k \ln x dx \qquad \int x^k \arcsin x dx$$

$$\int x^k \arctan x dx \qquad \int x^k \arccos x dx$$

we choose $dv = x^k dx$.

Examples:

1. $\int \ln x dx = \begin{vmatrix} u = \ln x & du = \dfrac{1}{x} dx \\ dv = dx & v = \int dx = x \end{vmatrix} = x \ln x - \int \dfrac{1}{x} \cdot x dx = x \ln x - \int dx = x \ln x - x + C.$

2. $\int x \arctan x dx = \begin{vmatrix} u = \arctan x & du = \dfrac{1}{x^2+1} dx \\ dv = x dx & v = \int x dx = \dfrac{x^2}{2} \end{vmatrix} = \dfrac{x^2}{2} \arctan x - \int \dfrac{x^2}{2} \cdot \dfrac{1}{x^2+1} dx =$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{x^2+1}\right) dx =$$

$$= \frac{x^2}{2} \arctan x - \frac{1}{2} (x - \arctan x) + C.$$

2.3 *Integration of Rational Functions*

I. Integration of Simplest Rational Functions

1. $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$;

2. $\int \frac{1}{(ax+b)^m}dx = -\frac{1}{a(m-1)(ax+b)^{m-1}} + C$;

3. $\int \frac{(Ax+B)dx}{x^2+2ax+b} = |\text{complete the square in the denominator}| =$

$$= \int \frac{A(x+a)+B-aA}{(x+a)^2+b-a^2}dx = \int \frac{\frac{A}{2}\cdot 2(x+a)dx}{(x+a)^2+b-a^2} + \int \frac{(B-aA)dx}{(x+a)^2+(b-a^2)} =$$

$$= \frac{A}{2}\int \frac{d(x^2+2ax+b)}{x^2+2ax+b} + (B-aA)\int \frac{1}{(x+a)^2+(b-a^2)}d(x+a) =$$

$$= \begin{cases} \frac{A}{2}\ln|x^2+2ax+b| + \frac{(B-aA)}{\sqrt{b-a^2}}\arctan\frac{x+a}{\sqrt{b-a^2}} + C, & b-a^2>0; \\ \frac{A}{2}\ln|x^2+2ax+b| + \frac{(B-aA)}{2\sqrt{a^2-b}}\ln\left|\frac{x+a-\sqrt{a^2-b}}{x+a+\sqrt{a^2-b}}\right| + C, & b-a^2<0. \end{cases}$$

Examples:

1. $\int \frac{1}{5x-2}dx = \frac{1}{5}\ln|5x-2| + C$;

2. $\int \frac{1}{(3x+4)^5}dx = -\frac{1}{12(3x+4)^4} + C$;

3. $\int \frac{(3x+1)dx}{x^2+2x+5} = \int \frac{3(x+1)-2}{(x+1)^2+4}dx = \frac{3}{2}\int \frac{2(x+1)dx}{(x+1)^2+4} - 2\int \frac{dx}{(x+1)^2+4} =$

$$= \frac{3}{2}\int \frac{d(x^2+2x+5)}{x^2+2x+5} - 2\int \frac{d(x+1)}{(x+1)^2+4} = \frac{3}{2}\ln|x^2+2x+5| - \arctan\frac{x+1}{2} + C;$$

4. $\int \frac{xdx}{x^2-4x-5} = \int \frac{(x-2)+2dx}{(x-2)^2-9} = \frac{1}{2}\int \frac{2(x-2)dx}{(x-2)^2-9} + 2\int \frac{dx}{(x-2)^2-9} =$

$$= \frac{1}{2}\int \frac{d(x^2-4x-5)}{x^2-4x-5} + 2\int \frac{d(x-2)}{(x-2)^2-9} = \frac{1}{2}\ln|x^2-4x-5| + \frac{1}{3}\ln\left|\frac{x-5}{x+1}\right| + C;$$

II. Integration of Rational Functions

If we have to compute the integral $\int \frac{P_n(x)}{Q_m(x)}dx$, where $P_n(x)$, $Q_m(x)$, $n<m$ are the polynomials, the fraction $\frac{P_n(x)}{Q_m(x)}$ needs to be expressed in *partial fractions* and reduced to the three simplest types of integrals of rational functions.

Example. $\int \frac{x^2+2}{x^2(x+1)}dx$

The integrand $\frac{x^2+2}{x^2(x+1)}$ is a proper rational fraction. Let us use the partial-fraction decomposition.

$$\frac{x^2+2}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}=\frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}=\frac{(A+C)x^2+(A+B)x+B}{x^2(x+1)}$$

Whence, equating the numerators, we obtain the system of equations for determining the coefficients

$$\begin{array}{c|l} x^2 & A+C=1, \\ x & A+B=0, \\ 1 & \quad B=2. \end{array}$$

Solving the system we find $A=-2;\quad B=2;\quad C=3.$

Thus,

$$\int \frac{x^2+2}{x^2(x+1)}dx=\int\left(\frac{-2}{x}+\frac{2}{x^2}+\frac{3}{x+1}\right)dx=-2\ln|x|-\frac{2}{x}+3\ln|x+1|+C.$$

Note. If the given integrand $\frac{P_n(x)}{Q_m(x)}$ is an improper fraction ($n\geq m$), we represent it as a sum of a polynomial and the proper rational fraction.

Example.

$$\int \frac{x^4-2x^3-2x+2}{x^2+1}dx=$$

$$=\int\left(x^2-2x-1+\frac{3}{x^2+1}\right)dx=$$

$$=\frac{x^3}{3}-x^2-x+3\arctan x+C$$

$$\begin{array}{l|l} x^4-2x^3-2x+2 & x^2+1 \\ \underline{x^4+x^2} & x^2-2x-1 \\ \quad -2x^3-x^2-2x+2 & \\ \quad \underline{-2x^3-2x} & \\ \qquad -x^2+2 & \\ \qquad \underline{-x^2-1} & \\ \qquad\quad 3 & \end{array}$$

2.4 *Integration of Trigonometric Functions*

I. General Trigonometric Substitution

Integrals of the form $\int R(\sin x, \cos x)dx$ where is a rational function of $\sin x$ and $\cos x$ are reduced to integrals of rational expression by so-called *general trigonometric substitution*

$$\tan\frac{x}{2} = t \qquad (-\pi < x < \pi).$$

Express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ and t:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

And

$$x = 2\arctan t, \qquad dx = \frac{2dt}{1+t^2}.$$

Here $\sin x$, $\cos x$ and dx are expressed rationally in terms of t. By substituting the expressions obtained into the original integral we get an integral of a rational function

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}.$$

Examples.

1. $\int \frac{1}{\sin x}dx = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C;$

2. $\int \frac{1}{\cos x}dx = \int \frac{1}{\sin\left(\frac{\pi}{2} - x\right)}dx = -\ln\left|\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| + C;$

3. $\int \frac{dx}{2-3\cos x} = \int \frac{\frac{2dt}{1+t^2}}{2-3\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2(1+t^2)-3(1-t^2)}{1+t^2}} = \int \frac{2dt}{21+2t^2-3+3t^2} =$

$$=\int\frac{2dt}{5t^2-1}=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right|+C=\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}\tan\frac{x}{2}-1}{\sqrt{5}\tan\frac{x}{2}+1}\right|+C.$$

General trigonometric substitution enables us to calculate any integrals of the form $\int R(\sin x,\cos x)dx$, but it often leads to very cumbersome expressions. There are some cases when the aim can be achieved with the aid of more convenient substitutions.

1) $\int R(\sin x)\cos x dx=\left|\begin{matrix}\sin x=t\\ \cos x dx=dt\end{matrix}\right|=\int R(t)dt;$

2) $\int R(\cos x)\sin x dx=\left|\begin{matrix}\cos x=t\\ -\sin x dx=dt\end{matrix}\right|=-\int R(t)dt;$

3) $\int R(\tan x)dx=\left|\begin{matrix}\tan x=t\\ dx=\dfrac{dt}{t^2+1}\end{matrix}\right|=\int R(t)\dfrac{dt}{1+t^2};$

4) $\int R(\sin^{2n}x,\cos^{2m}x)dx=\left|\begin{matrix}\tan x=t \quad dx=\dfrac{dt}{t^2+1}\\ \sin^2 x=\dfrac{\tan^2 x}{1+\tan^2 x}=\dfrac{t^2}{1+t^2}\\ \cos^2 x=\dfrac{1}{1+\tan^2 x}=\dfrac{1}{1+t^2}\end{matrix}\right|=\int R\left(\left(\dfrac{t^2}{1+t^2}\right)^n,\left(\dfrac{1}{1+t^2}\right)^m\right)\dfrac{dt}{1+t^2}.$

Example.

1) $\int\dfrac{\sin^3 x dx}{2+\cos x}=\int\dfrac{\sin^2 x\sin x dx}{2+\cos x}=\int\dfrac{(1-\cos^2 x)\sin x dx}{2+\cos x}=\left|\begin{matrix}\cos x=t\\ -\sin x dx=dt\end{matrix}\right|=-\int\dfrac{(1-t^2)dt}{2+t}=$

$$=\int\frac{t^2-1}{t+2}dt=\int t-2+\frac{3}{t+2}dt=\frac{t^2}{2}-2t+3\ln|t+2|+C=\frac{\cos^2 x}{2}-2\cos x+3\ln|\cos x+2|+C;$$

2) $\int\dfrac{dx}{2-\sin^2 x}=\left|\begin{matrix}\tan x=t\\ dx=\dfrac{dt}{1+t^2}\end{matrix}\right|=\int\dfrac{dt}{\left(2-\dfrac{t^2}{1+t^2}\right)(1+t^2)}=\int\dfrac{dt}{2+t^2}=$

$$=\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C=\frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}}+C.$$

II. Integrals of the form $\int \sin^m x \cos^n x dx,$ where m, n are rational numbers.

• If the power *n* of the cosine is **odd** (the power *m* of the sine can be arbitrary), then the substitution $t = \sin x$ is used;

Example.

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^2 x \cos x dx = \int \sin^2 x (1 - \sin^2 x) \cos x dx = \begin{vmatrix} \sin x = t \\ \cos x dx = dt \end{vmatrix} =$$

$$= \int t^2 (1 - t^2) dt = \int (t^2 - t^4) dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

• If the power *m* of the sine is **odd**, then the substitution $t = \cos x$ is used.

Example.

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int (1 - \cos^2 x) \sin x dx = \begin{vmatrix} \cos x = t \\ -\sin x dx = dt \end{vmatrix} =$$

$$= -\int (1 - t^2) dt = \int (t^2 - 1) dt = \frac{t^3}{3} - t + C = \frac{\sin^3 x}{3} - \sin x + C.$$

• If both powers *m* and *n* are **even**, then use the *double angle formulas* to reduce the powers of the sine or cosine in the integrand

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

$$\int \sin^2 x \cos^2 x dx = \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) dx = \frac{1}{4} \int (1 - \cos^2 2x) dx =$$

$$= \frac{1}{4} \int \left(1 - \frac{1}{2}(1 + \cos 4x) \right) dx = \frac{1}{8} \int (1 - \cos 4x) dx = \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + C.$$

• If $m + n$ is **even** $\left(\frac{m+1}{2} + \frac{n-1}{2} \text{ an integer} \right)$, then the substitution $t = \tan x$ is used.

$$\int \frac{dx}{\sqrt[3]{\sin^{11} x \cos x}} = \begin{vmatrix} m = -\frac{11}{3}, \; n = -\frac{1}{3} \Rightarrow m + n = -\frac{11}{3} - \frac{1}{3} = \underbrace{-4}_{\text{even}} \\ \tan x = t, \qquad \frac{dx}{\cos^2 x} = dt \end{vmatrix} = \int \frac{dx}{\cos^4 x \sqrt[3]{\tan^{11} x}} =$$

$$= \int \frac{1 + t^2}{\sqrt[3]{t^{11}}} dt = \int \left(t^{-\frac{11}{3}} + t^{-\frac{5}{3}} \right) dt = -\frac{3}{8} t^{-\frac{8}{3}} - \frac{3}{2} t^{-\frac{2}{3}} + C = -\frac{3}{8\sqrt[3]{\tan^8 x}} - \frac{3}{2\sqrt[3]{\tan^2 x}} + C.$$

III. Integrals of the form $\int \tan^n x dx$ or $\int \cot^n x dx$, where n is positive integer.

We can reduce the power of the integrand using $\tan^2 x = \frac{1}{\cos^2 x} - 1$ or $\cot^2 x = \frac{1}{\sin^2 x} - 1$ and *the reduction formula*

$$\int \tan^n x dx = \int \tan^{n-2} x \tan^2 x dx = \int \tan^{n-2} x \left(\frac{1}{\cos^2 x} - 1 \right) dx =$$

$$= \int \tan^{n-2} x \frac{dx}{\cos^2 x} - \int \tan^{n-2} x dx = \int \tan^{n-2} x d(\tan x) - \int \tan^{n-2} x dx.$$

Example.

$$\int \tan^3 x dx = \int \tan x \tan^2 x dx = \int \tan x \left(\frac{1}{\cos^2 x} - 1 \right) dx = \int \tan x \frac{dx}{\cos^2 x} - \int \tan x dx =$$

$$= \int \tan x d(\tan x) - \int \frac{\sin x dx}{\cos x} = \frac{1}{2} \tan^2 x + \int \frac{d \cos x}{\cos x} = \frac{1}{2} \tan^2 x + \ln|\cos x| + C.$$

IV. Integrals of the form $\int \tan^n x \frac{dx}{\cos^{2m} x}$ or $\int \cot^n x \frac{dx}{\sin^{2m} x}$, where n and m are positive integer.

We can reduce the power of the integrand using $\frac{1}{\cos^2 x} = \tan^2 x + 1$ or $\frac{1}{\sin^2 x} = \cot^2 x + 1$.

Examples.

1. $\int \tan x \frac{dx}{\cos^4 x} = \int \tan x \frac{1}{\cos^2 x} \frac{dx}{\cos^2 x} = \int \tan x (\tan^2 x + 1) d \tan x = \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + C;$

2. $\int \frac{dx}{\sin^6 x} = \int \left(\frac{1}{\sin^2 x} \right)^2 \frac{dx}{\sin^2 x} = -\int (\cot^2 x + 1)^2 d \cot x = -\frac{\cot^5 x}{5} - \frac{2 \cot^3 x}{3} - \cot x + C.$

NOTE. Functions rationally depending on hyperbolic functions are integrated in the same fashion as trigonometric functions.

$$\cosh^2 x - \sinh^2 x = 1; \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x}; \quad 1 - \coth^2 x = \frac{1}{\sinh^2 x};$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x; \quad \sinh 2x = 2 \cosh x \sinh x;$$

$$\cosh 2x - 1 = 2 \sinh^2 x; \quad \cosh 2x + 1 = 2 \cosh^2 x;$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y; \quad \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y;$$

2.5 *Integration of Irrational Functions*

I. Integrals of the form $\int R(x, x^{\frac{m}{n}}, \ldots x^{\frac{p}{s}})dx$, where R is a rational function of its arguments, reduce to the integral of a rational function by means of substitution:

$$x = t^k, \qquad dx = kt^{k-1}dt,$$

where k be a common denominator of the fractions $\frac{m}{n}, \ldots \frac{p}{s}$.

Example.

$$\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1} = \int \frac{x^{\frac{1}{2}}dx}{x^{\frac{3}{4}}+1} = \left| \begin{array}{c} \text{the common denominator of} \\ \text{the fractions } \frac{1}{2}, \ \frac{3}{4} \text{ is } 4, \text{so} \\ x = t^4, \qquad dx = 4t^3dt \end{array} \right| = \int \frac{\sqrt{t^4}\,4t^3dt}{\sqrt[4]{t^{12}}+1} = 4\int \frac{t^5dt}{t^3+1} =$$

$$= 4\int \left(t^2 - \frac{t^2}{t^3+1} \right) dt = 4\int t^2dt - \frac{4}{3}\int \frac{dt^3}{t^3+1} = \frac{4}{3}t^3 - \frac{4}{3}\ln\left|t^3+1\right| + C = \frac{4}{3}\left(x^{\frac{3}{4}} - \ln\left| x^{\frac{3}{4}}+1 \right| \right) + C.$$

NOTE. For integrals of the form $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{s}} \right) dx$ we make the substitution $x = \left(\frac{ax+b}{cx+d}\right)^k$, where k be a common denominator of the fractions $\frac{m}{n}, \ldots \frac{p}{s}$.

II. Euler's Substitutions

Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})dx$ are reduced to the integral of a rational function of a new variable with the aid of one of the following substitutions:

- **First Euler Substitution**

$$\sqrt{ax^2+bx+c} = t \pm x\sqrt{a} \text{ if } a > 0.$$

For the sake of definiteness we take the plus sign in front of $x\sqrt{a}$. Then

$$ax^2+bx+c = \left(t + x\sqrt{a}\right)^2 = ax^2 + 2\sqrt{a}xt + t^2,$$

whence x is determined as a rational function of t:

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t}$$

thus, dx will be expressed rationally in terms of t. Lastly

$$\sqrt{ax^2 + bx + c} = t + \sqrt{a}\,\frac{t^2 - c}{b - 2\sqrt{a}t}.$$

Since x, dx and $\sqrt{ax^2 + bx + c}$ are expressed rationally in terms of t, the original integral is transformed into an integral of rational function of t.

Example. Calculate the integral $I = \int \frac{dx}{\sqrt{x^2 + 4}}$.

Here $a = 1 > 0$ therefore $\sqrt{x^2 + 4} = t - x$.

Then

$$x^2 + 4 = t^2 - 2xt + x^2$$

whence

$$x = \frac{t^2 - 4}{2t}, \quad dx = \frac{t^2 + 4}{2t^2}dt$$

and

$$\sqrt{x^2 + 4} = t - \frac{t^2 - 4}{2t} = \frac{t^2 + 4}{2t}.$$

Consequently

$$I = \int \frac{\frac{t^2 + 4}{2t^2}dt}{\frac{t^2 + 4}{2t}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|x + \sqrt{x^2 + 4}\right| + C.$$

- **Second Euler Substitution**

$$\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c} \text{ if } c > 0.$$

Then

$$ax^2 + bx + c = t^2x^2 + 2\sqrt{c}xt + c$$

(for the sake of definiteness we take the plus sign in front of $\sqrt{c}$),

whence x is determined as a rational function of t:

$$x = \frac{2\sqrt{c}t - b}{a - t^2}.$$

Since x, dx and $\sqrt{ax^2 + bx + c}$ are also expressed rationally in terms of t, the original integral is transformed into an integral of rational function of t.

Example. Calculate the integral $I = \int \frac{dx}{\sqrt{x^2 + 4}}$.

Here $c = 4 > 0$ therefore $\sqrt{x^2 + 4} = tx + 2$.

Then

$$x^2 + 4 = x^2 t^2 + 4xt + 4$$

whence

$$x = \frac{4t}{1 - t^2}, \quad dx = \frac{4t^2 + 4}{(1 - t^2)^2} dt$$

and

$$\sqrt{x^2 + 4} = \frac{2t^2 + 2}{1 - t^2}.$$

Consequently

$$I = \int \frac{\frac{4t^2 + 4}{(1 - t^2)^2} dt}{\frac{2t^2 + 2}{1 - t^2}} = 2\int \frac{dt}{1 - t^2} = \ln\left|\frac{t + 1}{t - 1}\right| + C = \ln\left|\frac{\frac{\sqrt{x^2 + 4} - 2}{x} + 1}{\frac{\sqrt{x^2 + 4} - 2}{x} - 1}\right| + C =$$

$$= \ln\left|\frac{\sqrt{x^2 + 4} - 2 + x}{\sqrt{x^2 + 4} - 2 - x}\right| + C = \ln\left|\frac{\left(\sqrt{x^2 + 4} - 2 + x\right)\left(\sqrt{x^2 + 4} - 2 - x\right)}{\left(\sqrt{x^2 + 4} - (2 + x)\right)\left(\sqrt{x^2 + 4} - (2 + x)\right)}\right| + C =$$

$$= \ln\left|\frac{2x^2 + 2x\sqrt{x^2 + 4}}{4x}\right| + C = \ln\left|x + \sqrt{x^2 + 4}\right| - \ln 2 + C = \ln\left|x + \sqrt{x^2 + 4}\right| + C_1.$$

Note. We have solved the integral $I = \int \frac{dx}{\sqrt{x^2 + 4}}$ in two ways by first and second Euler Substitutions. The results coincide with the tabular value however the second Euler Substitution leads to the more cumbersome transformations of expressions obtained.

- **Third Euler Substitution**

$\sqrt{ax^2+bx+c}=(x-\alpha)t$ if $ax^2+bx+c=a(x-\alpha)(x-\beta)$, where $\{\alpha,\beta\}\subset\mathbb{R}$.

Therefore

$$\left(\sqrt{a(x-\alpha)(x-\beta)}\right)^2=\left((x-\alpha)t\right)^2,$$

$$a(x-\alpha)(x-\beta)=(x-\alpha)^2t^2,$$

$$a(x-\beta)=(x-\alpha)t^2.$$

Whence we find x as a function of t:

$$x=\frac{a\beta-\alpha t^2}{a-t^2}.$$

Since x, dx and $\sqrt{ax^2+bx+c}$ depend rationally upon t, the original integral is transformed into an integral of rational function of t.

Example. Calculate the integral $I=\int\frac{dx}{\sqrt{x^2+3x-4}}$.

Since $x^2+3x-4=(x+4)(x-1)$, we put

$$\sqrt{(x+4)(x-1)}=(x+4)t.$$

Then

$$(x+4)(x-1)=(x+4)^2t^2,$$

$$(x-1)=(x+4)t^2,$$

$$x=\frac{1+4t^2}{1-t^2},\quad dx=\frac{10t}{(1-t^2)^2}\,dt$$

and

$$\sqrt{x^2+3x-4}=\frac{5t}{1-t^2}.$$

Putting the expressions obtained into the original integral, we have

$$I=\int\frac{10t(1-t^2)}{5t(1-t^2)^2}dt=2\int\frac{1}{1-t^2}dt=\ln\left|\frac{t+1}{t-1}\right|+C=\ln\left|\frac{\sqrt{\frac{x-1}{x+4}}+1}{\sqrt{\frac{x-1}{x+4}}-1}\right|+C=\ln\left|\frac{\sqrt{x-1}+\sqrt{x+4}}{\sqrt{x-1}-\sqrt{x+4}}\right|+C.$$

The Euler substitutions often lead to rather cumbersome calculations, therefore we apply them only when it is difficult to find another method for solving given integral. There are simpler methods for calculating some integrals of the form $\int R(x,\sqrt{ax^2+bx+c})dx$.

III. Integrals of the form $\int \frac{(Ax+B)dx}{\sqrt{\pm x^2+2ax+b}}$.

- $$\int \frac{(Ax+B)dx}{\sqrt{x^2+2ax+b}} = \left| \begin{matrix} \text{complete the square in the denominator} \\ x^2+2ax+b=(x+a)^2+(b-a^2) \end{matrix} \right| =$$

$$= \int \frac{A(x+a)+B-aA}{\sqrt{(x+a)^2+b-a^2}}dx = \int \frac{\frac{A}{2}\cdot 2(x+a)dx}{\sqrt{(x+a)^2+b-a^2}} + \int \frac{(B-aA)dx}{\sqrt{(x+a)^2+(b-a^2)}} =$$

$$= \frac{A}{2}\int \frac{d(x^2+2a+b)}{\sqrt{x^2+2a+b}} + (B-aA)\int \frac{1}{\sqrt{(x+a)^2+(b-a^2)}}d(x+a) =$$

$$= A\sqrt{x^2+2ax+b} + (B-aA)\ln\left|x+a+\sqrt{x^2+2ax+b}\right| + C \ .$$

Example. $$\int \frac{(5x+1)dx}{\sqrt{x^2+2x+3}} = \left| \begin{matrix} \text{complete the square in the denominator} \\ x^2+2x+3=(x+1)^2+2 \end{matrix} \right| =$$

$$= \int \frac{5(x+1)-4}{\sqrt{(x+1)^2+2}}dx = \frac{5}{2}\int \frac{2(x+1)dx}{\sqrt{(x+1)^2+2}} - 4\int \frac{dx}{\sqrt{(x+1)^2+2}} = \frac{5}{2}\int \frac{d(x^2+2x+3)}{\sqrt{x^2+2x+3}} -$$

$$-4\int \frac{1}{\sqrt{(x+1)^2+2}}d(x+1) = 5\sqrt{x^2+2x+3} - 4\ln\left|x+1+\sqrt{x^2+2x+3}\right| + C .$$

- $$\int \frac{(Ax+B)dx}{\sqrt{-x^2+2ax+b}} = \left| \begin{matrix} \text{complete the square in the denominator} \\ -x^2+2ax+b=-(x^2-2ax)+b=(b+a^2)-(x-a)^2 \end{matrix} \right| =$$

$$= \int \frac{A(x-a)+B+aA}{\sqrt{(b+a^2)-(x-a)^2}}dx = \int \frac{\frac{A}{2}\cdot 2(x-a)dx}{\sqrt{(b+a^2)-(x-a)^2}} + \int \frac{(B+aA)dx}{\sqrt{(b+a^2)-(x-a)^2}} =$$

$$= -\frac{A}{2}\int \frac{d(-x^2+2ax+b)}{\sqrt{-x^2+2ax+b}} + (B+aA)\int \frac{1}{\sqrt{(b+a^2)-(x-a)^2}}d(x-a) =$$

$$= -A\sqrt{-x^2+2ax+b} + (B+aA)\arcsin\frac{x-a}{\sqrt{b+a^2}} + C .$$

Example. $\int \frac{(x+3)dx}{\sqrt{-x^2+2x+7}} = \left| \begin{array}{l} \text{complete the square in the denominator} \\ -x^2+2x+7=-(x^2-2x)+7=8-(x-1)^2 \end{array} \right| =$

$$= \int \frac{(x-1)+4}{\sqrt{8-(x-1)^2}} dx = \frac{1}{2}\int \frac{2(x-1)dx}{\sqrt{8-(x-1)^2}} + 4\int \frac{dx}{\sqrt{8-(x-1)^2}} = -\frac{1}{2}\int \frac{d(-x^2+2x+7)}{\sqrt{-x^2+2x+7}} +$$

$$+4\int \frac{1}{\sqrt{8-(x-1)^2}} d(x-1) = -\sqrt{-x^2+2ax+b} + 4\arcsin\frac{x-1}{\sqrt{8}} + C.$$

IV. Integrals of the form $\int \frac{dx}{(x-\alpha)\sqrt{ax^2+2ax+b}}$ are transformed into an integral of type discussed in **III.** by means of the substitution $(x-\alpha) = \frac{1}{t}$.

Example.

$$\int \frac{dx}{x\sqrt{5x^2-2x+1}} = \left| \begin{array}{c} x=\frac{1}{t} \\ dx=-\frac{1}{t^2}dt \end{array} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{5}{t^2}-\frac{2}{t}+1}} = -\int \frac{dt}{t\sqrt{\frac{5}{t^2}-\frac{2}{t}+1}} = -\int \frac{dt}{\sqrt{t^2-2t+5}} =$$

$$= -\int \frac{dt}{\sqrt{(t-1)^2+4}} = -\ln\left|t-1+\sqrt{t^2-2t+5}\right| + C = -\ln\left|\frac{1}{x}-1+\sqrt{\frac{1}{x^2}-\frac{2}{x}+5}\right| + C =$$

$$= -\ln\left|\frac{1-x+\sqrt{5x^2-2x+1}}{x}\right| + C.$$

V. Integration of binomial differentials.

Definition. An expression of the form $x^m(a+bx^n)^p dx$, where m, n, p, a, b are constants is called *a binomial differential*.

Theorem 2.1 (Chebyshev's).

Integrals of the form $\int x^m(a+bx^n)^p dx$, where m, n, p are rational numbers, is reduced to an integral of a rational function **ONLY** in the following cases:

Case 1. if p is an integer. Then, if $p>0$, the integrand is expanded by the formula of the Newton binomial; but if $p<0$, then we make the substitution $x=t^k$, where k is a common denominator of the fractions m and n.

Case 2. if $\frac{m+1}{n}$ is an integer. Then, we put $t^s = (a+bx^n)$, where s is a denominator of the fraction p.

Case 3. if $\frac{m+1}{n} + p$ is an integer. Then, we make the substitution $t^s = \left(\frac{a+bx^n}{x^n}\right)$, where s is a denominator of the fraction p.

Examples.

1. $I = \int \frac{dx}{\sqrt{x}(\sqrt[4]{x}+1)^5} = \int x^{-\frac{1}{2}}(1+x^{\frac{1}{4}})^{-5}dx$.

Here $m = -\frac{1}{2}$, $n = \frac{1}{4}$, $p = -5$. Since p is integer we have Case 1.

We make the substitution $x = t^4$. Then $dx = 4t^3 dt$.

Hence,

$$I = \int (t^4)^{-\frac{1}{2}}(1+(t^4)^{\frac{1}{4}})^{-5}(4t^3)dt = \int \frac{4t^3 dt}{t^2(t+1)^5} = 4\int \frac{tdt}{(t+1)^5} = 4\int \frac{t+1-1}{(t+1)^5}dt =$$

$$= 4\int \frac{dt}{(t+1)^4} - 4\int \frac{dt}{(t+1)^5} = -\frac{4}{3(t+1)^3} + \frac{1}{(t+1)^4} + C.$$

Returning to x, we get

$$I = -\frac{4}{3(\sqrt[4]{x}+1)^3} + \frac{1}{(\sqrt[4]{x}+1)^4} + C.$$

2. $I = \int \frac{x^3 dx}{\sqrt{(1-x^2)^3}} = \int x^3(1-x^2)^{-\frac{3}{2}}dx$.

Here $m = 3$, $n = 2$, $p = -\frac{3}{2}$. Since $\frac{m+1}{n} = \frac{3+1}{2} = 2$ is integer we have Case 2.

$$1 - x^2 = t^2 \Rightarrow x = \sqrt{1-t^2};$$

$$-2xdx = 2tdt \Rightarrow dx = -\frac{tdt}{\sqrt{1-t^2}}.$$

Hence,

$$I = -\int \left(\sqrt{1-t^2}\right)^3 (t^2)^{-\frac{3}{2}} \frac{tdt}{\sqrt{1-t^2}} = -\int \frac{1-t^2}{t^2}dt = \int dt - \int \frac{dt}{t^2} = t + \frac{1}{t} + C.$$

Returning to x, we get

$$I=\sqrt{1-x^2}+\frac{1}{\sqrt{1-x^2}}+C.$$

3. $I=\int\frac{dx}{x^{11}\sqrt{1+x^4}}=\int x^{-11}(1+x^4)^{-\frac{1}{2}}dx.$

Here $m=-11,\ n=4,\ p=-\frac{1}{2}$. Since p and $\frac{m+1}{n}=\frac{-11+1}{4}=-\frac{5}{2}$ are fractions, but $\frac{m+1}{n}+p=-\frac{5}{2}-\frac{1}{2}=-3$ is integer we have Case 3.

$$t^2=\frac{1+x^4}{x^4}\Rightarrow x=\frac{1}{\sqrt[4]{t^2-1}};\qquad dx=-\frac{tdt}{2\sqrt[4]{(t^2-1)^5}}.$$

Hence,

$$I=\int\left(\frac{1}{\sqrt[4]{t^2-1}}\right)^{-11}\left(1+\left(\frac{1}{\sqrt[4]{t^2-1}}\right)^4\right)^{-\frac{1}{2}}\left(\frac{-tdt}{2\sqrt[4]{(t^2-1)^5}}\right)=-\frac{1}{2}\int(t^2-1)^{\frac{11}{4}}\left(\frac{t^2}{t^2-1}\right)^{-\frac{1}{2}}(t^2-1)^{-\frac{5}{4}}tdt=$$

$$=-\frac{1}{2}\int(t^2-1)^2dt=-\frac{t^5}{10}+\frac{t^3}{3}-\frac{t}{2}+C.$$

Returning to x, we get

$$I=-\frac{1}{10x^{10}}\sqrt{(1+x^4)^5}+\frac{1}{3x^6}\sqrt{(1+x^4)^3}-\frac{1}{2x^2}\sqrt{1+x^4}+C.$$

VI. Integration by Trigonometric or Hyperbolic Substitution

Integration of functions rationally depending on x and one of expressions $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ or $\sqrt{x^2-a^2}$ can be reduced to integrals of functions with respect to sine or cosine (ordinary or hyperbolic) by corresponding substitution.

1. For integrals of the form $\int R\left(x,\sqrt{a^2-x^2}\right)dx$ let us put

$$x=a\sin t\Rightarrow\sqrt{a^2-x^2}=\sqrt{a^2(1-\sin^2 t)}=a\cos t$$

or

$$x=a\tanh t\Rightarrow\sqrt{a^2-x^2}=\sqrt{a^2(1-\tanh^2 t)}=\frac{a}{\cosh t}.$$

2. For integrals of the form $\int R\left(x,\sqrt{a^2+x^2}\right)dx$ we use substitution

$$x = a\tan t \Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2 x)} = \frac{a}{\cos t}$$

or

$$x = a\sinh t \Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2(1+\sinh^2 x)} = a\cosh t.$$

3. Integrals of the form $\int R\left(x,\sqrt{x^2-a^2}\right)dx$ can be solved by means of substitution

$$x = \frac{a}{\sin t} \Rightarrow \sqrt{x^2-a^2} = \sqrt{a^2\left(\frac{1}{\sin^2 t}-1\right)} = a\cot t$$

or

$$x = \cosh t \Rightarrow \sqrt{x^2-a^2} = \sqrt{a^2\left(\cosh^2 t-1\right)} = a\sinh t.$$

Example. $I = \int \frac{dx}{x\sqrt{x^2+4}}$

Let us use the substitution

$$x = 2\tan t \Rightarrow \sqrt{4+x^2} = \sqrt{4(1+\tan^2 x)} = \frac{2}{\cos t}$$

and

$$dx = \frac{2}{\cos^2 t}dt.$$

Therefore

$$I = \int \frac{dx}{x\sqrt{x^2+4}} = \int \frac{\frac{2dt}{\cos^2 t}}{2\tan t\,\frac{2}{\cos t}} = \frac{1}{2}\int \frac{dt}{\sin t} = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C = -\frac{1}{4}\ln\left|\frac{1-\cos t}{1-\cos t}\right| + C =$$

$$= \frac{1}{4}\ln\left|\frac{(1-\cos t)^2}{\sin^2 t}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos t}{\sin t}\right| + C = \frac{1}{2}\ln\left|\frac{1}{\sin t}-\cot t\right| + C =$$

$$= \frac{1}{2}\ln\left|\sqrt{1+\frac{1}{\tan^2 t}}-\frac{1}{\tan t}\right| + C = \frac{1}{2}\ln\left|\sqrt{1+\frac{4}{x^2}}-\frac{2}{x}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C.$$

3. The Definite Integral

3.1 The Definite Integral and Its Properties

Let the function $y = f(x)$ be positive, defined and continuous on the interval $[a, b]$. Find the area between the graph of $y = f(x)$, x-axis and the lines $x = a,\ x = b$ (the area of *a curvilinear trapezoid*).

Let us find the area approximately. Partition the interval $[a, b]$ into small intervals by points $a = x_0,\ x_1,\ x_2, \ldots\ x_k,\ x_{k+1}, \ldots\ x_{n-1},\ x_n = b$. In each interval $[x_0, x_1], [x_1, x_2], \ldots [x_k, x_{k+1}], \ldots [x_{n-1}, x_n]$ take a point and denote them $\xi_0,\ \xi_1, \ldots\ \xi_k, \ldots\ \xi_{n-1}$. At each of these points calculate the value of the function $f(\xi_0), \ldots\ f(\xi_k), \ldots\ f(\xi_{n-1})$ (Fig. 1).

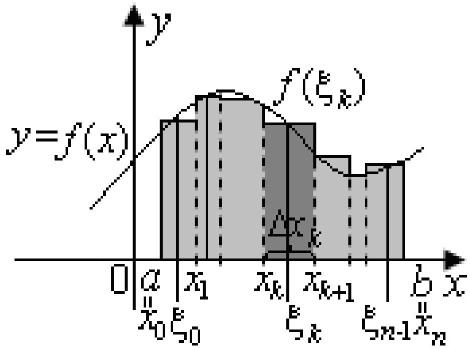


Figure 1.

Express the area as a combination of many vertically-oriented rectangles (the width= $\Delta x_k = x_{k+1} - x_k$, the height= $f(\xi_k)$)

$$S_n \approx \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k .$$

This sum is called *the integral sum* of the function $y = f(x)$ on the interval $[a, b]$.

If we chose the partition of $[a, b]$ small enough, then the area gets better (Fig. 2).

And as the width of rectangles approaches zero ($n \to \infty$), then the sum gives the area under the curve exactly. This idea leads to the concept of the definite integral.

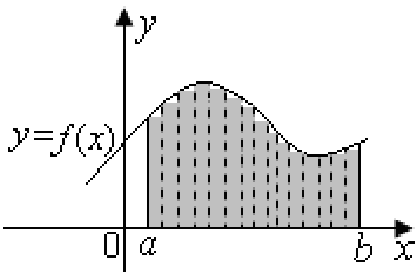


Figure 2.

Definition. If for any partition of the interval $[a, b]$ such that $\max \Delta x_k \to 0$ and for any choice of points ξ_k it exists the limit $\lim\limits_{\max \Delta x_k \to 0} S_n$, then that limit is called *the definite integral* of the function $f(x)$ from a to b and denoted by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k . \tag{3.1}$$

In this case the function $f(x)$ is called *integrable* on the interval $[a,b]$. The numbers *a* and *b* are called *the lower* and *the upper limits* of the integral and interval $[a,b]$ – *the interval of integration*.

Notes.

1. If $y = f(x)$ is positive on the interval $[a,b]$, then the area of a curvilinear trapezoid (Fig. 3) is

$$S = \int_a^b f(x)dx$$

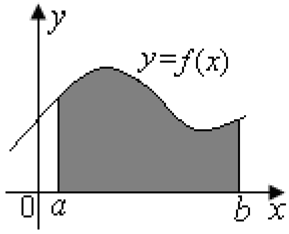


Figure 3.

2. If f is a constant function defined by $y = K$ for every point from $[a,b]$, then

$$\int_a^b K dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} K \cdot \Delta x_k = K \lim_{\max \Delta x_k \to 0} \underbrace{\sum_{k=0}^{n-1} \Delta x_k}_{\text{the length of the interval } [a,b]} = K(b-a).$$

Properties of the Definite Integral:

Theorem 3.1

If a function f is continuous on $[a,b]$, then it is integrable on this interval.

A proof of statement may be found in texts on advanced calculus.

Theorem 3.2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \quad \int_a^a f(x)dx = 0. \quad (3.2)$$

A proof of property follows from the definition of definite integral.

The second equality is natural from the geometric standpoint, because the length of the base of a curvilinear trapezoid is equal to zero; consequently, its area is zero too.

Theorem 3.3

$$\forall K \in \mathbb{R}, \quad K \neq 0: \int_a^b Kf(x)dx = K\int_a^b f(x)dx. \quad (3.3)$$

Proof. According to definition

$$\int_a^b Kf(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} Kf(\xi_k) \cdot \Delta x_k = K \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = K\int_a^b f(x)dx.$$

Theorem 3.4

$$\int_a^b \left(f_1(x) \pm f_2(x)\right)dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx. \qquad (3.4)$$

Proof. From the definition

$$\int_a^b \left(f_1(x) \pm f_2(x)\right)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} \left(f_1(\xi_k) \pm f_2(\xi_k)\right) \cdot \Delta x_k =$$

$$= \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f_1(\xi_k) \cdot \Delta x_k \pm \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f_2(\xi_k) \cdot \Delta x_k = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx.$$

Theorem 3.5

If $a < c < b$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \qquad (3.5)$$

Proof. Since the limit of the integral sum is independent of the partition, let us choose point c as one of the division points: $c = x_m$, $1 \le m \le n-1$.

Hence

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k) \cdot \Delta x_k = \lim_{\max \Delta x_k \to 0} \left(\sum_{k=0}^{m} f(\xi_k) \cdot \Delta x_k + \sum_{k=m+1}^{n-1} f(\xi_k) \cdot \Delta x_k \right) =$$

$$= \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{m} f(\xi_k) \cdot \Delta x_k + \lim_{\max \Delta x_k \to 0} \sum_{k=m+1}^{n-1} f(\xi_k) \cdot \Delta x_k = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Note. If $f(x) \ge 0$, this property is illustrated geometrically (Fig. 4).

The area of a curvilinear trapezoid with the base $[a,b]$ is a sum of areas of a curvilinear trapezoids with the base $[a,c]$ and with the base $[c,b]$.

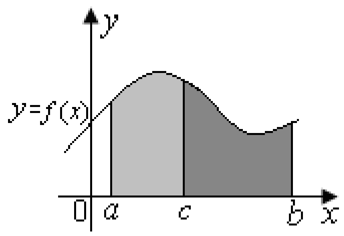


Figure 4.

Theorem 3.6

If the functions $y = f(x)$ and $y = g(x)$ satisfy the condition $f(x) \le g(x)$ on the interval $[a,b]$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx. \qquad (3.6)$$

Proof. Let us consider the difference

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b \left(g(x) - f(x)\right)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} \left(g(\xi_k) - f(\xi_k)\right) \cdot \Delta x_k .$$

Since $g(\xi_k) - f(\xi_k) \geq 0$, $\Delta x_k \geq 0$, each term of the sum is nonnegative, the entire sum is nonnegative, and its limit is nonnegative.

Thus

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq 0$$

or

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Note. If $f(x) \geq 0$, this property could be illustrated geometrically (Fig. 5).

The area of a curvilinear trapezoid under the function $y = f(x)$ is less than the area of a curvilinear trapezoid under the function $y = g(x)$.

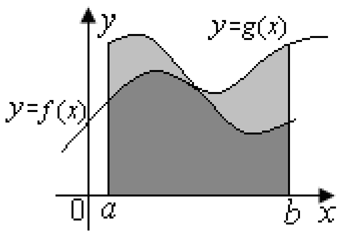


Figure 5.

Theorem 3.7

If m and M are the smallest and the greatest values of the function $y = f(x)$ on the interval $[a,b]$, $a \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a). \tag{3.7}$$

Proof. Since $m \leq f(x) \leq M$, we can use property 6 and note 2:

$$\underbrace{\int_a^b m dx}_{\Downarrow} \leq \int_a^b f(x)dx \leq \underbrace{\int_c^b M dx}_{\Downarrow}$$

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Note.

If $f(x) \geq 0$, this property is clearly illustrated geometrically (Fig. 6).

The area of a curvilinear trapezoid is between the areas of bigger and smaller rectangles.

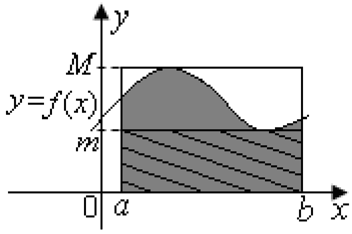


Figure 6.

Theorem 3.8 (Mean-value theorem)

If a function $f(x)$ is continuous on the interval $[a,b]$. then there exists a point $c \in [a,b]$ such that

$$\int_a^b f(x)dx = (b-a) f(c) . \qquad (3.8)$$

Proof. According to property 7 we have

$$m \le \frac{1}{b-a} \int_a^b f(x)dx \le M .$$

Whence

$$\frac{1}{b-a} \int_a^b f(x)dx = \mu ,$$

where $m \le \mu \le M$.

Since $f(x)$ is continuous, it takes on all intermediate values between m and M. Therefore, there exists a point $c \in [a,b]$ such that $\mu = f(c)$, and

$$\int_a^b f(x)dx = (b-a) f(c) .$$

3.3 Fundamental Theorem of Calculus (Newton-Leibniz Formula)

Let us consider the definite integral

$$\int_a^x f(t)dt ,$$

where the lower a limit is fixed and the upper limit x vary (to avoid confusion, we shall use *t* as the independent variable).

Then the value of the integral will vary as well and the integral is a function of upper limit

$$\Phi(x) = \int_a^x f(t)dt .$$

To obtain a geometric interpretation of $\Phi(x)$, suppose that $f(t) \geq 0$ for every t in $[a,b]$. In this case we have that $\Phi(x)$ is the area of the region under the graph of $f(t)$ from a to x (Fig. 7).

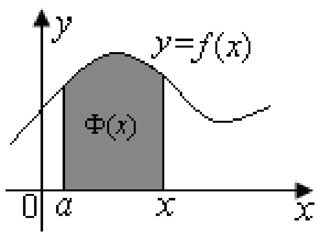


Figure 7.

Let us find the derivative of this function with respect to x.

Theorem 3.9

If function $f(x)$ is continuous function and $\Phi(x) = \int_a^x f(t)dt$, then we have

$$\Phi'(x) = \left(\int_a^x f(t)dt \right)' = f(x).$$

Thus, by definition of primitive (see 1.1 p. 4), $\Phi(x)$ is an antiderivative of $f(x)$.

A proof of statement may be found in [1].

Theorem 3.10 (Fundamental Theorem of Calculus)

Let function $F(x)$ is any antiderivative of function $f(x)$ on the interval $[a,b]$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a). \qquad (3.10)$$

(Newton–Leibniz Formula)

Proof. Let $F(x)$ be some antiderivative of $f(x)$. According the theorem 3.9, the function $\Phi(x) = \int_a^x f(t)dt$ is also an primitive of $f(x)$. From theorem 1.1 we know that the difference between them is a constant.

Thus for every x in $[a,b]$

$$\Phi(x) = F(x) + C$$

or

$$\int_a^x f(t)dt = F(x) + C.$$

Let us put $x = a$ and use the result of theorem 3.2

$$\int_a^a f(t)dt = F(a) + C\,,$$

$$0 = F(a) + C \Rightarrow C = -F(a)\,.$$

Hence,

$$\int_a^x f(t)dt = F(x) - F(a)\,.$$

Finally, we substitute b for x and obtain Newton–Leibniz formula:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example:

$$\int_1^2 (2x+1)dx = \left| \begin{matrix} \text{the primitive} \\ \text{for } (2x+1) \text{ is} \\ (x^2+x) \end{matrix} \right| = (x^2+x)\Big|_1^2 = (2^2+2) - (1^2+1) = 4.$$

3.4 Techniques of Evaluating Definite Integrals

I. Integration by Parts

The method of integration by parts developed for indefinite integrals may also be used to evaluate a definite integral.

Let functions $u(x)$ and $v(x)$ be differentiable. Then

$$\int_a^b u dv = (uv)\Big|_a^b - \int_a^b v du\,. \qquad (3.11)$$

Examples.

1. $$\int_0^1 xe^{2x}dx = \left| \begin{matrix} u = x & du = dx \\ dv = e^{2x}dx & v = \frac{1}{2}e^{2x} \end{matrix} \right| = \frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{2}\int_0^1 e^{2x}dx = \frac{1}{2}xe^{2x}\Big|_0^1 - \frac{1}{4}e^{2x}\Big|_0^1 =$$

$$= \frac{1}{2}\cdot 1 \cdot e^{2\cdot 1} - \frac{1}{2}\cdot 0 \cdot e^{2\cdot 0} - \left(\frac{1}{4}e^{2\cdot 1} - \frac{1}{4}e^{2\cdot 0}\right) = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{e^2+1}{4}.$$

2. $\int_1^3 x\ln x dx = \begin{vmatrix} u=\ln x & du=\frac{1}{x}dx \\ dv=xdx & v=\frac{x^2}{2} \end{vmatrix} = \frac{x^2}{2}\ln x\Big|_1^3 - \int_1^3 \frac{1}{x}\cdot\frac{x^2}{2}dx = \frac{x^2}{2}\ln x\Big|_1^3 - \frac{1}{2}\int_1^3 xdx =$

$$= \frac{x^2}{2}\ln x\Big|_1^3 - \frac{x^2}{4}\Big|_1^3 = \frac{3^2}{2}\ln 3 - \frac{1^2}{2}\ln 1 - \left(\frac{3^2}{4} - \frac{1^2}{4}\right) = 9\ln\sqrt{3} - 2.$$

II. Integration by the Substitution

The method of substitution is also useful when calculating a definite integral. We could use this idea to find an antiderivative and then apply the Newton–Leibniz formula.

Another method, which is often shorter, is to change the limits of integration. In this case we do not need to return to the old variable

Let the function $f(x)$ be continuous on the interval $[a,b]$ and let us evaluate the integral

$$\int_a^b f(x)dx.$$

Let us make a substitution $x=\varphi(u)$, where u is a new variable. The function $\varphi(u)$ is such that

1. $\varphi(\alpha)=a$ and $\varphi(\beta)=b$;
2. $\varphi(u)$ and $\varphi'(u)$ are continuous on $[\alpha,\beta]$;
3. $f(\varphi(u))$ is defined and continuous on $[\alpha,\beta]$.

Hence,

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(u))\varphi'(u)du. \qquad (3.12)$$

Examples.

1. Evaluate the integral $\int_1^3 \frac{\sqrt{x}}{1+x}dx$.

Make a substitution $x=t^2$, $dx=2tdt$.

Determine the new limits

if $x=1$, then $t=1$,

$$\text{if } x = 3 \text{, then } t = \sqrt{3}.$$

Thus

$$\int_1^3 \frac{\sqrt{x}}{1+x}dx = \int_1^{\sqrt{3}} \frac{2t^2}{1+t^2}dt = 2\int_1^{\sqrt{3}} \left(1 - \frac{1}{1+t^2}\right)dt == 2(t - \arctan t)\Big|_1^{\sqrt{3}} = 2\left(\sqrt{3} - 1 + \frac{\pi}{12}\right).$$

2. Compute the integral $\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{2+\cos x}$.

Apply the substitution

$$t = \cos x \Rightarrow x = \arccos t, \ -\sin x dx = dt.$$

Determine the new limits

$$\text{if } x = 0 \text{, then } t = 1,$$

$$\text{if } x = \frac{\pi}{2} \text{, then } t = 0.$$

Thus

$$\int_0^{\frac{\pi}{2}} \frac{\sin x dx}{2+\cos x} = -\int_1^0 \frac{dt}{2+t} = \int_0^1 \frac{dt}{2+t} = \ln|1+t|\Big|_0^1 = \ln|2| - \ln|1| = \ln 2.$$

4. Improper Integrals

Previously we studied the definite integral of a function $f(x)$ for the case when $f(x)$ is a *bounded* function defined on a *closed* interval $[a,b]$. Is it possible to integrate functions over infinite intervals? Could we integrate unbounded functions? Let us consider a notion of integral, called improper integral, in a few cases.

4.1 Improper Integrals with Infinite Limits

A definite integral, that has either or both limits infinite:

$$\int_a^{+\infty} f(x)dx,\quad \int_{-\infty}^{b} f(x)dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)dx,$$

is called an improper integral of the first type.

Let $f(x)$ be defined on $[a,+\infty]$ and integrable on $[a,b]$ for all $b>a$. If there exists a finite limit

$$\lim_{b\to+\infty}\int_a^b f(x)dx,$$

then the improper integral $\int_a^{+\infty} f(x)dx$ is called *convergent* and

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx. \qquad (4.1)$$

If such a limit is not finite then the improper integral *does not exist* and is called *divergent*.

The geometric meaning of an improper integral is obvious when the function $f(x)$ is positive. Since the integral $\int_a^b f(x)dx$ expresses the area of curvilinear trapezoid we can consider the improper integral $\int_a^{+\infty} f(x)dx$ as an area of unbounded region lying between the lines $y=f(x)$, $x=a$ and x-axis (Fig. 8).

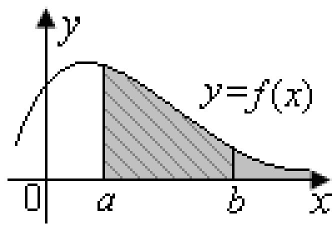


Figure 8.

Similarly, we define the improper interval over other infinite intervals:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx, \tag{4.2}$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)dx + \lim_{b\to+\infty}\int_{c}^{b} f(x)dx, \tag{4.3}$$

where c is any number ($c=0$ is often convenient). Note that this requires both of the limits to be finite in order for the integral to be also convergent. If either of two limits does not exist then the integral is divergent.

Examples.

1. Find out at which values of m the integral $\int_{1}^{+\infty}\frac{1}{x^m}dx$ is convergent and at which it is divergent.

If $m<1$, then $1-m>0$ and

$$\int_{1}^{+\infty}\frac{1}{x^m}dx = \lim_{b\to+\infty}\int_{1}^{b} x^{-m}dx = \lim_{b\to+\infty}\left.\frac{x^{1-m}}{1-m}\right|_{1}^{b} = \lim_{b\to+\infty}\frac{1}{1-m}\left(b^{1-m}-1\right) = +\infty.$$

If $m=1$, then

$$\int_{1}^{+\infty}\frac{1}{x}dx = \lim_{b\to+\infty}\int_{1}^{b}\frac{1}{x}dx = \lim_{b\to+\infty}\ln|x|\Big|_{1}^{b} = \lim_{b\to+\infty}\left(\ln|b|-\ln 1\right) = +\infty.$$

If $m>1$, then $m-1>0$ and

$$\int_{1}^{+\infty}\frac{1}{x^m}dx = \lim_{b\to+\infty}\int_{1}^{b} x^{-m}dx = \lim_{b\to+\infty}\left.\frac{-1}{(m-1)x^{m-1}}\right|_{1}^{b} = \lim_{b\to+\infty}\frac{-1}{m-1}\left(\frac{1}{b^{m-1}}-1\right) = \frac{1}{m-1}.$$

Consequently, the integral $\int_{1}^{+\infty}\frac{1}{x^m}dx$ converges if $m>1$ and it diverges when $m\le 1$.

2. Calculate $\int_{-\infty}^{+\infty}\frac{1}{x^2+1}dx$.

According to the definition

$$\int_{-\infty}^{+\infty}\frac{1}{x^2+1}dx = \int_{-\infty}^{0}\frac{1}{x^2+1}dx + \int_{0}^{+\infty}\frac{1}{x^2+1}dx = \lim_{a\to-\infty}\int_{a}^{0}\frac{1}{x^2+1}dx + \lim_{b\to+\infty}\int_{0}^{b}\frac{1}{x^2+1}dx =$$

$$= \lim_{a\to-\infty}\arctan x\Big|_{a}^{0} + \lim_{b\to+\infty}\arctan x\Big|_{0}^{b} = -\lim_{a\to-\infty}\arctan a + \lim_{b\to+\infty}\arctan b = \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

In some cases it is sufficient to determine whether the integral converges or diverges, and estimate the value. The following test can help us.

Comparison test.

1. Let functions $f(x)$ and $\varphi(x)$ be defined for all $x \geq a$ and integrable on each interval $[a,b]$ for all $b > a$. If $0 \leq f(x) \leq \varphi(x)$ for all $x \geq a$, then from convergence of the integral $\int_a^{+\infty} \varphi(x)dx$ it follows that the integral $\int_a^{+\infty} f(x)dx$ is convergent, and $\int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} \varphi(x)dx$; from divergence of the integral $\int_a^{+\infty} f(x)dx$ it follows that the integral $\int_a^{+\infty} \varphi(x)dx$ is also divergent.

2. Let function $f(x)$ be defined for all $x \geq a$. If the integral $\int_a^{+\infty} |f(x)|dx$ converges, then the integral $\int_a^{+\infty} f(x)dx$ also converges and is called *absolutely convergent*.

If the integral $\int_a^{+\infty} f(x)dx$ converges, and $\int_a^{+\infty} |f(x)|dx$ diverges, then the integral $\int_a^{+\infty} f(x)dx$ is called *conditionally convergent*.

Examples.

1. Investigate the integral $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ for convergence.

Since

$$\frac{1}{x^2(1+e^x)} < \frac{1}{x^2} \text{ for } x \geq 1,$$

and

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{b\to+\infty} \int_1^b \frac{1}{x^2}dx = -\lim_{b\to+\infty} \frac{1}{x}\Bigg|_1^b = -\lim_{b\to+\infty}\left(\frac{1}{b} - 1\right) = 1,$$

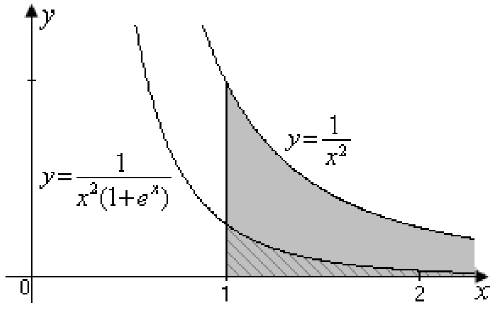


Figure 9.

we obtain that the integral $\int_1^{+\infty} \frac{dx}{x^2(1+e^x)}$ converges ant its value is less than 1 (Fig. 9).

2. Find out whether the integral $\int\limits_1^{+\infty} \frac{x+1}{\sqrt{x^3}} dx$ converges

It will be noted that

$$\frac{x+1}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}.$$

But

$$\int\limits_1^{+\infty} \frac{1}{\sqrt{x}} dx = \lim_{b \to +\infty} 2\sqrt{x}\Big|_1^b = +\infty.$$

Whence the original integral is divergent.

3. Investigate the convergence of the integral $\int\limits_1^{+\infty} \frac{\sin x}{x^2} dx$.

Since

$$\left|\frac{\sin x}{x^2}\right| \le \left|\frac{1}{x^2}\right| = \frac{1}{x^2} \text{ for all } x \ge 1$$

and

$$\int\limits_1^{+\infty} \frac{1}{x^2} dx = \lim_{b \to +\infty} \frac{-1}{x}\Big|_1^b = 1,$$

it follows that the integral $\int\limits_1^{+\infty} \left|\frac{\sin x}{x^2}\right| dx$ converges and $\int\limits_1^{+\infty} \frac{\sin x}{x^2} dx$ is absolutely convergent.

4.2 Improper Integrals of Discontinuous Functions

Definite integral that has an integrand that approaches infinity at one or more points in the range of integration is called an improper integral of the second type.

If the function $f(x)$ is defined for all $a \le x < b$, integrable on any interval $[a, b-\varepsilon]$, $0 < \varepsilon < b-a$ and unbounded to the left of the point b.

Let us consider

$$\int\limits_a^b f(x)dx = \lim_{\varepsilon \to 0+} \int\limits_a^{b-\varepsilon} f(x)dx. \qquad (4.4)$$

If this limit is existent and finite, then the improper integral is called *convergent*. Otherwise, it is called divergent.

Analogously, if the integrand $f(x)$ is unbounded to the right from the point a, then

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+}\int_{a+\varepsilon}^b f(x)dx. \qquad (4.5)$$

Finally, if the function is unbounded in the neighborhood of an interior point c of the interval $[a,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\varepsilon_1\to 0+}\int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2\to 0+}\int_{c+\varepsilon_2}^b f(x)dx. \qquad (4.6)$$

Examples.

1. Find out at which values of m the integral $\int_0^1 \frac{1}{x^m}dx$ is convergent and at which it is divergent. The integrand $\frac{1}{x^m}$ is defined for all $0 < x \le 1$ and unbounded to the right of the point 0.

If $m<1$, then $1-m>0$ and

$$\int_0^1 \frac{1}{x^m}dx = \lim_{\varepsilon\to 0}\int_{0+\varepsilon}^1 \frac{1}{x^m}dx = \lim_{\varepsilon\to 0}\left.\frac{x^{1-m}}{1-m}\right|_{0+\varepsilon}^1 = \lim_{\varepsilon\to 0}\frac{1}{1-m}\left(1-(0+\varepsilon)^{1-m}\right) = \frac{1}{1-m}.$$

If $m=1$, then $\int_0^1 \frac{1}{x}dx = \lim_{\varepsilon\to 0}\int_{0+\varepsilon}^1 \frac{1}{x}dx = \lim_{\varepsilon\to 0} \ln|x|\,\|_{0+\varepsilon}^1 = \lim_{\varepsilon\to 0}\left(\ln 1 - \ln|0+\varepsilon|\right) = +\infty$.

If $m>1$, then $m-1>0$ and

$$\int_0^1 \frac{1}{x^m}dx = \lim_{\varepsilon\to 0}\int_{0+\varepsilon}^1 \frac{1}{x^m}dx = \lim_{\varepsilon\to 0}\left.\frac{-1}{(m-1)x^{m-1}}\right|_{0+\varepsilon}^1 = \lim_{\varepsilon\to 0}\frac{-1}{m-1}\left(1-\frac{1}{(0+\varepsilon)^{m-1}}\right) = +\infty.$$

Consequently, the integral $\int_1^0 \frac{1}{x^m}dx$ converges if $m<1$ and it diverges when $m\ge 1$.

2. Investigate the integral $\int_{e^{-1}}^1 \frac{dx}{x\ln^3 x}$ for convergence.

The function $\frac{1}{x\ln^3 x}$ is unbounded to the left of the point 1.

$$\int_{e^{-1}}^1 \frac{dx}{x\ln^3 x} = \lim_{\varepsilon\to 0}\int_{e^{-1}}^{1-\varepsilon}\frac{d\ln x}{\ln^3 x} = \lim_{\varepsilon\to 0}\left.\frac{-1}{2\ln^2 x}\right|_{e^{-1}}^{1-\varepsilon} = \lim_{\varepsilon\to 0}\left(\frac{-1}{2\ln^2(1-\varepsilon)} - \frac{-1}{2\ln^2 e^{-1}}\right) = \frac{1}{2}.$$

Therefore the integral converges.

For the functions defined and positive on the interval convergence tests are analogous to the comparison tests for improper integrals with infinite limits.

Comparison test.

1. Let functions $f(x)$ and $\varphi(x)$ be defined on the interval $[a,b)$ and discontinuous at the point b. If $0 \le f(x) \le \varphi(x)$ at all points of interval $[a,b)$, then from convergence of the integral $\int_a^b \varphi(x)dx$ it follows that the integral $\int_a^b f(x)dx$ is convergent; from divergence of the integral $\int_a^b f(x)dx$ it follows that the integral $\int_a^b \varphi(x)dx$ is also divergent.

2. Let $f(x)$ be an alternating function on the interval $[a,b]$ and discontinuous only at the point b. If the integral $\int_a^b |f(x)|dx$ converges, then the integral $\int_a^b f(x)dx$ also converges and is called *absolutely convergent*.

If the integral $\int_a^b f(x)dx$ converges, and $\int_a^b |f(x)|dx$ diverges, then the integral $\int_a^b f(x)dx$ is called *conditionally convergent*.

Analogous tests are also valid for improper integrals $\int_a^b f(x)dx$, where $f(x)$ is unbounded to the right from the point a.

Example.

Investigate the integral $\int_0^1 \frac{\cos^2 x dx}{\sqrt[3]{1-x}}$ for convergence.

The integrand is unbounded to the right of the point 1.

Since $|\cos x| < 1$, we have $0 < \left|\frac{\cos^2 x}{\sqrt[3]{1-x}}\right| < \frac{1}{\sqrt[3]{1-x}}$. The integral $\int_0^1 \frac{dx}{\sqrt[3]{1-x}}$ is convergent according the first example of this chapter. Hence, the original integral converges.

5. Application of the Definite Integral

5.1 The Area of a Region

I. *The Area of a Curvilinear Trapezoid*

Let the function $y = f(x)$ be positive, defined and continuous on the interval $[a,b]$.

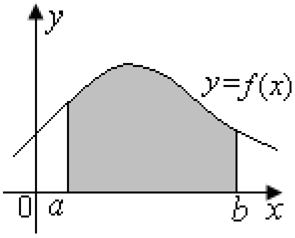


Figure 10.

As we know from the chapter 3.1 the area between the graph of $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ (Fig. 10)

$$S = \int_a^b f(x)dx. \qquad (5.1)$$

Example.

Compute the area of the region bounded by $y = e^x$, x-axis and the lines $x = -1$ and $x = 1$ (Fig.11).

Let us use the formula (5.1):

$$S = \int_{-1}^{1} e^x dx = e^x \Big|_{-1}^{1} = e - e^{-1} = \frac{e^2 - 1}{e} \text{ (units}^2\text{)}.$$

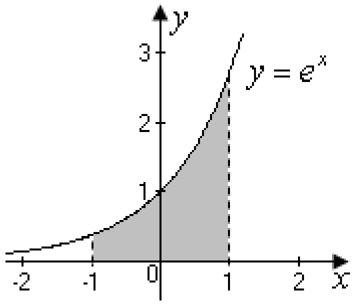


Figure 11.

If the curvilinear trapezoid is bounded by the curve represented by equations in parametric form

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} \quad t_1 \le t \le t_2$$

and

$$x(t_1) = a, \quad x(t_2) = b.$$

Let us use the formula (5.1) to compute the area

$$S = \int_a^b f(x)dx = \int_a^b ydx.$$

Change the variable in the integral

$$x = x(t), \qquad dx = x'(t)dt, \qquad t_1 \le t \le t_2,$$
$$y = f(x) = f(x(t)) = y(t).$$

Hence

$$S = \int_{t_1}^{t_2} y(t)x'(t)dt. \qquad (5.2)$$

Example.

Compute the area of the region bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Fig.12).

Let us use the parametric equations of ellipse

$$\begin{cases} x = a\cos t, \\ y = b\sin t. \end{cases}$$

Since the region is symmetric about the coordinate axes, we compute the area of one quarter. Here x varies from 0 to a, and so t varies between $t_1 = \frac{\pi}{2}$ and $t_2 = 0$.

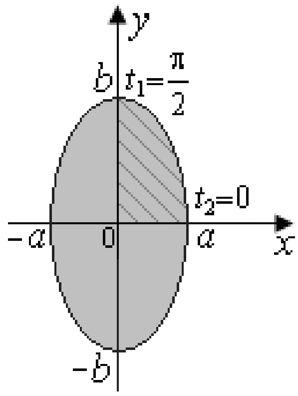


Figure 12.

According to the formula (5.2):

$$S = 4\int_{\frac{\pi}{2}}^{0} b\sin t(a\cos t)'dt = 4\int_{\frac{\pi}{2}}^{0} a\sin t(-b\sin t)dt = 4ab\int_{0}^{\frac{\pi}{2}} \sin^2 t dt =$$

$$= 2ab\int_{0}^{\frac{\pi}{2}} (1-\cos 2t)dt = 2ab\left(t - \frac{\sin 2t}{2}\right)\Bigg|_0^{\frac{\pi}{2}} = 2ab\left(\frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + \frac{\sin 0}{2}\right) = \pi ab \text{ (units}^2\text{)}.$$

Thus, the area of the region bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$S = \pi ab. \qquad (5.3)$$

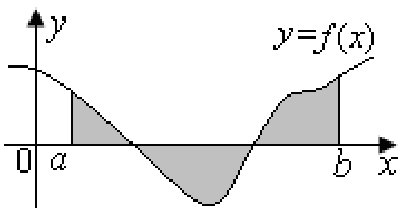


Figure 13.

If $f(x)$ changes sign on the interval $[a,b]$ a finite number of times (Fig.13), then

$$S = \int_a^b |f(x)|dx$$

II. ***The Area Between Two Curves***

Let the functions $y = f(x)$ and $y = g(x)$ be positive, defined and continuous on the interval $[a,b]$ and for every $x \in [a,b]$ $g(x) \le f(x)$.

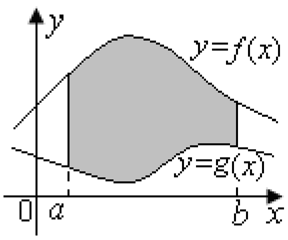


Figure 14.

Then the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ (Fig. 14) is

$$S = \int_a^b f(x)dx - \int_a^b g(x)dx,$$

$$S = \int_a^b (f(x) - g(x))dx. \qquad (5.4)$$

Example.

Evaluate the area of the region between the curves

$$y = 4 - x^2 \text{ and } y = x^2 - 2x + 1.$$

Solving the system of equation

$$\begin{cases} y = 4 - x^2, \\ y = x^2 - 2x + 1, \end{cases}$$

find the abscissas of the points of intersection of the curves. Then eliminating y we obtain

$$4 - x^2 = x^2 - 2x + 1,$$

whence $x_1 = -1$ and $x_2 = 2$.

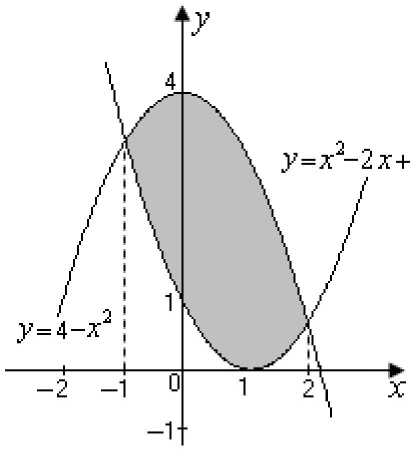


Figure 15.

As it seen from the figure 15, $4 - x^2 \ge x^2 - 2x + 1$ on the interval $[-1,2]$.

Consequently,

$$S = \int_{-1}^{2} ((4 - x^2) - (x^2 - 2x + 1))dx = \int_{-1}^{2} (3 - 2x^2 + 2x))dx = \left(3x - \frac{2x^3}{3} + x^2\right)\Bigg|_{-1}^{2} =$$

$$= \left(6 - \frac{16}{3} + 4\right) - \left(-3 + \frac{2}{3} + 1\right) = 6 \text{ (units}^2\text{)}.$$

III. ***The Area of a Curvilinear Sector in Polar Coordinates***

Consider a curve defined in polar coordinates by the equation

$$\rho = \rho(\varphi), \quad \alpha \le \varphi \le \beta,$$

where $\rho(\varphi)$ is a continuous function for $\varphi \in [\alpha, \beta]$.

Let us find the area inside of polar curve $\rho = \rho(\varphi)$ between the radius vectors $\varphi = \alpha$ and $\varphi = \beta$. The idea is the same as with the area of a curvilinear trapezoid: find an approximation that approaches the true value.

Partition the sector $[\alpha, \beta]$ into small subsectors by radius vectors $\alpha = \varphi_0,\ \varphi_1,\ \ldots\ \varphi_k,\ \varphi_{k+1}, \ldots\ \varphi_{n-1},\ \varphi_n = \beta$. In each part $[\varphi_k, \varphi_{k+1}]$, $k = 0, \ldots (n-1)$ take an angle ξ_k and calculate the value of the function $\rho(\xi_k)$ (Fig. 13).

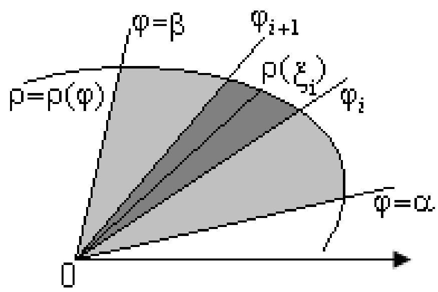


Figure 13.

We approximate the region using sectors of circles

$$S_k = \frac{1}{2}\rho^2(\xi_k)(\varphi_{k+1} - \varphi_k) = \frac{1}{2}\rho^2(\xi_k)\Delta\varphi_k,\ k = 0, \ldots (n-1).$$

Thus, the sum

$$S = \sum_{k=0}^{n-1} S_k = \sum_{k=0}^{n-1} \frac{1}{2}\rho^2(\xi_k)\Delta\varphi_k$$

give the approximation of the area of the region.

Since this sum is an integral sum, its limit as $\max \Delta\varphi_i \to 0$, is the definite integral, and we obtain the formula for the area of a curvilinear sector

$$S = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\varphi)d\varphi. \tag{5.5}$$

Example.

Find the area of a region enclosed by the portion of Archimedean spiral $\rho = \varphi,\ 0 \le \varphi \le \dfrac{3\pi}{2}$ (Fig. 14).

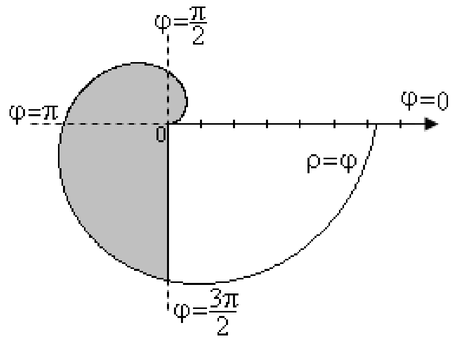


Figure 14.

Use the formula (5.5)

$$S = \frac{1}{2}\int_0^{\frac{3\pi}{2}} \varphi^2 d\varphi = \left.\frac{\varphi^2}{3}\right|_0^{\frac{3\pi}{2}} = \frac{(3\pi)^3}{3 \cdot 8} = \frac{9\pi^3}{8} \quad (\text{units}^2).$$

5.2 The Arc Length of a Curve

I. ***The Arc Length of a Curve in Rectangular Coordinates***

Let us find the length of the arc of a curve between points A and B. The curve is given by the equation $y = f(x)$ such that functions $f(x)$ and $f'(x)$ are continuous on the interval $[a,b]$.

Divide the interval up into n subintervals by the points

$$A = M_0, M_1, \ldots M_k, M_{k+1}, \ldots M_{n-1}, M_n = B.$$

Approximately the length of the curve is a sum of segments connecting these points (Fig. 15)

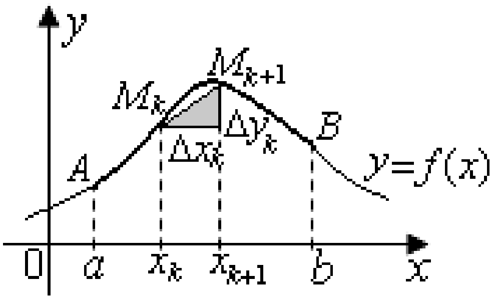


Figure 15.

$$L_{AB} \approx \sum_{k=0}^{n-1} |M_k M_{k+1}|.$$

The length of each segment we can find using Pythagorean theorem

$$|M_k M_{k+1}| = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}.$$

Since, by the Lagrange's theorem

$$\frac{\Delta y_k}{\Delta x_k} = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = f'(\xi_k), \quad x_k < \xi_k < x_{k+1},$$

we have

$$|M_k M_{k+1}| = \Delta x_k \sqrt{1 + \left(f'(\xi_k)\right)^2}$$

and

$$L_{AB} \approx \sum_{k=0}^{n-1} \Delta x_k \sqrt{1 + \left(f'(\xi_k)\right)^2}.$$

Therefore, this is integral sum for the continuous function $\sqrt{1 + (f'(x))^2}$ and a limit as $\max \Delta x_k \to 0$ give us the formula for computing the length of arc

$$L_{AB} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \qquad (5.6)$$

Example.

Evaluate the length the curve $y=\sqrt{(x-1)^3}$ between the points $(1,0)$ and $(5,8)$.

Find the derivative of the function

$$y'=\left(\sqrt{(x-1)^3}\right)'=\frac{3}{2}\sqrt{x-1}.$$

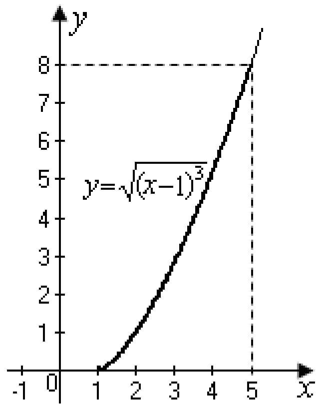


Figure 16.

Hence,

$$L=\int_1^5\sqrt{1+\left(\frac{3}{2}\sqrt{x-1}\right)^2}\,dx=\int_1^5\sqrt{\frac{9}{4}x-\frac{5}{4}}\,dx=\frac{2}{3}\cdot\frac{4}{9}\sqrt{\left(\frac{9}{4}x-\frac{5}{4}\right)^3}\Bigg|_1^5=$$

$$=\frac{8}{27}\left(\sqrt{\left(\frac{45}{4}-\frac{5}{4}\right)^3}-\sqrt{\left(\frac{9}{4}-\frac{5}{4}\right)^3}\right)=\frac{8}{27}(10\sqrt{10}-1)\quad\text{(units).}$$

II. ***The Arc Length of a Curve Represented Parametrically***

Let a curve be given by the equations in the parametric form

$$x=x(t),\quad y=y(t),$$

and the derivatives $x'(t),\ y'(t)$ be continuous on the interval $[t_1,t_2]$.

In this case we can use formula (5.6), where

$$f'(x)=\frac{dy}{dx}=\frac{y'_t}{x'_t}=\frac{y'(t)}{x'(t)},\quad dx=x'(t)dt\quad\text{and}\quad x(t_1)=a,\quad x(t_2)=b.$$

Hence,

$$L_{AB}=\int_{t_1}^{t_2}\sqrt{1+\left(\frac{y'(t)}{x'(t)}\right)^2}\,x'(t)dt.$$

Finally,

$$L_{AB}=\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt. \qquad (5.7)$$

Note: If the **space** curve is represented parametrically

$$x=x(t),\quad y=y(t),\quad x=x(t),\quad t\in[t_1,t_2],$$

then

$$L_{AB}=\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}\,dt.$$

Example.

Find the length of one arc of cycloid $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} t \in [0, 2\pi]$.

Let us find the half of curve as $t \in [0, \pi]$.

Differentiating with respect to t, we obtain

$$\begin{cases} x' = a(t - \sin t)' = a(1 - \cos t), \\ y' = a(1 - \cos t)' = a \sin t. \end{cases}$$

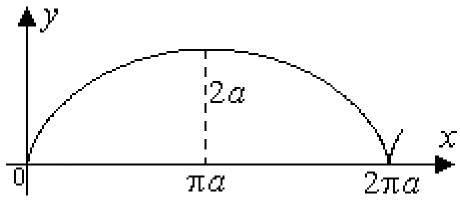


Figure 17.

Hence,

$$L = 2\int_0^{\pi} \sqrt{\left(a(1-\cos t)\right)^2 + \left(a \sin t\right)^2}\, dt = 2\int_0^{\pi} \sqrt{a^2(1 - 2\cos t + \cos^2 t + \sin^2 t)}\, dt =$$

$$= 2a\int_0^{\pi} \sqrt{2 - 2\cos t}\, dt = 2a\int_0^{\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt = 4a\int_0^{\pi} \left|\sin \frac{t}{2}\right| dt = -8a \cos\frac{t}{2}\Big|_0^{\pi} =$$

$$= -8a \cos\frac{\pi}{2} + 8a \cos 0 = 8a \quad \text{(units)}.$$

III. ***The Arc Length of a Curve in Polar Coordinates***

If a smooth curve is given by the equation $\rho = \rho(\varphi)$, $\alpha < \varphi < \beta$, in polar coordinates.

Let us use the formulas for converting polar coordinates to Cartesian coordinates

$$\begin{cases} x = \rho \cos \varphi, \\ y = \rho \sin \varphi. \end{cases}$$

Since $\rho = \rho(\varphi)$, we put this expression in place of ρ and obtain

$$\begin{cases} x = \rho(\varphi) \cos \varphi, \\ y = \rho(\varphi) \sin \varphi, \end{cases} \quad \alpha < \varphi < \beta.$$

These equations are regarded as parametric equations of the curve. Applying formula (5.7) we obtain

$$L_{AB} = \int_{\alpha}^{\beta} \sqrt{\left(x'(\varphi)\right)^2 + \left(y'(\varphi)\right)^2}\, d\varphi = \int_{\alpha}^{\beta} \sqrt{\left((\rho(\varphi)\cos\varphi)'\right)^2 + \left((\rho(\varphi)\sin\varphi)'\right)^2}\, d\varphi =$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\rho'(\varphi)\cos\varphi - \rho(\varphi)\sin\varphi\right)^2 + \left(\rho'(\varphi)\sin\varphi + \rho(\varphi)\cos\varphi\right)^2}\, d\varphi =$$

$$= \int_{\alpha}^{\beta} \sqrt{(\rho')^2 \cos^2\varphi - 2\rho\rho' \sin\varphi\cos\varphi + \rho^2 \sin^2\varphi + (\rho')^2 \sin^2\varphi + 2\rho\rho' \sin\varphi\cos\varphi + \rho^2 \cos^2\varphi}\, d\varphi =$$

$$= \int_{\alpha}^{\beta} \sqrt{(\rho')^2\left(\cos^2\varphi + \sin^2\varphi\right) + \rho^2\left(\sin^2\varphi + \cos^2\varphi\right)}\, d\varphi = \int_{\alpha}^{\beta} \sqrt{(\rho')^2 + \rho^2}\, d\varphi.$$

Hence,

$$L_{AB} = \int_{\alpha}^{\beta} \sqrt{\left(\rho'(\varphi)\right)^2 + \left(\rho(\varphi)\right)^2}\, d\varphi. \tag{5.8}$$

Example.

Find the length of the cardioid $\rho = a(1 - \cos\varphi)$ (Fig. 18).

This curve is symmetrical about polar axis, that's why we varying the angle from 0 to π and multiplying the integral by 2. Here, $\rho' = a\sin\varphi$.

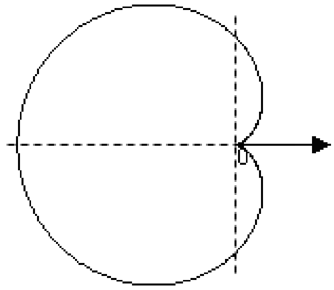


Figure 18.

Hence,

$$L_{AB} = 2\int_{0}^{\pi} \sqrt{(a\sin\varphi)^2 + \left(a(1-\cos\varphi)\right)^2}\, d\varphi =$$

$$= 2a\int_{0}^{\pi} \sqrt{\sin^2\varphi + 1 - 2\cos\varphi + \cos^2\varphi}\, d\varphi = 2a\int_{0}^{\pi} \sqrt{2 - 2\cos\varphi}\, d\varphi = 2a\int_{0}^{\pi} \sqrt{4\sin^2\frac{\varphi}{2}}\, d\varphi =$$

$$= 4a\int_{0}^{\pi} \sin\frac{\varphi}{2}\, d\varphi = -8a\cos\frac{\varphi}{2}\Big|_{0}^{\pi} = -8a\cos\frac{\pi}{2} + 8a\cos 0 = 8a \text{ (units).}$$

5.3 Volume of a Solid

I. *Volume of a Solid From the Areas of Parallel Sections*

Suppose we have a solid. Assume that we know the area of any section of the solid by the plane perpendicular to the x-axis (Fig. 19) and this area is a function of x: $S = S(x)$.

Cut the solid by planes $x = a,\ x = x_1, \dots\ x = x_k$, $x = x_{k+1}, \dots\ x = b$ into n layers. Each layer is a cylindrical body, which volume is a product of the area of the base $(S = S(\xi_k))$ and the altitude (Δx_k):

$$V_k = S(\xi_k)\Delta x_k .$$

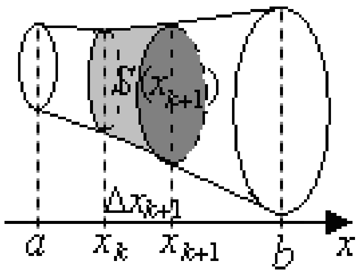


Figure 19.

The volume of all the cylinders will be

$$V \approx \sum_{k=0}^{n-1} V_k = \sum_{k=0}^{n-1} S(\xi_k)\Delta x_k .$$

It is the integral sum of the continuous function $S(x)$ on the interval $a \le x \le b$ and, finally, we obtain the formula for the volume of a solid

$$V = \int_a^b S(x)dx. \qquad (5.9)$$

Example.

Evaluate the volume of ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (Fig. 20).

Let us make a section of ellipsoid by the plane $x = x_0$ parallel to the *yz*-plane. Here we have the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x_0^{\ 2}}{a^2}$$

or

$$\frac{y^2}{b^2\left(1 - \dfrac{x_0^{\ 2}}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \dfrac{x_0^{\ 2}}{a^2}\right)} = 1.$$

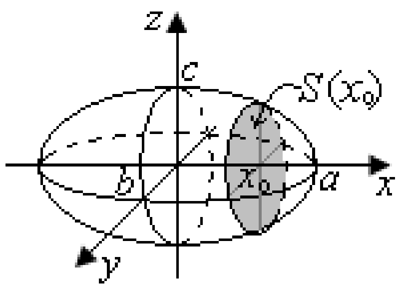


Figure 20.

According to formula (5.3), the area of this ellipse is

$$S(x_0)=\pi b\sqrt{1-\frac{x_0{}^2}{a^2}}\cdot c\sqrt{1-\frac{x_0{}^2}{a^2}}=\pi bc\left(1-\frac{x_0{}^2}{a^2}\right).$$

Hence, the volume of ellipsoid is

$$V=\int_{-a}^{a}S(x)dx=\pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx=\pi bc\left(x-\frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a}=\frac{4}{3}\pi abc\ \text{(units)}^3.$$

II. The Volume of a Solid of Revolution

Consider the solid generated by revolution about the x-axis of the curvilinear trapezoid bounded by the curve $y=f(x)$ ($f(x)\geq 0$), the x-axis and the straight lines $x=a$ and $x=b$ (Fig. 21).

An arbitrary section of this solid made by plane perpendicular to the x-axis is a circle of radius $f(x)$ and its area is

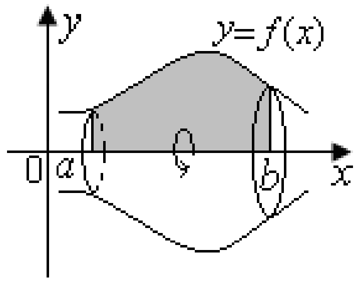


Figure 21.

$$S(x)=\pi(f(x))^2.$$

Let us use formula (5.9) and obtain the formula of volume of a solid of revolution about the x-axis

$$V_{ox}=\pi\int_a^b(f(x))^2dx. \tag{5.10}$$

Example.

Find the volume of a solid obtained by revolving about the x-axis of the figure bounded by the first arc of the sinusoid $y=\sin x$ (Fig. 22).

$$V_{ox}=\pi\int_0^{\pi}(\sin x)^2dx=\frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)dx=$$

$$=\frac{\pi}{2}\left(x-\frac{\sin 2x}{2}\right)\Bigg|_0^{\pi}=\frac{\pi^2}{2}\ \text{(units)}^3.$$

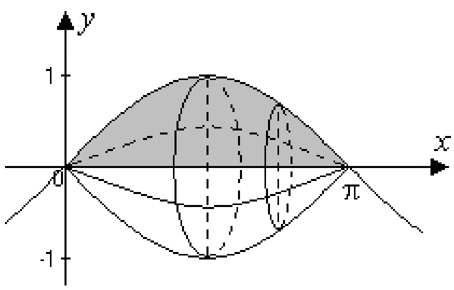


Figure 22.

Let us consider the solid of revolution about the y-axis of the curvilinear trapezoid bounded by the curve $y = f(x)$, the x-axis and the straight lines $x = a$ and $x = b$ (Fig. 23).

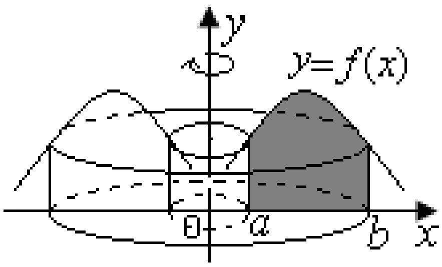


Figure 23.

The volume of a solid of revolution about the y-axis

$$V_{oy} = 2\pi\int_a^b x\, f(x)dx. \qquad (5.11)$$

Example.

The figure bounded by the arc of the sinusoid $y = \sin x$, the x-axis and the straight line $x = \frac{\pi}{2}$ revolves about the y-axis (Fig. 24). Compute the volume of the solid of revolution thus obtained.

$$V_{oy} = 2\pi\int_0^{\frac{\pi}{2}} x\sin x dx = \begin{vmatrix} u = x & du = dx \\ dv = \sin x dx & v = -\cos x \end{vmatrix} =$$

$$= 2\pi\left(-x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx \right) =$$

$$= 2\pi \sin x\Big|_0^{\frac{\pi}{2}} = 2\pi \text{ (units)}^3.$$

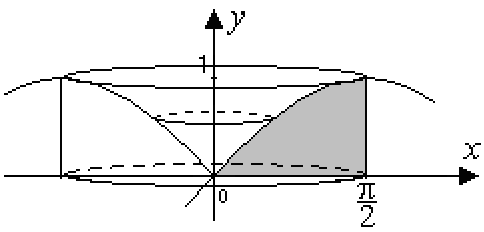


Figure 24.

If the solid of revolution is generated by the rotation of the curvilinear trapezoid bounded by the parametric curve $\begin{cases} x = x(t), \\ y = y(t), \end{cases}$ $t_1 \le t \le t_2$, then

$$V_{ox} = \pi\int_{t_1}^{t_2} y^2(t)x'(t)dt. \qquad (5.12)$$

$$V_{oy} = 2\pi\int_{t_1}^{t_2} x(t)\, y(t)x'(t)dt. \qquad (5.13)$$

The volume of the solid of revolution of polar curve $\rho = \rho(\varphi)$, $\alpha \le t \le \beta$, about the polar-axis is

$$V_\rho = \frac{2}{3}\pi\int_\alpha^\beta \rho^3(\varphi)\sin\varphi d\varphi. \qquad (5.14)$$

5.4 The Surface of a Solid of Revolution

Let us consider the arc of the smooth nonnegative function $y = f(x)$ and the surface generated by revolving this arc about the x-axis (Fig. 25). Determine the area of this surface.

Subdivide the interval into n parts by the points

$A = M_0, M_1, \ldots M_k, M_{k+1}, \ldots M_{n-1}, M_n = B$.

Draw the chords $AM_1, \ldots M_k M_{k+1}, \ldots M_{n-1}B$, whose lengths are determined as follows (see 5.2 I)

$$\Delta S_k = \Delta x_k \sqrt{1 + \left(f'(\xi_k)\right)^2}$$

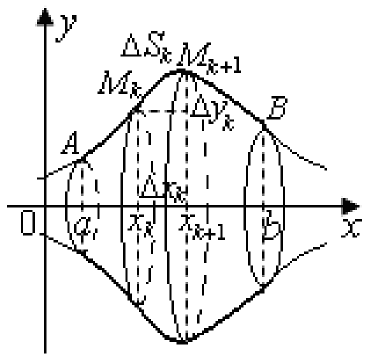


Figure 25.

Each chord of length describes (during the rotation) a truncated cone whose surface is

$$\Delta P_k = 2\pi \frac{f(x_{k+1}) - f(x_k)}{2} \Delta S_k = \pi\left(f(x_{k+1}) - f(x_k)\right)\sqrt{1 + \left(f'(\xi_k)\right)^2}\,\Delta x_k .$$

Thus, the surface describes by the broken line is equal to the sum

$$P_{ox} \approx \sum_{k=0}^{n-1} \Delta P_k = \sum_{k=0}^{n-1} \pi\left(f(x_{k+1}) - f(x_k)\right)\sqrt{1 + \left(f'(\xi_k)\right)^2}\,\Delta x_k .$$

The limit of this sum, when the largest segment ΔS_k approaches zero gives a formula of the area of the surface of revolution

$$P_{ox} = \lim_{\Delta S_k \to 0} \sum_{k=0}^{n-1} \pi\left(f(x_{k+1}) - f(x_k)\right)\sqrt{1 + \left(f'(\xi_k)\right)^2}\,\Delta x_k = \lim_{\Delta x_k \to 0} \sum_{k=0}^{n-1} 2\pi f(\xi_k)\sqrt{1 + \left(f'(\xi_k)\right)^2}\,\Delta x_k .$$

$$P_{ox} = 2\pi \int_a^b f(x)\sqrt{1 + \left(f'(x)\right)^2}\,dx. \qquad (5.15)$$

The surface generated by revolving of the arc about the y-axis

$$P_{oy} = 2\pi \int_a^b x\sqrt{1 + \left(f'(x)\right)^2}\,dx. \qquad (5.16)$$

If the surface of revolution is generated by the rotation of the parametric curve $x = x(t),\ y = y(t),\ t_1 \le t \le t_2$, then

$$P_{ox} = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt, \qquad P_{oy} = 2\pi \int_{t_1}^{t_2} x(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt \qquad (5.17)$$

Example.

1. Compute the surface of revolution of the curve $y = x^3$ between the straight lines $x = -\frac{2}{3}$ and $x = \frac{2}{3}$.

Let us use the formula (5.17).

$$P_{ox} = 2\cdot 2\pi\int_0^{\frac{2}{3}} x^3\sqrt{1+((x^3)')^2}\,dx = 4\pi\int_0^{\frac{2}{3}} x^3\sqrt{1+9x^4}\,dx =$$

$$= \frac{\pi}{9}\int_0^{\frac{2}{3}}\sqrt{1+9x^4}\,d(1+9x^4) = \frac{\pi}{9}\cdot\frac{2}{3}\sqrt{(1+9x^4)^3}\Big|_0^{\frac{2}{3}} =$$

$$= \frac{196\pi}{729}\ (\text{units})^2.$$

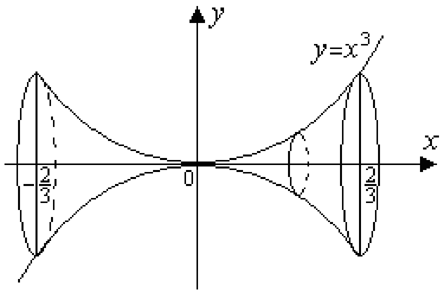


Figure 26.

2. Find the surface of revolution of the first arc of cycloid

$$\begin{cases} x = a(t-\sin t), \\ y = a(1-\cos t), \end{cases} t\in[0, 2\pi].$$

Let us use the formula (5.17).

$$P_{ox} = 2\pi\int_0^{2\pi} a(1-\cos t)\sqrt{\left(a(1-\cos t)\right)^2 + \left(a\sin t\right)^2}\,dt =$$

$$= 2\pi a^2\int_0^{2\pi}(1-\cos t)\sqrt{2-2\cos t}\,dt =$$

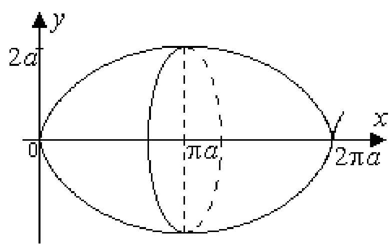


Figure 27.

$$= 2\pi a^2\int_0^{2\pi}(1-\cos t)\sqrt{4\sin^2\frac{t}{2}}\,dt = 8\pi a^2\int_0^{2\pi}\sin^3\frac{t}{2}\,dt = 8\pi a^2\int_0^{2\pi}\left(1-\cos^2\frac{t}{2}\right)\sin\frac{t}{2}\,dt =$$

$$= -16\pi a^2\int_0^{2\pi}\left(1-\cos^2\frac{t}{2}\right)d\left(\cos\frac{t}{2}\right) = -16\pi a^2\left(\cos\frac{t}{2} - \frac{\cos^3\frac{t}{2}}{3}\right)\Bigg|_0^{2\pi} = \frac{64\pi a^2}{3}\ (\text{units})^2.$$

5.5 Physical Application of the Definite Integral

I. Work of the Variable Force

Suppose a force F moves an object along the x-axis, and the direction of the force coincides with the direction of motion. Let us determine the work done by the force F as the body is moved from the point $x = a$ to the point $x = b$.

The work done by a constant force in moving an object a distance is equal to the product of the force and the distance moved. That is, if the force F is constant, then

$$W = F(b - a).$$

But in most cases the applied force is not constant, but varies depending on the position of material point. Assume that the force $F(x)$ varies continuously from a to b.

In order to find the total work divide the interval $[a, b]$ into n arbitrary parts by points $a = x_0,\ x_1,\ x_2,\ \ldots,\ x_n = b$ of length $\Delta x_1,\ \Delta x_2,\ \ldots,\ \Delta x_n$. In each part (x_k, x_{k+1}), $k = 0, 1, \ldots n$ choose an arbitrary point ξ_k and evaluate the work of the force on each part $W_k = F(\xi_k)\Delta x_k$, $k = 0, 1, \ldots n$.

Hence, the total work is approximately

$$W \approx \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n-1} F(\xi_k)\Delta x_k .$$

Obviously, this expression is an integral sum of the function $F(x)$ on the interval $[a, b]$. The limit of this sum as $\max \Delta x_k \to 0$ exists and leads to the work of the force $F(x)$ over the path from the point $x = a$ to the point $x = b$

$$W = \int_a^b F(x)dx. \qquad (5.18)$$

Example.

The compression S of a helical spring is proportional to the applied force F. Compute the work of the force F when the spring is compressed 5 cm, if a force of one kilogram is required to compress it 1 cm.

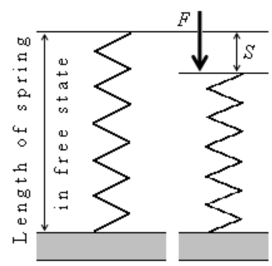


Figure 28.

It is given that the force F and the distance covered S are connected by the relation $F = kS$, where k is a constant.

Let us express S in meters and F in kilograms. When $S = 0{,}01$, $F = 1$, that is, $1 = k \cdot 0{,}01$, whence $k = 100$, $F = 100S$.

By formula (5.18) we have

$$W = \int_0^{0,05} 100 S dS = 100 \frac{S^2}{2}\Big|_0^{0,05} = 0{,}125 \text{ kilogram-meter.}$$

II. Mass, Coordinates of the Centre of Gravity and Moments of Inertia

Suppose on an *xy*-plane there is a system of material points

$$P(x_1,\ y_1),\ P(x_2,\ y_2)\ldots,\ P(x_n,\ y_n)$$

with masses $m_1,\ m_2,\ \ldots,\ m_n$.

The product $x_k m_k$ and $y_k m_k$ are called the *static moments* of the mass relative to the *y*- and *x*-axis. According to well-known formulas from mechanics, the coordinates of the centre of gravity of this material system will be defined by the formulas

$$x_c = \frac{\sum_{k=1}^{n} x_k m_k}{\sum_{k=1}^{n} m_k}, \quad y_c = \frac{\sum_{k=1}^{n} y_k m_k}{\sum_{k=1}^{n} m_k}. \tag{5.19}$$

Rotational inertia is a property of an object which can be rotated. It is also known as *moment of inertia*. It is also sometimes called the *second moment of mass.* It is possible to calculate the total rotational inertia for the system of material points

about the *x*-axis $\quad I_x = \sum_{k=1}^{n} x_k^2 m_k$,

about the *y*-axis $\quad I_y = \sum_{k=1}^{n} y_k^2 m_k$, (5.20)

about the origin $\quad I_0 = \sum_{k=1}^{n} \left(x_k^2 + y_k^2\right) m_k$.

We use these formulas in finding physical characteristics of various objects.

1. The mass, the centre of gravity and moments of inertia of a material line

Consider the arc of the material curve $y = f(x)$, $a \le x \le b$, and let linear density (mass per unit lenght) of this material curve be γ. We assume that linear density is the same in all points of the line. Objects whose mass is uniformly distributed throughout the object are called **homogeneous**.

Divide the interval $[a,b]$ into n parts by points $x_1,\ x_2,\ \ldots,\ x_n$. This partition divide the curve into n parts of length $\Delta l_1,\ \Delta l_2,\ \ldots,\ \Delta l_n$. The masses of these parts are $m_1 = \gamma\Delta l_1,\ m_2 = \gamma\Delta l_2,\ \ldots,\ m_n = \gamma\Delta l_n$. Choose the point ξ_k in each part (x_k, x_{k+1}), $k = 0, 1, \ldots n$.

Then the total mass is

$$m \approx \sum_{k=1}^{n} m_k = \sum_{k=1}^{n} \gamma\Delta l_k = \sum_{k=1}^{n} \gamma\Delta x_k \sqrt{1+\left(f'(\xi_k)\right)^2}\,.$$

Hence, **the mass of a material line** is

$$m = \int_a^b \gamma\sqrt{1+\left(f'(x)\right)^2}\,dx. \qquad (5.21)$$

According to formulas (5.19) and (5.20) we obtain

$$x_c \approx \frac{\sum_{k=1}^{n}\gamma\xi_k\Delta l_k}{\sum_{k=1}^{n}\gamma\Delta l_k} = \frac{\sum_{k=1}^{n}\xi_k\Delta x_k\sqrt{1+\left(f'(\xi_k)\right)^2}}{\sum_{k=1}^{n}\Delta x_k\sqrt{1+\left(f'(\xi_k)\right)^2}}\,.$$

$$y_c \approx \frac{\sum_{k=1}^{n}\gamma f(\xi_k)\Delta l_k}{\sum_{k=1}^{n}\gamma\Delta l_k} = \frac{\sum_{k=1}^{n} f(\xi_k)\Delta x_k\sqrt{1+\left(f'(\xi_k)\right)^2}}{\sum_{k=1}^{n}\Delta x_k\sqrt{1+\left(f'(\xi_k)\right)^2}}\,.$$

That’s leads to formulas of **the centre of gravity of a material line**

$$x_c = \frac{\int_a^b x\sqrt{1+\left(f'(x)\right)^2}\,dx}{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx} = \frac{M_{oy}}{m}, \quad y_c = \frac{\int_a^b f(x)\sqrt{1+\left(f'(x)\right)^2}\,dx}{\int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx} = \frac{M_{ox}}{m}. \qquad (5.22)$$

Here, M_{oy} and M_{ox} are the static moments of the curve relative to the y- and x-axis.

Moments of inertia of a material line

about the x-axis $$I_x = \int_a^b f^2(x)\sqrt{1+\left(f'(x)\right)^2}\,dx, \qquad (5.23)$$

about the y-axis $$I_y = \int_a^b x^2\sqrt{1+\left(f'(x)\right)^2}\,dx, \qquad (5.24)$$

about the origin $$I_o = \int_a^b \left(x^2 + f^2(x)\right)\sqrt{1+\left(f'(x)\right)^2}\,dx. \qquad (5.25)$$

Example.

Determine the coordinates of the center of gravity of a homogeneous arc of curve $y = a\cosh\frac{x}{a}$, $-a \le x \le a$ (Fig. 29).

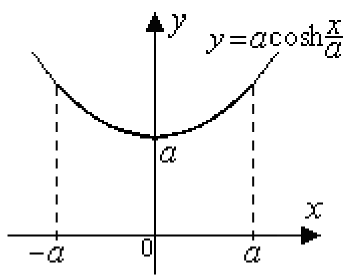


Figure 29.

Since the arc is symmetric about the y-axis, the center is on the y-axis, that is $x_c = 0$.

By the second of formulas (5.22)

$$m = \int_{-a}^{a}\sqrt{1+\left((a\cosh\frac{x}{a})'\right)^2}\,dx = \int_{-a}^{a}\sqrt{1+\left(\sinh\frac{x}{a}\right)^2}\,dx = 2\int_0^a \cosh\frac{x}{a}dx = 2a\sinh\frac{x}{a}\Big|_0^a = 2a\sinh 1.$$

$$M_{ox} = \int_{-a}^{a} a\cosh\frac{x}{a}\sqrt{1+\left((a\cosh\frac{x}{a})'\right)^2}\,dx = \int_{-a}^{a} a\cosh\frac{x}{a}\sqrt{1+\left(\sinh\frac{x}{a}\right)^2}\,dx =$$

$$= \int_{-a}^{a} a\cosh\frac{x}{a}\sqrt{1+\left(\sinh\frac{x}{a}\right)^2}\,dx = 2a\int_0^a \cosh^2\frac{x}{a}dx = a\int_0^a\left(1+\cosh\frac{2x}{a}\right)dx =$$

$$= a\left(x+\frac{a}{2}\sinh\frac{2x}{a}\right)\Big|_0^a = a\left(a+\frac{a}{2}\sinh 2\right).$$

Hence,

$$y_c = \frac{M_{ox}}{m} = \frac{a\left(a+\frac{a}{2}\sinh 2\right)}{2a\sinh 1} = \frac{a(1+\sinh 2)}{4\sinh 1} \approx 1{,}18a.$$

Finally, the center of gravity is

$$\left(0, \frac{a(1+\sinh 2)}{4\sinh 1}\right).$$

2. The mass and the centre of gravity of a material plane figure

Let us consider a curvilinear trapezoid bounded by the line $y = f(x)$, $a \le x \le b$, which is a material plane figure (lamina). Suppose that lamina is homogeneous, that is, the area density (mass per unit area) is constant γ.

The mass of a material plane figure

$$m = \int_a^b \gamma f(x)dx. \tag{5.26}$$

The centre of gravity of a material plane figure

$$x_c = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx} = \frac{M_{oy}}{m}, \quad y_c = \frac{\frac{1}{2}\int_a^b f^2(x)dx}{\int_a^b f(x)dx} = \frac{M_{ox}}{m}. \tag{5.27}$$

Here, M_{oy} and M_{ox} are the static moments of the material plane figure relative to the *y*- and *x*-axis.

Moments of inertia of a material plane figure

about the *x*-axis $$I_x = \int_a^b f^2(x)\sqrt{1+\left(f'(x)\right)^2}dx, \tag{5.28}$$

about the *y*-axis $$I_y = \int_a^b x^2\sqrt{1+\left(f'(x)\right)^2}dx, \tag{5.29}$$

about the origin $$I_o = \int_a^b \left(x^2 + f^2(x)\right)\sqrt{1+\left(f'(x)\right)^2}dx. \tag{5.30}$$

Example.

Find the coordinates of the center of gravity of homogeneous lamina bounded by the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and lines $x = 0,\ y = 0$.

Since the figure is symmetric about the bisector of first quarter, the center is on the line $y = x$, that is $x_c = y_c$.

Let us apply the formulas (5.27)

$$M_{oy} = \int_0^a x\left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(ax - 2\sqrt{a}\sqrt{x^3} + x^2\right)dx =$$

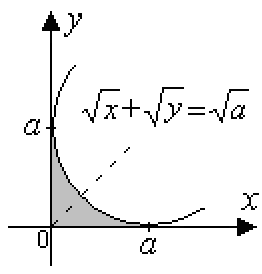


Figure 30.

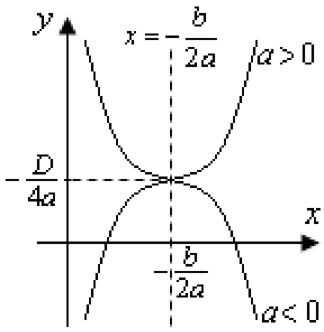
$$= \left(\frac{ax^2}{2} - \frac{4}{5}\sqrt{a}\sqrt{x^5} + \frac{x^3}{3} \right)\Bigg|_0^a = \frac{a^3}{30}.$$

$$m = \int_0^a \left(\sqrt{a} - \sqrt{x}\right)^2 dx = \int_0^a \left(a - 2\sqrt{a}\sqrt{x} + x\right) dx = \frac{a^2}{6}.$$

Hence,

$$x_c = y_c = \frac{M_{oy}}{m} = \frac{a}{5}.$$

Appendix 1. Graphs of Certain Functions in Cartesian Coordinates

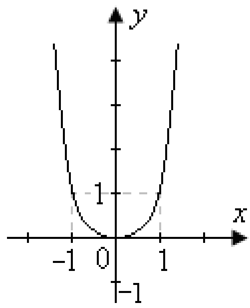


$y = ax^2 + bx + c,\ a \neq 0$

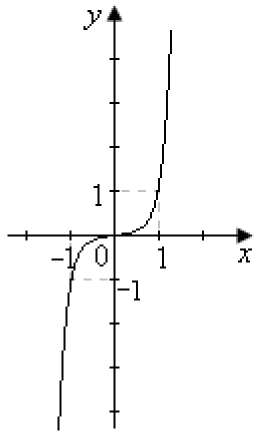
(Parabola)

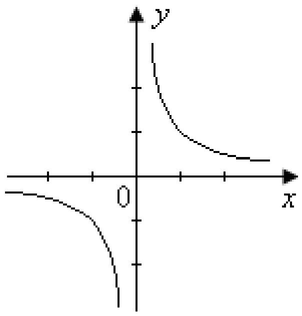
Power Function

$y = x^n$, ***n*** is even (2, 4, 6…)



$y = x^n$, ***n*** is odd (3, 5, 7…)



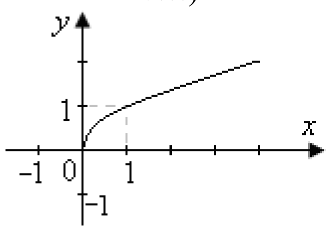


Hyperbola

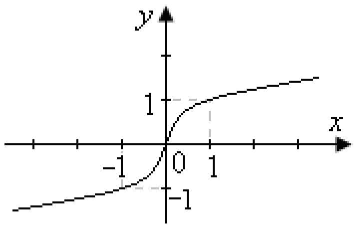
$y = \frac{k}{x}$

Inverse Power Functions

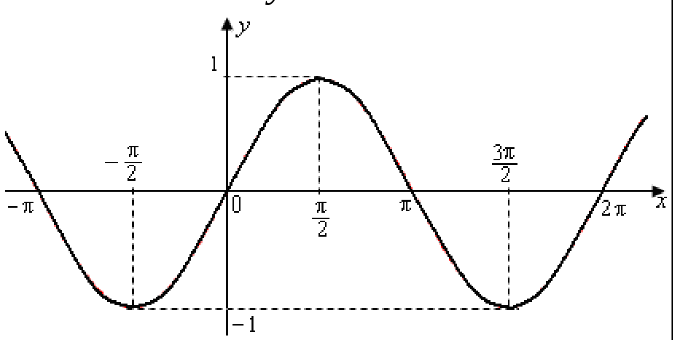
$y = \sqrt[n]{x}$, ***n*** is even (2, 4, 6…)



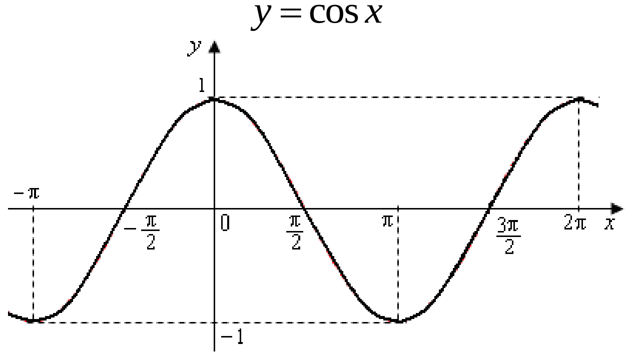
$y = \sqrt[n]{x}$, ***n*** is odd (3, 5, 7…)



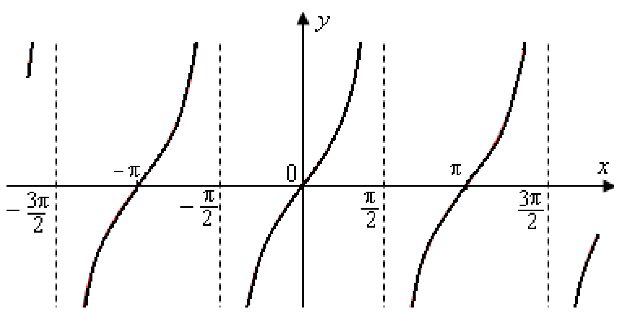
$y = \sin x$



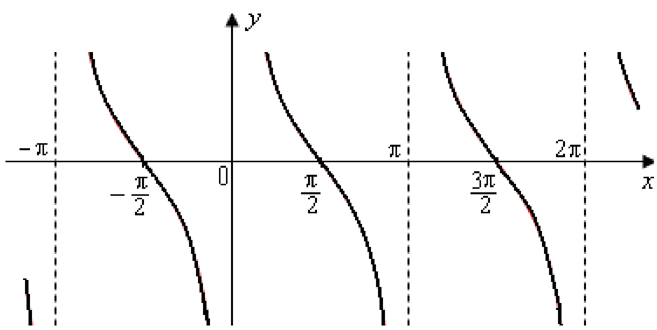
$y = \cos x$



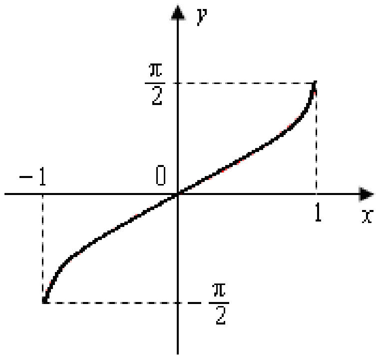
$y = \tan x$



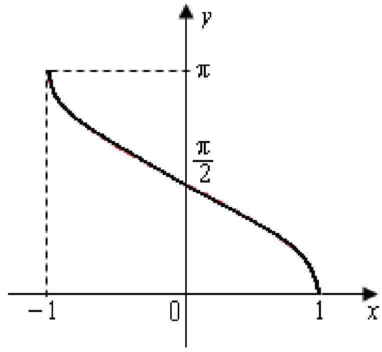
$y = \cot x$



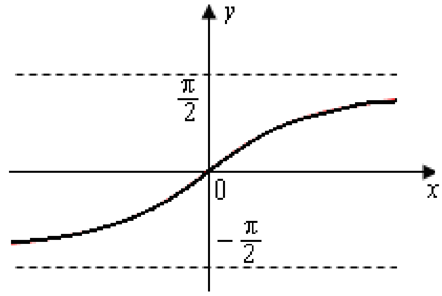
$y = \arcsin x$



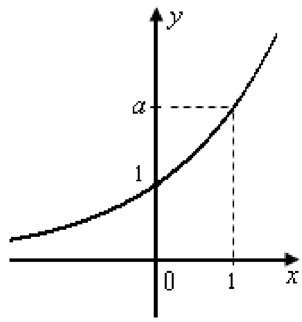
$y = \arccos x$



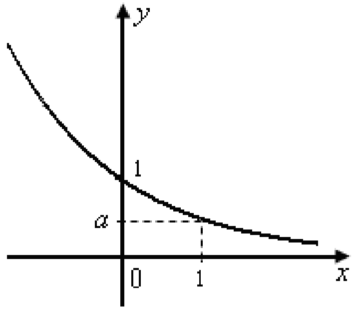
$y = \arctan x$



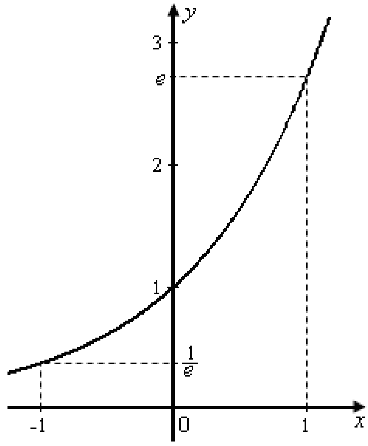
$y = a^x,\ a > 1$



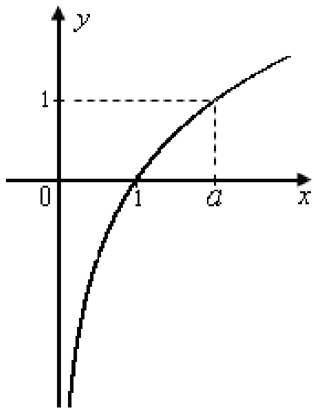
$y = a^x,\ 0 < a < 1$



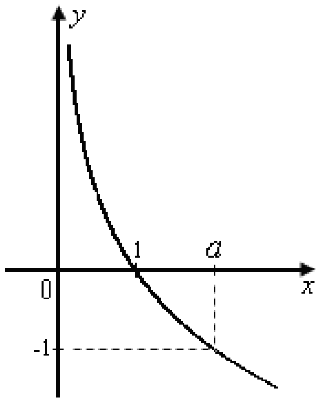
$y = e^x$



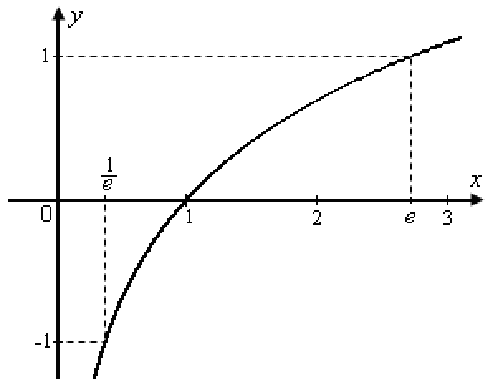
$y = \log_a x,\ a > 1$



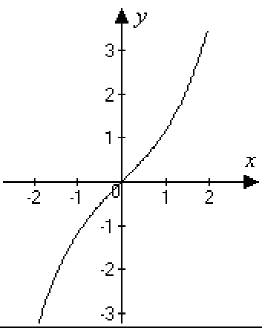
$y = \log_a x,\ 0 < a < 1$



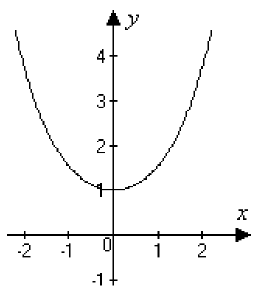
$y = \ln x$



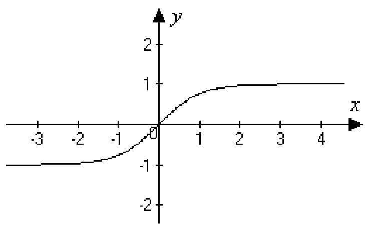
$y = \sinh x$



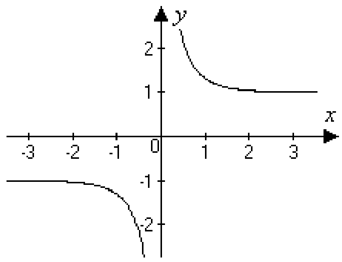
$y = \cosh x$



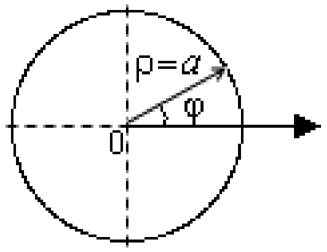
$y = \tanh x$



$y = \coth x$



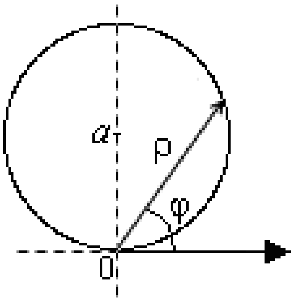
Appendix 3. Graphs of Certain Functions in Polar Coordinates



Circle

$x^2 + y^2 = a^2$

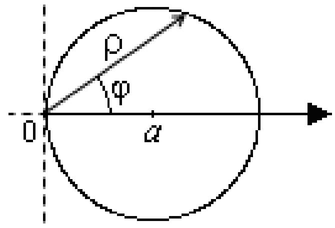
$\rho = a$



Circle

$x^2 + y^2 = 2ay$

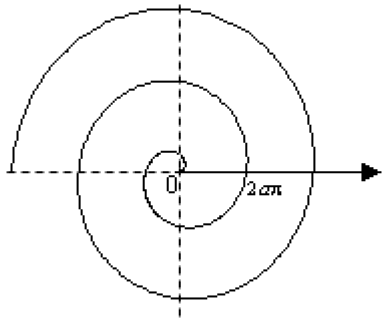
$\rho = 2a \sin\varphi$



Circle

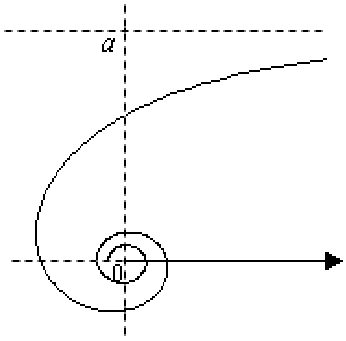
$x^2 + y^2 = 2ax$

$\rho = 2a\cos\varphi$



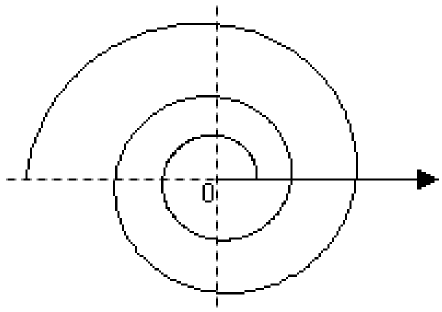
Archimedean spiral

$\rho = a\varphi$



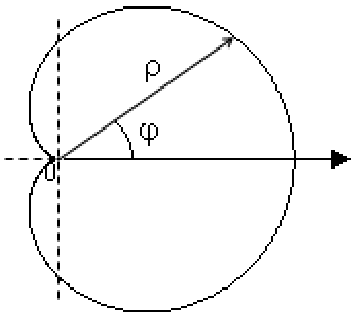
Hyperbolic spiral

$\rho = \frac{a}{\varphi}$



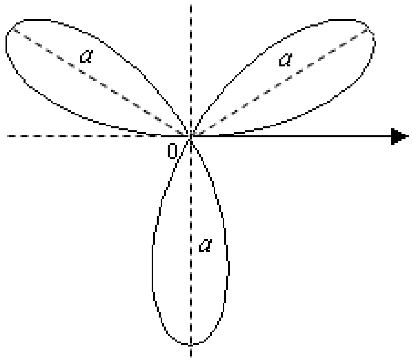
Logarithmic spiral

$\rho = e^{a\varphi}$



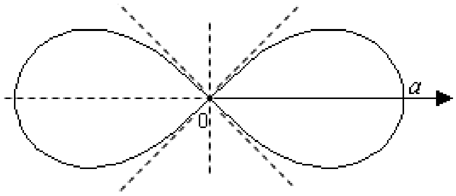
Cardioid

$\rho = a(1 + \cos\varphi)$



Triple-petaled rose

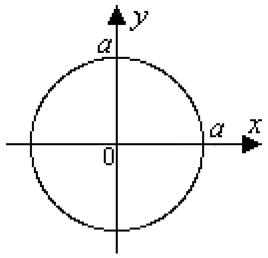
$\rho = a \sin 3\varphi$



Lemniscate of Bernoulli

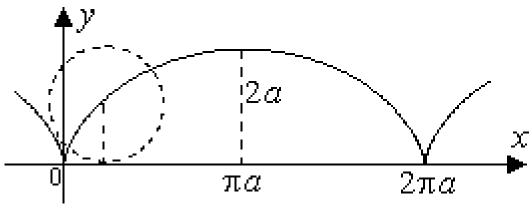
$\rho = a\sqrt{\cos 2\varphi}$

Appendix 3. Graphs of Certain Functions in Parametric Form



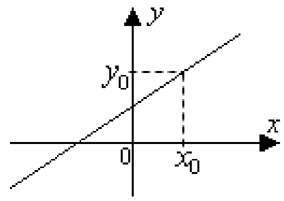
Circle

$$\begin{cases} x = a\cos t, \\ y = a\sin t, \end{cases} t \in [0, 2\pi]$$



Cycloid

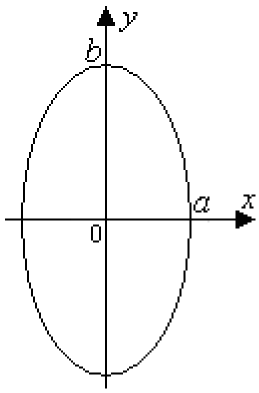
$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} t \in [0, 2\pi]$$



Straight line

$$y = kx + b$$

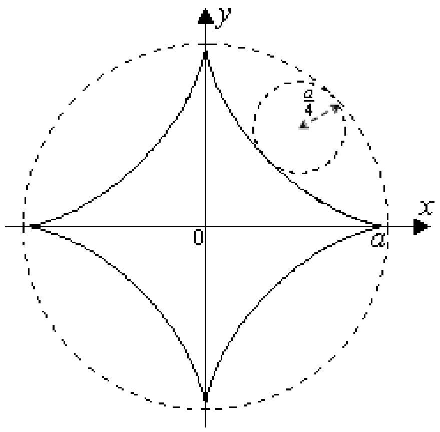
$$\begin{cases} x = lt + x_0, \\ y = mt + y_0. \end{cases}$$



Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

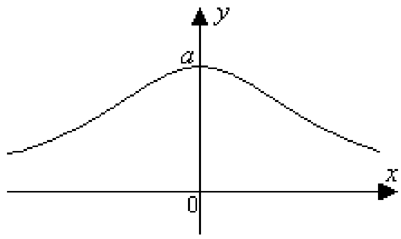
$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} t \in [0, 2\pi]$$



Astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

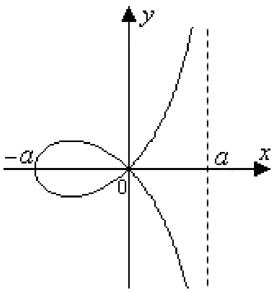
$$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t, \end{cases} t \in [0, 2\pi]$$



Witch of Agnesi

$$y(x^2 + a^2) = a^3$$

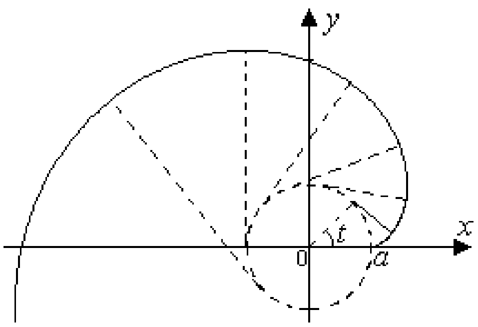
$$\begin{cases} x = at, \\ y = \dfrac{a}{t^2 + 1}. \end{cases}$$



Strophoid

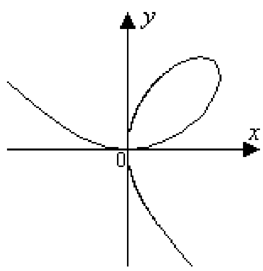
$$y^2(a - x) = x^2(a + x)$$

$$\begin{cases} x = a\dfrac{t^2 - 1}{t^2 + 1}, \\ y = at\dfrac{t^2 - 1}{t^2 + 1}. \end{cases}$$



Involute of a circle

$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t), \end{cases} t \in [0, 2\pi]$$



Folium of Decartes

$$y^3 + x^3 = 3axy$$

$$\begin{cases} x = \dfrac{3at}{t^3 + 1}, \\ y = \dfrac{3at^2}{t^3 + 1}. \end{cases}$$

Appendix 4. The table of derivatives. Properties of derivatives

$C' = 0 \ \ \forall C \in \mathbb{R};$		$(x)' = 1;$
$(x^n)' = nx^{n-1};$	$\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2};$	$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}};$
$(e^x)' = e^x;$		$(a^x)' = a^x \ln a;$
$(\ln x)' = \dfrac{1}{x};$		$(\log_a x)' = \dfrac{1}{x \ln a};$
$(\sin x)' = \cos x;$		$(\cos x)' = -\sin x;$
$(\tan x)' = \dfrac{1}{\cos^2 x};$		$(\cot x)' = -\dfrac{1}{\sin^2 x};$
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}};$		$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}};$
$(\arctan x)' = \dfrac{1}{1+x^2};$		$(\text{arccot} x)' = -\dfrac{1}{1+x^2};$
$(\sinh x)' = \cosh x;$		$(\cosh x)' = \sinh x;$
$(\tanh x)' = \dfrac{1}{\cosh^2 x};$		$(\coth x)' = -\dfrac{1}{\sinh^2 x};$

1. $\forall C \in \mathbb{R} \ \ (C \cdot f)' = C \cdot f';$
2. $(f + g)' = f' + g';$
3. $(f \cdot g)' = f' \cdot g + f \cdot g';$
4. if $g(x) \neq 0, \ \left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g - f \cdot g'}{g^2}.$
5. Chain Rule $\left(f(g(x))\right)'_x = f'_g(g(x)) \cdot g'_x(x).$

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