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T.V. MATVIEIEVA, M.V. CHURSANOVA, F.M. GAREEVA

SOLVING PROBLEMS IN ELECTROSTATICS

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as a textbook for foreign students
of higher educational institutions

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Reviewers: Slipukhina I. A., Doctor of Pedagogical Sciences, Professor, Chief Researcher of the National Center "Junior Academy of Science of Ukraine"
Voytovych I. S., Doctor of Pedagogical Sciences, Professor, Head of the Department of Information and Communication Technologies and Methods of Teaching Informatics, Rivne State University of Humanities

Editor-in-Chief: Kotovskyi V.Y. – Head of the Department of General Physics and Modelling of Physical Processes, D.Sc., Professor

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The textbook is compiled in accordance with the current syllabus of the Physics course on the chapter "Electrostatics" for the students of engineering and pedagogical specialties of all forms of learning at higher education institutions.

It contains brief theoretical information that the student must know to solve problems in electrostatics, methodical guidelines, examples of solving basic typical problems and problems for independent work.

Much attention is paid to the use of the differentiation and integration method (hereinafter, the DI-method).

The "Appendices" provide variants for homework that can be used to carry out independent and control work, and also the scheme of gradual increase in complexity in the formation of the "Electric field" concept and the comparison tables of the DI-method based calculation of the moment of inertia and the electric field.

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PREFACE

The textbook is compiled in accordance with the current syllabus of the Physics course on the chapter “Electrostatics” for the students of all forms of learning at higher education institutions.

The main goal of the textbook is to help students learn how to solve physical problems in Electrostatics on their own. Six topics of practical lessons are considered in the textbook, each topic contains:

– basic theoretical questions that a student should know to solve problems on electric field;
– a list of tasks that determine the normative level of knowledge;
– brief theoretical information;
– methodical guidelines for solving problems;
– examples of solving basic typical problems;
– tasks for independent work of different types and levels of complexity.

To assist teachers, there are variants for homework in the “Appendices” section that can be used for independent and control tasks.

The textbook contains examples of solving basic typical problems and tasks for independent work.

During compilation of examples and problems for independent work, popular collections of problems in General physics have been used [1 – 7].

GENERAL GUIDELINES FOR SOLVING PROBLEMS

Each branch of physics uses its own specific methods and techniques for solving problems. However, there is a general approach that makes it easier to solve problems.

Most problems in physics are solved in three stages: the first one is physical (construction of a physical model), the second one is mathematical and the third is the stage of analysis.

1. *The physical stage* involves analysis of the phenomena described in the problem and finding out the physical laws to which they are subject. This stage requires:

– to find out what is given in the problem, trying to imagine the described events and to understand the essence of the task;
– to write briefly the data of the problem in SI units. For the convenience of records and calculations, very large and very small values should be written in standard form ($N \cdot 10^n$);
– make schematic drawings, graphs or electrical circuits illustrating the problem.

2. *The mathematical stage* is to make up and solve the necessary equations, which are mathematical expressions of the relevant physical laws and relations between quantities, taking into account the conditions in the problem. This stage requires obtaining a specific answer to the problem. Also, the following hints may be useful:

– if the required characteristics are not specified in the problem, but a body or a substance is specified, then it is necessary to know its characteristics (for example, relative permittivity, resistivity, etc.) which are taken from reference tables;
– if, in addition to the desired quantities, the equation contains other unknown quantities which can not be expressed through the given in the problem, they can be temporarily considered known, and then reduced in the process of further transformations;

- before solving the equations, one should check whether the system is complete, i.e. whether the number of equations corresponds to the number of unknown quantities that appear in them;
- solution of the system of equations should begin with the removal of those unknown quantities that are not in the question of the problem;
- in the vast majority of cases, the solution of equations should be presented in general (algebraic) form, and numerical values should be substituted at the final stage.

3. *The analysis stage* is most often associated with evaluation of the correctness of the answer. To do that,

- make sure that the result has the required units;
- analyze the obtained numerical result. For example, if in the result of calculations it is obtained that the mass of a person is 1000 kg, then it is evident that either physical model or calculations have error. If possible, check the order of magnitude of the obtained value using the reference data tables;
- the final numerical answer should be rounded (according to the rounding rules);
- solve the problem in another way and compare the obtained results.

Differentiation and integration method (DI-method)

When solving problems in physics at a technical university, it is expedient to use the method of differentiation and integration (**DI-method**), which is one of the most universal calculation methods. It is most often used in such sections of the general physics course as dynamics of rotational motion of a rigid body (calculation of a moment of inertia of a body about its rotation axis), electrostatics and electromagnetism (calculation of electric field, electric potential, magnetic field in certain point in space).

The **DI-method** is based on the two mathematical principles:

- The principle of possibility of presenting a physical law in a differential form;
- The superposition principle (if the physical quantities included in the law are additive).

The use of **DI-method** consists of two stages.

The first stage is to find the differential of the desired physical quantity. For that purpose, the studied physical body is divided into infinitesimal parts for which the desired quantity is already known.

The second stage is to summarize (integrate) contributions of all the infinitesimal elements of the desired quantity. At this stage, one should pay the most attention to the choice of integration variables and to the definition of limits of integration. To determine the integration variable, it is necessary to analyze in detail:

a) which variables the differential of the desired quantity depends on;

b) which variable is the most essential for the most convenient integration.

Then, all the other variables are expressed as a function of that variable. Next, the limits of integration are defined as extreme (limit) values of the integration variable. After calculating the definite integral, the numerical value of the desired quantity is obtained [8].

When teaching students to apply the **DI-method** for solving problems in physics, it is necessary to use such important learning principles as

- *the principle of initial knowledge* (the necessity of mathematical training of students on the skills of differentiation and integration);
- *the principle of consistency* (reliance of new material on the material studied before, consideration of new material in parts, fixation of students' attention on key issues, implementation of interdisciplinary relations).

Problems should be presented to students with gradual increase in complexity in the sequence from simple to complex. Such an approach displays that application of the **DI-method** is based on the fact that the desired physical quantity for some elementary (infinitesimal) part of the body is already known to students, and they can add (integrate) these quantities to find the desired quantity for the entire physical body of certain size, which is built of these elementary parts.

Let's explain these considerations with an example. If we know the mathematical expression for calculating the magnitude of electric field due to a point charge (a charged particle of infinitesimal size), we can use the **DI-method** to calculate the magnitude of electric field due to a charge distributed along the length of a rod (a physical body of finite or infinitely large size). Here, the point charge can be considered as the elementary part of the rod.

We offer the following algorithm for finding the magnitude of electric field using the **DI-method**:

1. Carefully read the text of the problem, determine which charged body is considered and which infinitesimal elements it can be divided into;
2. Make an illustration for the problem depicting
   - the given charged body,
   - the infinitesimal element of the body with charge *dq*,
   - the distance *r* from the elementary charge *dq* to the point in space where we need to find the electric field,
   - the direction of the electric field vector at this point, as well as its projection onto the coordinate axes;
3. Determine the elementary value of the electric field *dE* in the projections onto the coordinate axes:

$$dE_n = k \cdot \frac{dq}{r^2} \cdot e_{\vec{n}},$$

   where $e_{\vec{n}}$ is the projection of the unit vector $\vec{e}$ onto the *n*-th axis, $k = \frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9$ m/F.

   Note: if the body is symmetrical, it is sufficient to determine projection of the electric field vector onto the axis that coincides with the axis of symmetry, as far as projections onto other axes are mutually compensated by symmetry;
4. Determine the elementary charge *dq*:

   if the body is

- linear $dq = \tau dl = \frac{Q}{L} \cdot dl$,
- planar $dq = \sigma ds = \frac{Q}{S} \cdot ds$,
- volumetric $dq = \rho dv = \frac{Q}{V} \cdot dv$.

Note: here we take that the charge is distributed uniformly. If the charge distribution is not uniform, then the charge density should be considered as a function of distance (that is, $\tau = \tau(r)$, $\sigma = \sigma(r)$, $\rho = \rho(r)$);

5. Substitute the obtained dq into the formula for dE;
6. Express all the variables that are included in the obtained formula with only one variable which is the most convenient to integrate (for example, the angle φ between the direction of the vector $d\vec{E}$ and the normal to the surface of the body, or the solid angle Ω);
7. Write the calculation formula $E_n = \int dE_n$;
8. Determine the limits of integration, which depend on the parameters of the body (length L, radius R, angle φ, etc.);
9. Calculate integrals for each of the projections of the vector $d\vec{E}$ onto the coordinate axes;
10. Find the magnitude of the electric field at the given point:
$$E = \sqrt{\sum_n E_n^2};$$
11. Obtain the final result and analyze it.

In order to better understand application of the **DI-method** to determining the magnitude of electric field, we advise students to consider the “Scheme of gradual increase in complexity in the formation of the concept of electric field”, which is given in the Appendix C.

The material in the Appendix D helps students to draw analogies between the **DI-method** applications in different sections of physics. It focuses on the fact that using the known formula for a desired physical quantity in the simpler case (for an infinitesimal part of the body), it is possible to find the formula for this quantity in the more complex cases (for a body of finite or infinitely large size). Namely, the comparison of the **DI-method** based calculation of such physical quantities as the moment of inertia and the magnitude of electric field is made. It should be noted that in contrast to the moment of inertia, the electric field is a vector physical quantity. Therefore, when finding it, one must first calculate projections of the electric field vector onto the *x*, *y* and *z* axes, and only then one can calculate the magnitude of the desired quantity $E=\sqrt{E_x^2+E_y^2+E_z^2}$ (in cases where the system has a certain symmetry, the problem can be simplified). But the general scheme for solving problems on the moment of inertia and the electric field is still similar.

Gradual increase in the complexity of problem solving is very important when mastering such an indispensable and widely used method of solving problems as the **DI-method**. The application of this method expands students' understanding of the use of mathematics to solve problems in physics, promotes the development of complex thinking, carries an element of creativity and develops scientific physical thinking.

Topic 1. COULOMB'S LAW. INTERACTION OF CHARGED BODIES

What a student should know

1. Coulomb's law.
2. Electric charge conservation law.
3. Superposition principle for the electric fields.
4. A concept of:
   a) a point charge;
   b) linear, surface and volume charge densities;
   c) dielectric permittivity;
   d) electric constant.
5. Differentiation and integration method (**DI-method**).

Literature: [6, § 23.1 – 23.3]; [7, § 5.1 – 5.3]; [9, § 21]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: § 23, No 8, 9, 10, 13], examples 1.1, 1.2.

Homework: see Table A.1 on p. 138.

1.1. Brief theoretical information

Coulomb's law: the magnitude of force of interaction between two point charged bodies* is proportional to the product of absolute values of their charges and inversely proportional to the square of the distance between them:

$$F = k\frac{|Q_1||Q_2|}{\varepsilon_r r^2}, \qquad \mathbf{(1.1)}$$

where $k = 1/(4\pi\varepsilon_0) = 9\cdot10^9$ m/F is the Coulomb constant; ε_r is the relative permittivity of the medium (in vacuum or in the air $\varepsilon_r = 1$).

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* Further we consider concept of the "point charge" for simplicity.

Charges of the same sign repel one another, while charges with opposite signs attract one another.

The charge conservation law:

$$\sum_{i=1}^{n} Q_i = \text{const},$$

where $\sum_{i=1}^{n} Q_i$ is the algebraic sum of charges in the isolated system of bodies; n is the number of charged bodies.

Linear charge density:

$$\tau = \frac{Q}{l},$$

where l is the length of a uniformly charged thread-like body.

Surface charge density:

$$\sigma = \frac{Q}{S},$$

where S is the surface area of a uniformly charged body.

Volume charge density:

$$\rho = \frac{Q}{V},$$

where V is the volume of a uniformly charged body.

1.2. Methodical guidelines

1. The Coulomb's law-related problems consider charged point bodies interacting with each other and with other bodies. When solving problems, it is necessary to carefully analyze and take into account all the forces (not only the Coulomb's forces) exerted on each charge given in the problem.

2. If electric field is produced by a system of charges, the resulting force is determined according to the principle of superposition of fields. One should remember that the force is a vector quantity, so the signs of the individual charges (q_i) of the system must be taken into account during calculations, as they determine directions of the vectors $\vec{F}$ (see example 1.1).

3. Formula (1.1) allows to calculate the force of interaction between point charges. If a charged body cannot be considered as a single point (the charge is distributed along a line, over a surface or throughout a volume, and the geometric dimensions cannot be neglected), the method of differentiation and integration (**DI-method**) is used.

1.2.1. Interaction of point charges

Example 1.1

The distance between two point charges $Q_1 = 10^{-6}$ C and $Q_2 =$ $= -Q_1$ is 10 cm. Find the force F exerted on a point charge $Q = 0.1$ μC distanced by $r_1 = 6$ cm from the first charge and by $r_2 =$ $= 8$ cm from the second charge.

Given:
$Q_1 = 10^{-6}$ C
$Q_2 = -Q_1$
$d = 0.1$ m
$Q = 10^{-7}$ C
$r_1 = 0.06$ m
$r_2 = 0.08$ m

F – ?

I. Physical model

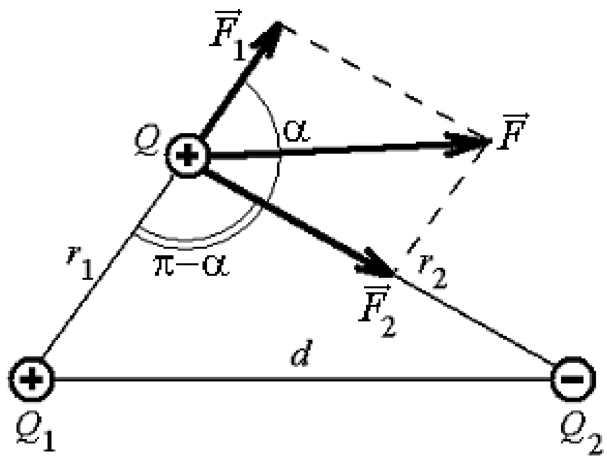

Figure 1.1

II. Mathematical model

1. According to the superposition principle for electric fields, each charge produces electric field regardless of the presence of other charges nearby. Therefore, the force $\vec{F}$ exerted on the point

charge Q can be found as vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$ by fields produced by each charge separately: $\vec{F} = \vec{F}_1 + \vec{F}_2$.

2. Magnitudes of electric field forces in vacuum[1] due to the first and the second charge are found using the Coulomb's law[2]:

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{QQ_1}{r_1^2} = k\frac{QQ_1}{r_1^2},$$

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{QQ_2}{r_2^2} = k\frac{QQ_2}{r_2^2}. \qquad \textbf{(1.2)}$$

3. Magnitude of the net force is found by the cosine theorem:

$$F = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos(\pi - \alpha)} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\alpha},$$

where α is the angle between the forces $\vec{F}_1$ and $\vec{F}_2$, which is 90° because the triangle formed by the charges in space is right-angled[3] (α = 90°, cos 90° = 0), then

$$F = \sqrt{F_1^2 + F_2^2}. \qquad \textbf{(1.3)}$$

4. Substituting expressions (1.2) into (1.3), we obtain the force on the point charge Q:

$$F = \sqrt{\left(\frac{kQQ_1}{r_1^2}\right)^2 + \left(\frac{kQQ_2}{r_2^2}\right)^2} = kQ\sqrt{\frac{Q_1^2}{r_1^4} + \frac{Q_2^2}{r_2^4}}.$$

III. Numerical calculations:

$$F = 9\cdot 10^9 \cdot 10^{-7}\sqrt{\frac{10^{-12}}{1.296\cdot 10^{-5}} + \frac{10^{-12}}{4.096\cdot 10^{-5}}} \approx 0.3(\text{N}).$$

Answer: F = 0.3 N.

---

[1] If text of the problem does not specify the medium, it is assumed that the charges are in vacuum (ε_r = 1).

[2] One should remember that charges of the same sign repel and charges of different signs attract (see Fig. 1.1).

[3] Because it is the Egyptian triangle.

Example 1.2

Two identical point charges with equal values $Q_1 = Q_2 = 1\ \mu C$ are fixed in space and separated by a distance $d = 2$ cm. A particle with negative charge of value $Q = 2\ \mu C$ and mass $m = 4$ g is free to move and lies initially at rest on the perpendicular bisector of the two fixed charges a distance x from the midpoint between those charges. Show that if x is small compared with d, the motion of Q is simple harmonic along the perpendicular bisector. Determine the period of that motion.

Given:
$Q_1 = Q_2 = 1\ \mu C = 10^{-6}$ C
$|Q| = 2\ \mu C = 2 \cdot 10^{-6}$ C, $Q < 0$
$d = 2 \cdot 10^{-2}$ m
$x << d$
$m = 4$ g $= 4 \cdot 10^{-3}$ kg

T – ?

I. Physical model

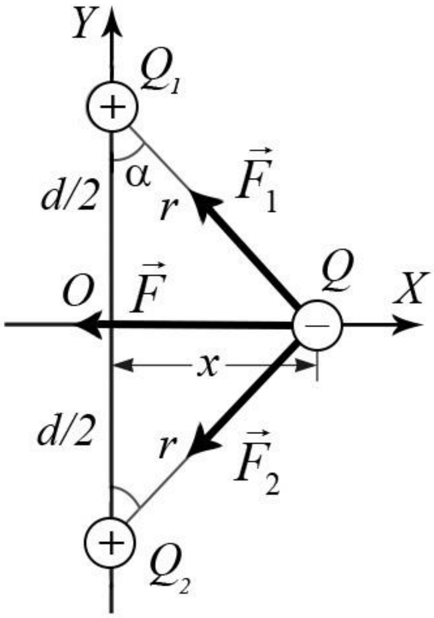

Figure 1.2

II. Mathematical model

1. In order to obtain the law of motion of the charge Q, we need to find the net force $\vec{F}$ exerted on that charge. Considering the given conditions, we can neglect forces of gravitation and air resistance comparing to the electric forces on the particle Q. It experiences electric force from the charges Q_1 and Q_2 and, according to the superposition principle, the net force is the vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$ exerted by fields produced by each of them separately:

$$\vec{F} = \vec{F}_1 + \vec{F}_2.$$

2. As far as we have point charges, electric forces of interaction between them can be found with the Coulomb's law:

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{QQ_1}{r^2} = k\frac{QQ_1}{r^2},$$

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{QQ_2}{r^2} = k\frac{QQ_2}{r^2} = k\frac{QQ_1}{r^2}, \qquad \textbf{(1.4)}$$

where k is the Coulomb constant, r is the distance from the charge Q to the charge Q_1 and from the charge Q to the charge Q_2 which is the same because Q lies the perpendicular bisector of the charges Q_1 and Q_2.

The negative charge Q is attracted by the positive charges Q_1 and Q_2, so the forces $\vec{F}_1$ and $\vec{F}_2$ are directed towards those charges along the lines joining Q and each of them (Fig. 1.2).

3. Let's choose the coordinate system with axis OY along the line joining the charges Q_1 and Q_2 and axis OX along the perpendicular bisector between them (see Fig. 1.2). Let's denote α the angle that the distance r makes with the axis OY. We choose this angle because at the given condition $x << d$ we can consider α being small. Also, α is the same both for the distance to the charge Q_1 and to the charge Q_2 due to the symmetry. Then, we can write projections of the net force onto the coordinate axes taking magnitudes of its components from formulas (1.4):

$$F_x = F_{1x} + F_{2x} = -F_1 \sin\alpha - F_2 \sin\alpha = -2k\frac{QQ_1}{r^2}\sin\alpha,$$

$$F_y = F_{1y} + F_{2y} = F_1 \cos\alpha - F_2 \cos\alpha = 0;$$

$$F = F_x = -2k\frac{QQ_1}{r^2}\sin\alpha. \qquad \textbf{(1.5)}$$

We see that the y-component of the net force is zero, which means that the net force is directed along the axis OX and motion of the charge Q takes place along the axis OX. The negative sign shows that direction of the net force is opposite to the direction of displacement along the axis OX.

The distance r is the hypotenuse of the right triangle where $r = \dfrac{x}{\sin\alpha}$ (see Fig 1.2), and we can write formula (1.5) as

$$F = -2k\frac{QQ_1}{x^2}\sin^3\alpha. \qquad \textbf{(1.6)}$$

Now let's find $\sin\alpha$. As α is small, we use the small angle approximation:

$$\sin\alpha \approx \tan\alpha = \frac{x}{d/2} = \frac{2x}{d}.$$

Then, formula (1.6) can be written as

$$F \approx -2k\frac{QQ_1}{x^2}\frac{8x^3}{d^3} = -16k\frac{QQ_1}{d^3}x. \qquad \textbf{(1.7)}$$

We have obtained that the net force on the charge Q is quasi-elastic: it is directly proportional to the displacement x of the charge and acts in the direction opposite to that of displacement. That proves that it can be the restoring force and cause simple harmonic motion along the axis OX.

4. Now we can write the law of motion of the charge Q using the Newton's second law with the net force given by formula (1.7) and taking into account that motion is only along OX:

$$ma = F = -16k\frac{QQ_1}{d^3}x, \qquad mx'' = -16k\frac{QQ_1}{d^3}x,$$

where we take acceleration a as the second derivative of the displacement x along the axis OX. Continuing mathematical transformations, we can obtain the equation of harmonic oscillator:

$$x'' + 16k\frac{QQ_1}{md^3}x = 0. \qquad \textbf{(1.8)}$$

In the equation of harmonic oscillator (1.8), the multiple of the variable x is the square of the natural frequency of oscillations:

$$\omega_0^2 = 16k\frac{QQ_1}{md^3}. \qquad \textbf{(1.9)}$$

Taking (1.9), the period of the simple harmonic motion is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{md^3}{16kQQ_1}}.$$

III. Numerical calculations:

$$T = 2\pi\sqrt{\frac{4\cdot10^{-3}\cdot8\cdot10^{-6}}{16\cdot9\cdot10^{9}\cdot10^{-6}\cdot2\cdot10^{-6}}} \approx 2.1\cdot10^{-3}\,(\text{s}).$$

Answer: The motion of the charge Q is simple harmonic along the axis OX because the net force on it can be approximated as quasi-elastic $F = -16k\dfrac{QQ_1}{d^3}x$ and it is directed along OX. The period of motion is $T = 2.1\cdot10^{-3}$ s.

1.2.2. Interaction between a point charge and a charge uniformly distributed along a thin rod

Example 1.3

A thin long rod is uniformly charged with a linear charge density $\tau = 10^{-5}$ C/m. A point charge $Q_0 = 10^{-8}$ C is located at a distance $a = 0.2$ m on the perpendicular to the axis of the rod built from the rod's end. Find the force F of interaction between the charged rod and the point charge.

Given:
$\tau = 10^{-5}$C/m
$a = 0.2$ m
$Q_0 = 10^{-8}$ C

F – ?

I. Physical model

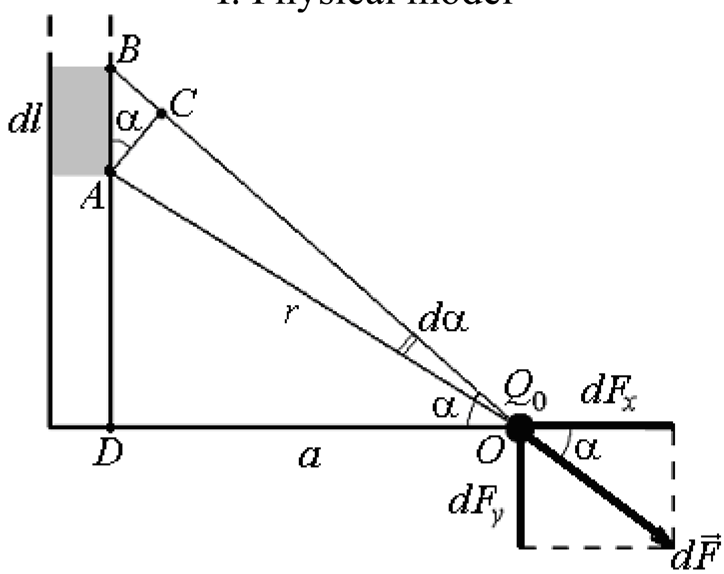

Figure 1.3

II. Mathematical model

Let's use the **DI-method** (see p. 8).

1. Take an infinitesimal rod element of length *dl* carrying a charge $dq = \tau dl$ (Fig. 1.3), which interacts with the charge Q_0 located at the point *O*. According to the Coulomb's law, the magnitude of force of interaction of these two point charges *dq* and Q_0 is equal to:

$$dF = \frac{Q_0 \tau dl}{4\pi\varepsilon_0 |AO|^2}, \qquad \textbf{(1.10)}$$

where $|AO| = \frac{|OD|}{\cos\alpha} = \frac{a}{\cos\alpha}$.

2. Make some geometric transformations according to Fig. 1.3*:

$$dl = \frac{|AC|}{\cos\alpha} = \frac{|AO| d\alpha}{\cos\alpha} = \frac{a d\alpha}{\cos^2\alpha}. \qquad \textbf{(1.11)}$$

3. Substituting expression (1.11) into formula (1.10), we obtain:

$$dF = \frac{Q_0 \tau d\alpha}{4\pi\varepsilon_0 a} = \frac{kQ_0 \tau d\alpha}{a}.$$

4. Since the force is a vector quantity, we must consider its two components dF_x and dF_y (Fig. 1.3):

$$dF_x = dF\cos\alpha = \frac{kQ_0\tau}{a}\cos\alpha d\alpha,$$

$$\textbf{(1.12)}$$

$$dF_y = dF\sin\alpha = \frac{kQ_0\tau}{a}\sin\alpha d\alpha.$$

5. Integrating expressions (1.12) within the integration limits from 0 to $\pi/2$, we obtain:

$$F_x = \int_0^{\pi/2} \frac{kQ_0\tau}{a}\cos\alpha d\alpha = \frac{kQ_0\tau}{a}(\sin\frac{\pi}{2} - \sin 0) = \frac{kQ_0\tau}{a},$$

$$F_y = \int_0^{\pi/2} \frac{kQ_0\tau}{a}\sin\alpha d\alpha = -\frac{kQ_0\tau}{a}(\cos\frac{\pi}{2} - \cos 0) = \frac{kQ_0\tau}{a}.$$

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*Neglecting the angle $d\alpha$ ($d\alpha \to 0$), the angles *BAC* and *AOD* are equal.

6. Since $F_x = F_y$, then, due to the superposition principle for electric fields, the magnitude of the resultant force is found by the cosine theorem:

$$F = \sqrt{F_x^2 + F_y^2 + 2F_xF_y\cos\beta},$$

where β = 90°, cos 90° = 0 (Fig. 1.2). Therefore,

$$F = \sqrt{F_x^2 + F_y^2} = F_x\sqrt{2} = \frac{kQ_0\tau\sqrt{2}}{a}.$$

III. Numerical calculations:

$$F = \frac{\sqrt{2}\cdot 9\cdot 10^9\cdot 10^{-8}\cdot 10^{-5}}{0.2} = 64\cdot 10^{-4}\,(\text{N}).$$

Answer: F = 6.4 mN.

1.2.3. Interaction between linearly distributed charges

Example 1.4

Two identical thin rods of length 2a carry equal charges Q uniformly distributed along their lengths. The rods lie along the x axis with their centers separated by a distance b. Find the magnitude of the force exerted by the left rod on the right one.

Given:
Q
a
b
F – ?

I. Physical model

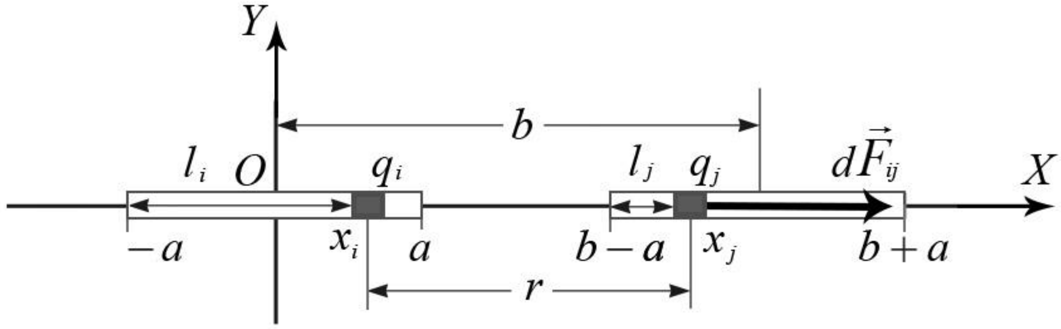


Figure 1.4

II. Mathematical model

1. The problem refers to the electrostatic interaction of two bodies, both having electric charge uniformly distributed along the length. In such a case we have to use the differentiation and integration method twice. Firstly, we take an arbitrary infinitesimal element of the left rod and find the force exerted by it on the right rod similarly to the case of interaction between a point charge and a linearly distributed charge. Secondly, we calculate the total force exerted by the left rod on the right rod by integrating the force by one element over the entire rod length.

2. Let's choose the coordinate system with axis OX passing along the rods and origin in the middle of the left rod (see Fig. 1.4). Take an arbitrary element of the left rod having infinitesimal length dl_i and carrying infinitesimal charge q_i. Let its coordinate along the axis OX be x_i. Similarly, we take an arbitrary element of the right rod with infinitesimal length dl_j and infinitesimal charge q_j and coordinate x_j along the axis OX.

According to the Coulomb's law, the magnitude of force of interaction between these two infinitesimal elements can by calculated as interaction of two point charges:

$$F_{ij} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{(x_j - x_i)^2}, \qquad \textbf{(1.13)}$$

where $r = x_j - x_i$ is the distance between the two chosen infinitesimal elements (Fig. 1.4).

3. Now take that the element of the left rod q_i and its coordinate x_i are fixed, that is considered as constant parameters. While the element of the right rod q_j and its coordinate x_j vary along the length l_j of the right rod. To write down this variation, we can use the linear density of the charge distribution along the rod which equals to the entire charge over the entire length

$$\tau = \frac{Q}{2a}, \qquad \textbf{(1.14)}$$

and then $q_j = \tau dl_j$. Note that the two rods are identical and τ is the same both for the left and the right rod.

From Fig. 1.4 we can find that position of the element q_j on the right rod $l_j = x_j - (b - a)$, where $(b - a)$ is the constant coordinate of the beginning of the rod. Then the differential d$l_j = dx_j$. Taking into account the aforementioned thinking, we can write formula (1.13) as

$$F_{ij} = \frac{1}{4\pi\varepsilon_0}\frac{q_i \tau dl_j}{(x_j - x_i)^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_i \tau dx_j}{(x_j - x_i)^2}. \quad \textbf{(1.15)}$$

4. Now we can calculate the force F_i exerted by one i-th element of left rod on the entire right rod by integrating expression (1.15) with respect to the variable x_j withing the limits given by coordinates of the beginning and the end of the right rod: from $(b - a)$ to $(b + a)$. This is the first iteration of the **DI-method** where x_i is fixed while x_j varies along the length of the right rod:

$$F_i = \int_{b-a}^{b+a} \frac{1}{4\pi\varepsilon_0}\frac{q_i \tau dx_j}{(x_j - x_i)^2} = \frac{1}{4\pi\varepsilon_0} q_i \tau \int_{b-a}^{b+a} \frac{d(x_j - x_i)}{(x_j - x_i)^2} =$$
$$= -\frac{1}{4\pi\varepsilon_0} q_i \tau \left(\frac{1}{b + a - x_i} - \frac{1}{b - a - x_i} \right). \quad \textbf{(1.16)}$$

5. To calculate the total force F exerted by the left rod on the right rod we should make the second iteration of the **DI-method** where the elementary charge q_i and its coordinate x_i vary along the length l_i of the left rod and $q_i = \tau dl_i$. Also, $l_i = x_j - (-a)$ and d$l_i = dx_i$. With these considerations, the result of formula (1.16) can be written as

$$F_i = -\frac{1}{4\pi\varepsilon_0}\tau^2 \left(\frac{1}{b + a - x_i} - \frac{1}{b - a - x_i} \right) dx_i \quad \textbf{(1.17)}$$

and integrated with respect to x_i varying withing the limits given by coordinates of the beginning and the end of the left rod: from $-a$ to a. By integrating (1.17) we obtain

$$F = \int_{-a}^{a} -\frac{1}{4\pi\varepsilon_0}\tau^2 \left(\frac{1}{b + a - x_i} - \frac{1}{b - a - x_i} \right) dx_i =$$
$$= \frac{1}{4\pi\varepsilon_0}\tau^2 \int_{-a}^{a} \left(\frac{1}{x_i - (b + a)} - \frac{1}{x_i - (b - a)} \right) dx_i =$$

$$=\frac{1}{4\pi\varepsilon_0}\tau^2\left(\ln\left(x_i-(b+a)\right)\Big|_{-a}^{+a}-\ln\left(x_i-(b-a)\right)\Big|_{-a}^{+a}\right)=$$

$$=\frac{1}{4\pi\varepsilon_0}\tau^2\left(\ln\left(\frac{-b}{-2a-b}\right)-\ln\left(\frac{2a-b}{-b}\right)\right)=$$

$$=\frac{1}{4\pi\varepsilon_0}\tau^2\ln\left(\frac{b^2}{(2a+b)(-2a+b)}\right)=\frac{1}{4\pi\varepsilon_0}\tau^2\ln\left(\frac{b^2}{b^2-4a^2}\right).$$

With regard to formula (1.14) we can write the final answer:

$$F=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2a}\right)^2\ln\left(\frac{b^2}{b^2-4a^2}\right).$$

Answer: the magnitude of the force exerted by the left rod on the right one is $F=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{2a}\right)^2\ln\left(\frac{b^2}{b^2-4a^2}\right)$.

1.3. Problems for independent work

Interaction of point charges

1.1. Determine the force of interaction of two point charges $Q_1=Q_2=1\text{C}$ in vacuum at a distance r = 1 m from each other.

1.2. Two balls with masses m = 0.1 g each are suspended at one point with threads of length l = 20 cm each. When given equal electric charges, the balls move away from each other so that the angle between the threads reaches the value of $\alpha=60^\circ$. Find the charge given to each ball.

1.3. Two equally charged balls are suspended at one point with threads of equal length. Then the balls are immersed into oil with density $\rho_0=8\cdot10^2$kg/m^3. Determine the relative permittivity ε_r of the oil if the angle between the threads remains unchanged after the immersion of the balls into the oil. Density of material of the balls is $\rho_b = 1.6 \cdot 10^3$ kg/m^3.

1.4. There are two balls of masses m = 1 g each. What charge Q should be given to the balls so that the force of mutual repulsion of charges would balance the force of mutual gravitational attraction of the balls? Consider the balls as material points.

1.5. In the elementary theory of the hydrogen atom, it is assumed that an electron is rotating about a nucleus in a circular orbit. Determine the electron's velocity v if the radius of its orbit is r = = 53 pm. Determine the electron's frequency of rotation.

1.6. Point charges Q, $2Q$, $3Q$, $4Q$, $5Q$, $6Q$ ($Q = 0.1$ μC) are located at the vertices of a regular hexagon with side $r = 10$ cm. Find the force F acting on a point charge Q located in the hexagon's plane and equidistant from its vertices.

1.7. Two identical conductive charged balls are separated by a distance r = 60 cm. The repulsive force F_1 between the balls is $70 \cdot 10^{-6}$ N. After the balls have been brought into contact with each other and separated by the previous distance, the repulsive force increases to $F_2 = 1.6 \cdot 10^{-4}$ N. Find the charges Q_1 and Q_2 of the balls before the contact. Diameter of the balls is considered to be much smaller than the distance between them.

1.8. Two identical conductive charged balls are separated by a distance r = 30 cm. The attraction force F_1 between the balls is 90 μN. After the balls have been brought into contact with each other and separated by the previous distance, they start to repel with the force F_2 = 160 μN. Find the charges Q_1 and Q_2 of the balls before the contact. Diameter of the balls is considered to be much smaller than the distance between them.

1.9. Two positive point charges Q and $4Q$ are fixed at a distance l = 60 cm from each other. Determine at what point on the line passing through the charges one should place the third charge Q_1 so that it will be in equilibrium. Determine the sign that charge must have in order for the equilibrium to be stable, considering the charges can move only along the line connecting them.

1.10. The distance between free charges Q_1 = 180 nC and Q_2 = = 720 nC is 60 cm. Determine at what point on the line passing through the charges one should place the third charge Q_3 so that the system of charge will be in equilibrium. Determine magnitude and sign of that charge. Will that equilibrium be stable?

1.11. Three identical charges Q = 1nC each are located at the vertices of an equilateral triangle. What negative charge Q_1 should be placed in the center of the triangle so that the force of attraction balances the force of mutual repulsion of the charges? Will that equilibrium be stable?

1.12. There are identical charges Q = 0.3 nC each at the vertices of a square. What negative charge Q_1 should be placed in the center of the square so that the force of mutual repulsion of the positive charges balances the force of attraction of the negative charge?

1.13. What mass m_p should a proton have in order for the force of electrostatic repulsion of the two protons to be equal to the force of their gravitational attraction?

1.14. There are two systems of point charges $q_1, q_2, ..., q_i, ..., q_{N_1}$ and $q'_1, q'_2, ..., q'_k, ..., q'_{N_2}$ fixed at points with radii-vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_i, ..., \vec{r}_{N_1}$ and $\vec{r}'_1, \vec{r}'_2, ..., \vec{r}'_k, ..., \vec{r}'_{N_2}$. Find the force $\vec{F}$ exerted by the charge system q'_k on the charge system q_i.

1.15. Calculate the ratio of electrostatic and gravitational forces of interaction between two electrons and between two protons. At what value of the specific charge *q/m* of the particle would these forces have the same magnitude?

1.16. What the force of interaction between two copper balls would be if the total charge of all electrons contained in them differed by 1% from the total charge of all nuclei? The mass of each ball is 1 g, the distance between them is 1 m.

1.17. Two small equally charged balls of mass m each are suspended at one point with silk threads of length l. The distance between the balls is $x << l$. Find the rate of leakage of charge dq/dt from each ball if the distance between them reduces with the speed $v = a/\sqrt{x}$, where a is constant.

1.18. Two positive charges q_1 and q_2 are located at points with radii-vectors $\vec{r}_1$ and $\vec{r}_2$. Find the negative charge q_3 and the radius vector $\vec{r}_3$ of the point where it must be placed so that the force on each of these three charges is zero.

Interaction of a point charge with a charge uniformly distributed along a rod and a ring

1.19. A thin rod of length l = 10 cm is uniformly charged with a linear charge density τ = 10^3 nC/m. A point charge Q = 100 nC is located at the prolongation of the rod's axis at a distance a = 20 cm from its end. Determine the force F of interaction between the rod and the point charge.

1.20. A long thin rod is uniformly charged with a linear charge density τ = 10^4 nC/m. A point charge Q = 10 nC is located at the prolongation of the rod's axis at a distance a = 20 cm from its end. Determine the force F of interaction between the rod and the point charge.

1.21. Electric charge is uniformly distributed with a linear density τ = 10^3 nC/m along a thin semicircle of radius R = 10 cm. A point charge Q = 20 nC is placed in the center of the semicircle. Determine the force F of interaction between the point charge and the charged semicircle.

1.22. A thin thread of length l = 20 cm is uniformly charged with a linear density τ = 10 nC/m. A point charge Q = 1 nC is located at a distance a = 10 cm from the thread on the perpendicular bisector of the thread. Determine the force F exerted on the charge by the thread.

1.23. A thin long rod is uniformly charged with a linear density τ = 10^4 nC/m. Find the force exerted on a point charge Q = 10 nC, which is located at a distance a = 20 cm from the rod on its perpendicular bisector.

1.24. A thin infinitely long thread is bent making an angle of 90^0. Electric charge is uniformly distributed along the thread with a linear density τ = 1 μC/m. Determine the force F exerted on a point charge Q = 0.1 μCl, which is located on the prolongation of one of the sides of the angle and distant by a = 50 cm from the vertex of the angle.

1.25. Electric charge $Q = 10^2$ nC is uniformly distributed along a thin ring of radius R = 10 cm. A point charge Q_1 = 10 nC is located on the perpendicular to the plane of the ring drawn from its center. Determine the force F exerted on the point charge Q_1 by the charged ring if the charge is distant from the center of the ring by: 1) l_1 = 20 cm; 2) l_2 = 2 m.

1.26. Electric charge is uniformly distributed with a linear charge density τ = 1 nC/m along a thin ring of radius R = 10 cm. A point charge Q_0 = 400 nC is located in the center of the ring. Determine the force F stretching the ring. Neglect interaction between the ring's own charges.

1.27. A charge q is distributed throughout a body of volume V with a volumetric charge density $\rho = \rho(\vec{r})$; a charge q' is distributed throughout a body of volume V' with a volumetric charge density $\rho = \rho(\vec{r}\,')$. Write an expression for the force $\vec{F}$ exerted by the charge q' on the charge q.

1.28. What force is exerted on an electron in a cavity inside of a charged spherical layer if the volumetric charge density ρ in the layer depends only on the distance from its center?

1.29. A thin ring of wire of radius R = 100 mm carries an electric charge Q = 50 μC. What will the increase in the force stretching the ring be if a point charge Q_0 = 7 μC is placed in the center of the ring?

Topic 2. ELECTRIC FIELD. ELECTRIC DISPLACEMENT FIELD

What a student should know

1. Electric field, units for the electric field.
2. Electric displacement field.
3. Relation between the electric field vector $\vec{E}$ and the electric displacement vector $\vec{D}$.
4. Electric field due to a point charge and due a uniform charge distribution along an infinite line and over a plane.
5. Flux of the electric field vector.
6. Flux of the electric displacement vector.
7. Gauss's law for conductors and dielectrics.
8. Superposition principle for electric fields.
9. A concept of surface density of bound charges.
10. Differentiation and integration method (**DI-method**).

Literature: [6, § 23.4 – 23.6, § 24.1 – 24.4]; [7, § 5.4 – 5.6, 6.1 – 6.4]; [9, § 22.1 – 22.5, § 23]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: § 23 No 25, 36, 37; § 24 No 10, 11], examples 2.1 – 2.5.

Homework: see Table A.2 on p. 138.

2.1. Brief theoretical information

Electric field vector:

$$\vec{E} = \frac{\vec{F}}{q},$$

where $\vec{F}$ is the force exerted by the field on a charge q introduced into an arbitrary point in the field.

Gauss's law for conductors: flux of the vector $\vec{E}$ through a closed surface is equal to the algebraic sum of charges* enclosed within this surface divided by the electric constant ε_0:

$$\oint_S E_n dS = \frac{1}{\varepsilon_0}\sum q_i .$$

Gauss's law for dielectrics: flux of the vector $\vec{D}$ through a closed surface is equal to the algebraic sum of free charges enclosed within this surface:

$$\oint_S D_n dS = \sum q_i , \qquad \textbf{(2.1)}$$

where D_n is the projection of the electric displacement vector onto the normal to the surface.

Electric field due to a point charge q at a distance r from it:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad \textbf{(2.2)}$$

Superposition principle for electric fields: at any point in space, the total electric field $\vec{E}$ due to a group of source charges equals the vector (geometric) sum of the electric fields of all the charges:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n. \qquad \textbf{(2.3)}$$

In the case of two electric fields E_1 and E_2, the magnitude of resultant electric field is

$$E = \sqrt{E_1^2 + E_2^2 + 2E_1E_2\cos\alpha}. \qquad \textbf{(2.4)}$$

According to the superposition principle, the electric field vector $\vec{E}$ in a dielectric material equals to the geometric sum of the field due to free charges $\vec{E}_0$ and the field due to bound charges $\vec{E}'$:

$$\vec{E} = \vec{E}_0 + \vec{E}'. \qquad \textbf{(2.5)}$$

Surface density of bound charges:

$$\sigma' = \varepsilon_0\left(\varepsilon_r - 1\right)E_n, \qquad \textbf{(2.6)}$$

where E_n is the normal component of the electric field vector.

*free charges

Electric field due to an infinitely long uniformly charged thread at the points external to the thread:

$$E = \frac{\tau}{2\pi\varepsilon_0\varepsilon_r a},$$

where a is the distance from the point to the thread, τ is the linear density of the charge distribution along the thread.

Electric field due to an infinite uniformly charged plane:

$$E = \frac{\sigma}{2\varepsilon_0\varepsilon_r}, \qquad \textbf{(2.7)}$$

where σ is the surface density of the charge distribution over the plane.

Relation between the electric field vector $\vec{E}$ and electric displacement vector $\vec{D}$:

$$\vec{D} = \varepsilon_r\varepsilon_0\vec{E}.$$

Flux of the electric displacement vector:

a) in the case of a uniform field:

$$\Delta N_D = D\Delta S\cos\alpha;$$

b) in the case of a inhomogeneous field and any surface:

$$\Delta N_D = \int_S D_n dS,$$

where D_n is the projection of the vector $\vec{D}$ onto the normal direction to the surface element of area dS.

2.2. Methodical guidelines

1. Electric field due to a system of charges is found according to the superposition principle.

2. In the case when electric charge is uniformly distributed over a thread (rod, ring, sphere, ball), the **DI-method** is used.

3. The resultant electric field produced by two charges is conveniently determined using the geometric method of vector addition (formula 2.3). However, in the case of a system of three or

more charges, it is necessary to use the algebraic method, i.e., to express vectors through their projections onto the axes of pre-chosen coordinate system.

4. To calculate electric field in dielectric materials, one should use the following two methods.

The first method, called the *superposition method*, is based on the superposition principle (formula 2.5). First, the field of free charges $\vec{E}_0$ is determined, and then the field of bound charges $\vec{E}'$ is determined. Next, the field in the dielectric material is calculated by formula (2.5). However, some difficulties may arise if using this method, for example, in determining σ' in formula (2.6), which depends on E_n (it may be unknown) and in determining $\vec{E}'$. The **DI-method** is used for that.

The second method is based on determining the electric displacement vector $\vec{D}$ with the Gauss's law (formula 2.1), and then the electric field $\vec{E}$ in the dielectric can be found by formula (2.7). The second method is called *the Gaussian method*.

5. It should be noted that in most problems on calculation of the electric field in dielectric materials, the following simplifications are applied:

- the dielectric material is considered homogeneous and isotropic;
- boundaries of the dielectric material are perpendicular to the electric field lines (coincide with the equipotential surfaces).

2.2.1. Electric field due to a charge distribution over a spherical surface

Example 2.1

Electric charge is uniformly distributed over a surface of a hemisphere with surface charge density σ = 1 nC/m^2. Find the electric field E in the geometric center of the hemisphere.

Given:
$\sigma = 1$ nC/m^2 = 10^{-9} C/m^2
$\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m
E – ?

I. Physical model

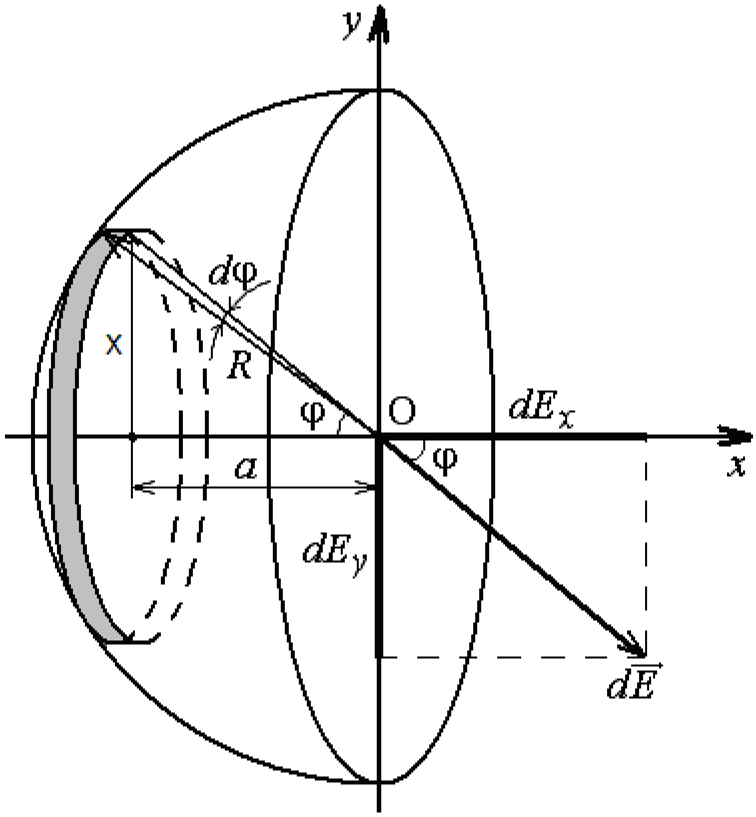


Figure 2.1

II. Mathematical model

Let's use the **DI-method**.

1. Divide the hemisphere into infinitesimal thin rings carrying an infinitesimal charge $dq = \sigma dS$ (σ is the surface charge density). Such a charge can be considered as a point charge. Then, the formula for the electric field due to the point charge dq at a distance r from that charge, can be written as:

$$dE = \frac{\sigma dS}{4\pi\varepsilon_r\varepsilon_0 r^2}, \qquad \textbf{(2.8)}$$

where ε_r = 1 (the medium is not specified the problem, so it is assumed that the charge is in vacuum).

2. Since electric field is a vector quantity, we decompose it into two components: dE_x along the normal to the plane of the ring and dE_y parallel to the plane of the ring.

3. Add these components for all elements of the ring. Due to the symmetry, the components that are parallel to the plane of the ring in total give zero, while the vertical components are determined by the formula:

$$dE_x = dE\cos\varphi, \qquad \textbf{(2.9)}$$

where $\cos\varphi = a/R$ (Fig. 2.1).

4. Substitute formula (2.9) into equation (2.8):

$$dE_x = \frac{\sigma a dS}{4\pi\varepsilon_0 R^3},$$

where $dS = 2\pi x R d\varphi$ ($2\pi x$ is the length of the ring; $Rd\varphi$ is the thickness of the ring).

Therefore,

$$dE_x = \frac{a\sigma 2\pi x d\varphi}{4\pi\varepsilon_0 R^2} = \frac{a\sigma x d\varphi}{2\varepsilon_0 R^2}. \qquad \textbf{(2.10)}$$

Distances: $a = R\cos\varphi$, $x = R\sin\varphi$. Then

$$dE_x = \frac{\cos\varphi\sin\varphi\sigma R^2 d\varphi}{2\varepsilon_0 R^2} = \frac{\cos\varphi\sin\varphi\sigma d\varphi}{2\varepsilon_0}. \qquad \textbf{(2.11)}$$

5. Integrating expression (2.11) within the integration limits from zero (the farthest ring) to $\pi/2$ (the nearest ring) we obtain:

$$E_x = \frac{\sigma}{2\varepsilon_0}\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi d\varphi = \frac{\sigma}{2\varepsilon_0}\int_0^{\frac{\pi}{2}} 1/2\sin 2\varphi d\varphi = \frac{\sigma}{2\varepsilon_0}\left(-\frac{1}{2}\cos\varphi\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\sigma}{4\varepsilon_0}.$$

III. Numerical calculations:

$$E_x = \frac{10^{-9}}{4\cdot 8.85\cdot 10^{-12}} = 28.3 \text{ (V/m)}.$$

Answer: $E_x = 28.3$ V/m.

2.2.2. Electric field due to a charged plane

Example 2.2

Two infinite parallel plates charged with surface densities σ_1 *and* σ_2 *are located in vacuum at a distance d from each other. The OX axis is directed perpendicular to the plate, the origin is on the plate* σ_1*. Plot dependences of the x-component* E_x *of the electric field vector on the coordinate x. Consider the following cases: a)* $\sigma_1 = \sigma$, $\sigma_2 = 2\sigma$*; b)* $\sigma_1 = \sigma$, $\sigma_2 = -\sigma$.

Given:

d

a) $\sigma_1 = \sigma$, $\sigma_2 = 2\sigma$

b) $\sigma_1 = \sigma$, $\sigma_2 = -\sigma$

$E_x(x)$ – ?

I. Physical model

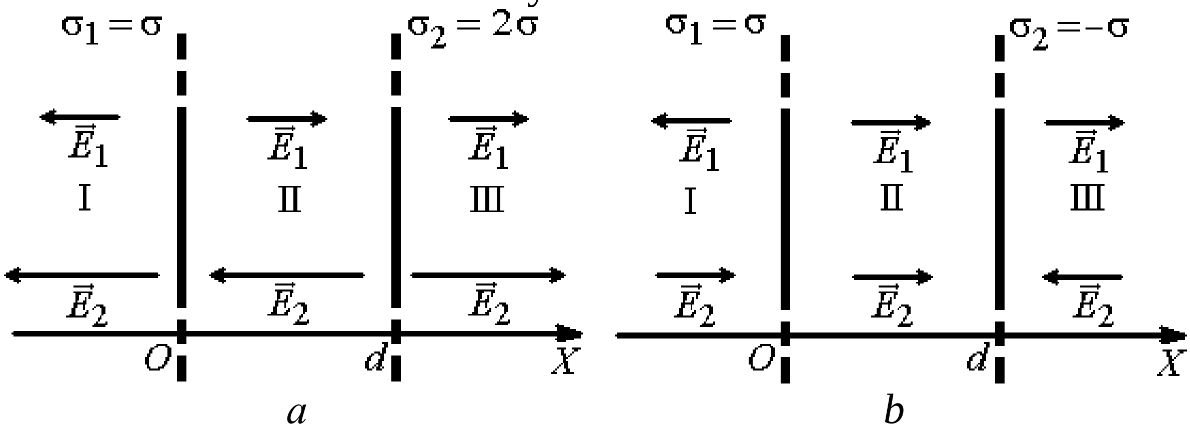


Figure 2.2

II. Mathematical model

1. Infinite plates divide space into three regions: I – to the left of the plate; II – between the plates; III – to the right of the plate (Fig. 2.2, *a*, *b*). In these regions, each of the plates creates its own electric field (the plate σ_1 creates the field $\vec{E}_1$, and the plate σ_2 creates the field $\vec{E}_2$). Magnitudes of these electric fields are determined by the formula

$$E = \frac{|\sigma|}{2\varepsilon_0}.$$

2. Directions of the vectors $\vec{E}_1$ and $\vec{E}_2$ for the two given cases are shown in Fig. 2.2, *a*, *b*, respectively. In each of the three regions, vector of the resulting electric field is determined by the principle of superposition

$$\vec{E} = \vec{E}_1 + \vec{E}_2.$$

Taking into account directions and magnitudes of the vectors $\vec{E}_1$ and $\vec{E}_2$, we can find *OX*-projections of the resultant vectors $\vec{E}_{\mathrm{I}}$, $\vec{E}_{\mathrm{II}}$, $\vec{E}_{\mathrm{III}}$ in each of the regions I, II, III:

a) $\sigma_1 = \sigma$, $\sigma_2 = 2\sigma$ (Fig. 2.2, *a*):

$$E_{\mathrm{I}} = -E_1 - E_2 = -\frac{\sigma}{2\varepsilon_0} - \frac{2\sigma}{2\varepsilon_0} = -\frac{3\sigma}{2\varepsilon_0};$$

$$E_{\mathrm{II}} = E_1 - E_2 = \frac{\sigma}{2\varepsilon_0} - \frac{2\sigma}{2\varepsilon_0} = -\frac{\sigma}{2\varepsilon_0};$$

$$E_{\mathrm{III}} = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{2\sigma}{2\varepsilon_0} = \frac{3\sigma}{2\varepsilon_0}.$$

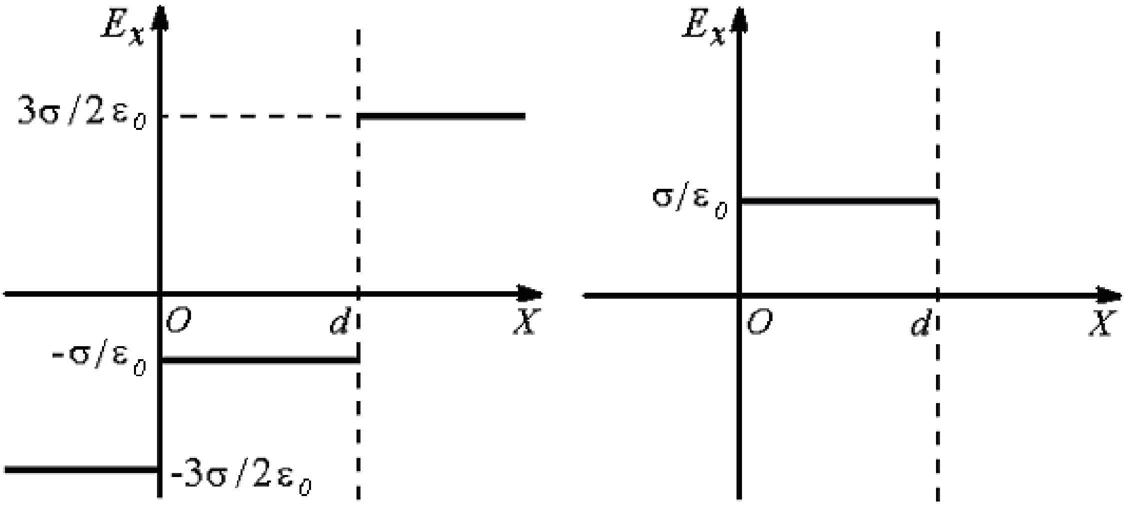
b) $\sigma_1 = \sigma$, $\sigma_2 = -\sigma$ (Fig. 2.2, *b*):

$$E_{\mathrm{I}} = -E_1 + E_2 = -\frac{\sigma}{2\varepsilon_0} + \frac{|\sigma|}{2\varepsilon_0} = 0;$$

$$E_{\mathrm{II}} = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{|\sigma|}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0};$$

$$E_{\mathrm{III}} = E_1 - E_2 = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0} = 0.$$

3. Plot graphs of the dependence $E_x(x)$, where E_x is the *OX*-projection of the electric field vector:



a *b*

Figure 2.3

Answer: the corresponding graphs are shown in Figure 2.3, *a*, *b*.

2.2.3. Electric force on a charge in electric field

Example 2.3

There are an infinitely long uniformly charged thread with a linear charge density $\tau_1 = 3\cdot10^{-7}$ C/m and a charged segment of length l = 20 cm with a linear charge density $\tau_2 = 2\cdot10^{-7}$ C/m placed in one plane perpendicularly to each other at a distance r_0 = 10 cm (Fig. 2.4). Determine the force of interaction between them.

Given:

$\tau_1 = 3\cdot10^{-7}$ C/m

$\tau_2 = 2\cdot10^{-7}$ C/m

l = 0.2 cm

r_0 = 0.1 cm

F – ?

I. Physical model

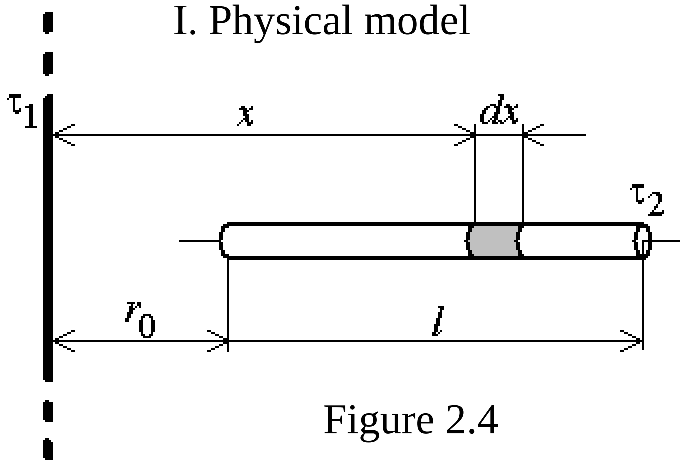


Figure 2.4

1. The physical system consists of two bodies: a thread and a segment.

The essence of the physical phenomenon is that the field produced by the thread exerts electric force on the charges of the segment. It is necessary to determine the force of this interaction.

2. In order to find the force exerted by the thread on the charge of the segment Q_2 we can use the relation between the electric force and the electric field:

$$F = qE.$$

Electric field $E(x)$ due to the infinitely long uniformly charged thread is not uniform, and the charge of the segment is not a point charge. Let's use the formula for the electric field due to the infinitely long uniformly charged thread:

$$E(x) = \frac{\tau_1}{2\pi\varepsilon_0 x}.$$

Then, let's use the **DI-method** to calculate the force.

II. Mathematical model

1. Consider the segment element of length dx carrying the charge $dq = \tau_2 dx$, which is located at the distance x from the thread (Fig. 2.4). We can consider the charge dq as a point charge. The force exerted on that charge is:

$$dF = Edq = \frac{\tau_1\tau_2}{2\pi\varepsilon_0 x}dx.$$

2. The force exerted on each segment element depends on the distance x from that element to the thread. Therefore, we take the distance x for the integration variable and integrate within the limits from r_0 to $r_0 + l$. So,

$$F = \int_{r_0}^{r_0+l} \frac{\tau_1\tau_2}{2\pi\varepsilon_0 x}\frac{dx}{x} = \frac{\tau_1\tau_2}{2\pi\varepsilon_0}\ln\left(1+\frac{l}{r_0}\right).$$

III. Numerical calculations

$$F = \frac{3\cdot10^{-7}\cdot2\cdot10^{-7}}{2\cdot3.14\cdot8.85\cdot10^{-12}}\ln\left(1+\frac{0.2}{0.1}\right) \approx 1.2\cdot10^{-3}(\text{N}).$$

Answer: $F \approx 1.2$ mN.

2.2.4. Electric field in dielectric materials

Example 2.4

Two infinite coaxial cylinders with radii R_1 = 5 cm and R_2 = = 10 cm are uniformly charged with surface densities σ_1 = 10 nC/m^2 and σ_2 = –3 nC/m^2. The space between the cylinders is filled with paraffin (ε_r = 2). Determine the magnitude of the electric field E at the points A, B, C, which are at distances r_1 = 2 cm, r_2 = 6 cm and r_3 = 15 cm from the axis of the cylinders.

Given:
$R_1 = 0.05$ m
$R_2 = 0.1$ m
$\sigma_1 = 10^{-8}$ C/m^2
$\sigma_2 = -3\cdot10^{-9}$ C/m^2
$\varepsilon_r = 2$
$r_1 = 0.02$ m
$r_2 = 0.06$ m
$r_3 = 0.15$ m

E_A - ?,
E_B - ?,
E_C - ?.

I. Physical model

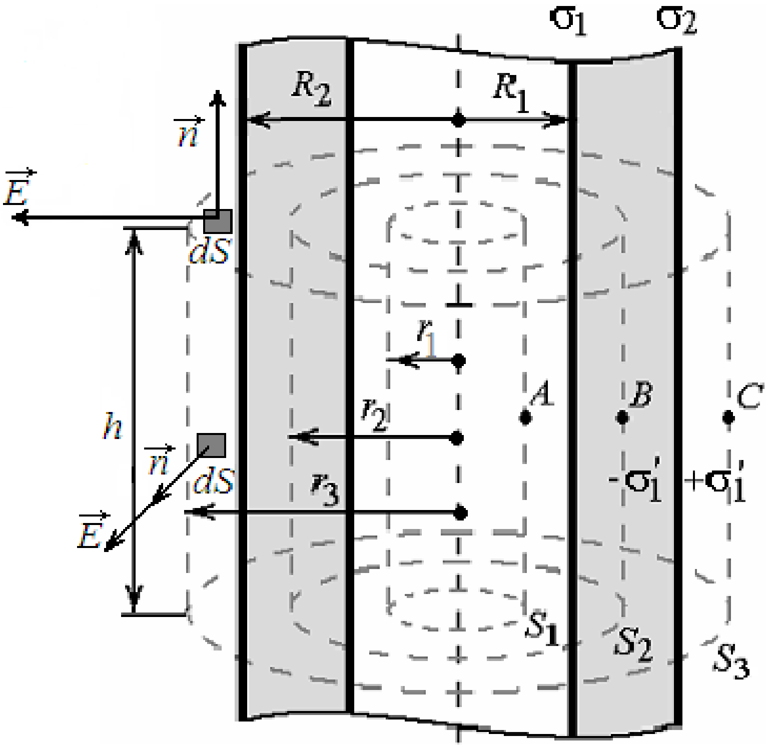


Figure 2.5

The total field is produced by free charges on cylinders with surface densities σ_1, σ_2 and by bound* charges arising from the polarization of paraffin, the densities of which are $-\sigma_1'$, $+\sigma_1'$ (Fig. 2.5).

*Electric field due to the bound charges is not zero only inside the dielectric material.

II. Mathematical model

The superposition method. To determine the electric field at the points *A, B, C,* we use the Gauss's law. For that, we draw auxiliary cylindrical surfaces S_1, S_2, S_3 passing through the points *A, B, C,* respectively, with the radii $r_1 < R_1$, $R_1 < r_2 < R_2$ and $r_3 > R_2$ (Fig. 2.5). Then the Gauss's law for each surface can be written as:

$$\oint_S \vec{E}d\vec{S} = \frac{\sum Q_i}{\varepsilon_0}. \tag{2.12}$$

Let's transform the left part of the Gauss's law:

$$\oint \vec{E}d\vec{S} = \int_{S_{1,2,3lateral}} E_n dS + 2\int_{S_{1,2,3ends}} E_n dS,$$

where E_n is the projection of the electric field vector onto the normal to the Gaussian surfaces.

Consider *ends* of the Gaussian cylinders (Fig. 2.5). For them, projection of the electric field vector onto the normal is zero ($E_n = 0$), then

$$\int_{S_{1,2,3ends}} E_n dS = 0, \text{ since } E_n = 0 \text{ (Fig. 2.5)}, dS \neq 0.$$

All points on the *lateral surface* are in the same conditions with respect to the charge, which allows us to consider E_n as a constant value (E_n = const), then

$$\int_{S_{1,2,3lat.}} E_n dS = E_n \int_{S_{1,2,3lat.}} dS = E_n S_{1,2,3lat.} = E_n 2\pi r \cdot h, \tag{2.13}$$

where r and h are the radius and the height of the auxiliary surface.

The sum of the charges enclosed by the auxiliary surface, according to expression (2.12), depends on the radius of that surface.

1. If $r_1 < R_1$, then $\sum Q_i = 0$, because there are no charges inside the charged cylinder. So,

$$\oint_{S_1} \vec{E}d\vec{S} = 0.$$

Therefore, the electric field $E = 0$ at the point *A*.

2. At the point *B*, the field is created by the charges inside the surface S_2 that have surface densities σ_1 and $-\sigma_1'$ (the field due to the charges $+\sigma_1'$ and σ_2 is zero). By the Gauss's law, we first determine

the field due to the charge σ_1 distributed over the surface of the cylinder of radius R_1:

$$\sum Q_i = \sigma_1 2\pi R_1 h. \quad \textbf{(2.14)}$$

Substitute expression (2.14) into equation (2.12). Taking into account the right part of the formula (2.13), we obtain:

$$E_2 2\pi r_2 h = \frac{\sigma_1 2\pi R_1 h}{\varepsilon_0};$$

$$E_2 r_2 = \frac{\sigma_1 R_1}{\varepsilon_0};$$

$$E_2 = \frac{\sigma_1 R_1}{\varepsilon_0}\frac{1}{r_2}.$$

3. Similarly we determine the field due to the bound charges of surface density σ_1' (they are negative):

$$E_2' = \frac{\sigma_1' R_1}{\varepsilon_0}\frac{1}{r_2}.$$

According to formula (2.6),

$$\sigma_1' = \varepsilon_0 \left(\varepsilon_r - 1\right) E\left(R_1\right). \quad \textbf{(2.15)}$$

It is important to note that in formula (2.15) $E(R_1)$ is the value of the total electric field in the dielectric material at the point R_1. That value is unknown. We need to find its relation with the total field E_B across the dielectric. Since the electric fields E_2 and E'_2 are inversely proportional to the distance r_2, the total values $E(R_1)$ and E_B must also meet that condition:

$$\frac{E\left(R_1\right)}{E_B} = \frac{r_2}{R_1}.$$

Then,

$$E_2' = \left(\varepsilon_r - 1\right) E_B.$$

Since, according to the formula (2.4, *b*),

$$E_B = E_2 - E_2',$$

therefore,

$$E_B = \frac{R_1 \sigma_1}{\varepsilon_0 \varepsilon_r}\frac{1}{r_2}. \quad \textbf{(2.16)}$$

4. At the point C, which is at the distance r_3 from the axis, the field is created only by free charges, the surface densities of which are σ_1 and σ_2. So, by the same algorithm we find the value of the electric field E_C:

$$E_C = \frac{R_1\sigma_1}{\varepsilon_0}\frac{1}{r_3} - \frac{R_2\sigma_2}{\varepsilon_0}\frac{1}{r_3}.$$

The Gaussian method.

1. Let's calculate electric field at the point B by the Gaussian method. According to the Gauss's law (2.1)

$$D2\pi r_2 h = 2\pi R_1\sigma_1 h,$$

so,

$$D = R_1\sigma_1 / r_2.$$

2. Next, according to the relation between the electric field $\vec{E}$ and the electric displacement field $\vec{D}$, we find the desired value:

$$E(r_2) = \frac{R_1\sigma_1}{\varepsilon_0\varepsilon_r}\frac{1}{r_2},$$

which coincides with the expression (2.16), found by the superposition method.

III. Numerical calculations:

$$E(r_2) = \frac{0,05\cdot 10^{-8}}{8.85\cdot 10^{-12}\cdot 2}\cdot\frac{1}{0.06} \approx 4.7\cdot 10^2\,(\text{V/m});$$

$$E(r_3) = \frac{0.05\cdot 10^{-8}}{8.85\cdot 10^{-12}}\cdot\frac{1}{0.15} - \frac{0.1\cdot 3\cdot 10^{-9}}{8.85\cdot 10^{-12}}\cdot\frac{1}{0.15} \approx 1.5\cdot 10^2\,(\text{V/m}).$$

Answer: $E(A) = 0$, $E(B) \approx 4.7\cdot 10^2$ (V/m), $E(C) \approx 1.5\cdot 10^2$ (V/m).

2.2.5. Flux of the electric field vector and the electric displacement vector

Example 2.5

Electric field is created by a point charge $q = 0.1$ µC. Determine the flux N_D of the electric displacement vector through a round surface of radius $R = 30$ cm. The charge is equidistant from the *boundary of the surface and is located at a distance a = 40 cm from its center.*

Given:

$q = 10^{-7}$ C

$R = 0.3$ m

$a = 0.4$ m

N_D – ?

I. Physical model

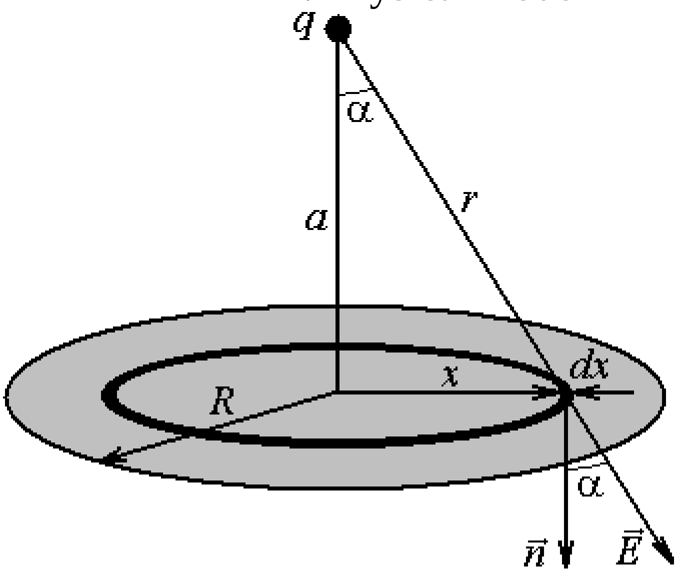


Figure 2.6

II. Mathematical model

1. The flux N_D of the electric displacement vector $\vec{D}$ can be found similarly to the flux of the electric field vector:

$$N_D = \int_S D_n dS, \text{ where } D_n = D\cos\alpha. \qquad \textbf{(2.17)}$$

The integral is taken over the area S; D_n is the projection of the electric displacement vector onto the normal direction to the surface element of area dS.

2. In vacuum, the electric displacement $\vec{D}$ is related to the electric field $\vec{E}$ by the following relation ($\varepsilon_r = 1$):

$$\vec{D} = \varepsilon_0 \vec{E}. \qquad \textbf{(2.18)}$$

3. Let's use the **DI-method** (see p. 8).

Divide the surface into infinitesimal rings. Choose one such ring of radius x and thickness dx. Then the infinitesimal area of the ring is equal to:

$$dS = 2\pi x dx.$$

Since the field due to the point charge has central symmetry, for any point on the ring the value of the electric field is the same and equals to:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = k\frac{Q}{a^2 + x^2}.$$

4. For the points on the ring, the angle α between the electric field vector $\vec{E}$ and the normal to the surface of the ring remains unchanged (Fig. 2.6), therefore,

$$\left(\vec{E};\vec{n}\right) = E\cos\alpha = E\frac{a}{\sqrt{a^2 + x^2}}. \qquad \textbf{(2.19)}$$

5. Substituting expressions (2.18) and (2.19) into formula (2.17), we obtain:

$$N_D = \int_S \varepsilon_0\left(\vec{E};\vec{n}\right)dS = \int_0^R \varepsilon_0 E\frac{a}{\sqrt{a^2 + x^2}}2\pi x dx = \int_0^R \frac{\varepsilon_0 a}{4\pi\varepsilon_0}\frac{Q}{\left(a^2 + x^2\right)^{3/2}}2\pi x dx.$$

Integrate this expression within the limits from 0 to R:

$$N_D = \frac{aQ}{4}\int_0^R \frac{d\left(a^2 + x^2\right)}{\left(a^2 + x^2\right)^{\frac{3}{2}}} = \frac{aQ}{4}(-2)\frac{1}{\sqrt{a^2 + x^2}}\Bigg|_0^R =$$

$$= -\frac{aQ}{2}\left(\frac{1}{\sqrt{a^2 + R^2}} - \frac{1}{a}\right) = \frac{aQ}{2}\left(\frac{1}{a} - \frac{1}{\sqrt{a^2 + R^2}}\right).$$

III. Numerical calculations:

$$N_D = \frac{0.4\cdot 10^{-7}}{2}\left(\frac{1}{0.4} - \frac{1}{\sqrt{0.4^2 + 0.3^2}}\right) = 10^{-8}\,(\text{C}).$$

Answer: N_D = 10 nC.

2.3. Problems for independent work

Electric field due to point charges

2.1. A distance d between two point charges $Q_1 = 8$ nC and $Q_2 = -5{,}3$ nC is 40 cm. Determine the electric field at the point in the middle between the two charges. What would the magnitude of the electric field be if the second charge were positive?

2.2. Electric field is created by two point charges $Q_1 = 10$ nC and $Q_2 = -20$ nC, which are at a distance $d = 20$ cm from each other. Determine the electric field E at the point distant from the first charge by $r_1 = 30$ cm and from the second charge by $r_2 = 50$ cm.

2.3. A distance d between two positive point charges $Q_1 = 9Q$ and $Q_2 = Q$ is 8 cm. At what distance from the first charge is the point where the resultant electric field E is zero? Where would that point be if the second charge were negative?

2.4. Two point charges $Q_1 = 2Q$ and $Q = -Q$ are at a distance d from each other. Find position of the point on the line passing through these charges where the resultant electric field E is zero.

2.5. Electric field is created by two point charges $Q_1 = 40$ nC and $Q_2 = -10$ nC, which are at a distance $d = 10$ cm from each other. Determine the electric field E at the point distant from the first charge by $r_1 = 12$ cm and from the second charge by $r_2 = 6$ cm.

2.6. Identical point charges q are located at the vertices of a regular hexagon with side a. Determine the electric field E in the center of the hexagon if: a) the sign of all the charges is the same, b) the signs of the neighboring charges are opposite.

2.7. N point charges $q_1, q_2, ..., q_i, ..., q_N$ are located in vacuum at points with radii-vectors $\vec{r}_1, \vec{r}_2, ..., \vec{r}_i, ..., \vec{r}_N$. Write expression for the electric field $\vec{E}$ at the point defined by the radius vector $\vec{r}$.

2.8. A positive point charge of 50 μC is located on the *XY* plane at the point with a radius-vector $\vec{r}_0 = 2\vec{i} + 3\vec{j}$, where $\vec{i}$ and $\vec{j}$ are the unit vectors of the *OX* and *OY* axes. Find direction and magnitude of the electric field vector at the point with radius-vector $\vec{r}_E = 8\vec{i} - 5\vec{j}$. Here $\vec{r}_0$ and $\vec{r}$ are measured in meters.

2.9. There are point charges q and $-q$ at the vertices of a square with a diagonal equal to $2l$. Find magnitude of the electric field at the point located at the distance x from the center of the square and symmetrical about its vertices.

Electric field due to a charge distributed over a circle and a sphere

2.10. A metal sphere of radius $R = 10$ cm carries an electric charge $Q = 8$ nC. Determine the electric field E at the following points: a) at a distance $r_1 = 1$cm from the center of the sphere; b) on its surface; c) at a distance $r_2 = 15$ cm from the center of the sphere. Plot the E vs r graph.

2.11. Two concentric charged metal spheres with radii $R_1 = 6$ cm and $R_2 = 10$ cm carry charges $Q_1 = 1$ nC and $Q_2 = -0.5$ nC, respectively. Determine the electric field E at the points distant from the center of the sphere by $r_1 = 5$ cm, $r_2 = 9$ cm, $r_3 = 15$ cm. Plot the graph of dependance $E(r)$.

2.12. Electric charge is uniformly distributed along a thin circle of radius $R = 8$ cm with a linear charge density $\tau = 10$ nC/m. What is the magnitude of electric field at the point equidistant from all points of the circle by a distance r = 10 cm?

2.13. A charge q = 20 nC is uniformly distributed along a thin wire ring of radius R = 60 mm.

a) Taking the axis of the ring as OX, find the electric field $\vec{E}$ as a function of X (take the origin for X in the center of the ring);

b) investigate the following cases: $\vec{x} = 0$ and $|\vec{x}| >> r$;

c) determine the maximum value of the electric field magnitude E_m and the corresponding coordinates $\vec{x}_m$.

2.14. A charge q = 0.7 nC is uniformly distributed along a thin semicircle of radius R = 20 cm. Find magnitude of the electric field in the center of that semicircle.

2.15. A thin wire ring of radius R carries electric charge q. Find magnitude of the electric field on the axis of the ring as a function of the distance l to its center. Investigate the obtained dependence if $l >> R$. Determine the maximum value of the electric field magnitude and the corresponding distance l.

2.16. A point charge q is located in the center of a thin ring of radius R which carries a charge $-q$ uniformly distributed along its length. Find magnitude of the electric field on the axis of the ring at the point located at a distance x from its center, if $x >> R$.

2.17. A thin non-conductive ring of radius R is charged with a linear density $\lambda = \lambda_0 \cos\varphi$, where λ_0 is constant, φ is the azimuthal angle. Find magnitude of the electric field: a) in the center of the ring; b) on the axis of the ring depending on the distance x from its center. Investigate the obtained expression for $x >> R$.

2.18. A sphere of radius r is charged with a surface density $\sigma = \vec{a} \cdot \vec{r}$, where $\vec{a}$ is the constant vector; $\vec{r}$ is the radius-vector of a point on the surface of the sphere relative to its center. Find magnitude of the electric field in the center of the sphere.

Electric field due to a charged line

2.19. A charge Q = 500 nC is uniformly distributed over a surface of a straight metal rod with diameter d = 5 cm and length l = 4 m. Determine magnitude of the electric field E at the point located on the perpendicular bisector of the rod at a distance a = = 1 cm from its surface.

2.20. Electric charge is uniformly distributed along a length of an infinitely long straight conductor. Determine the linear charge density τ if the magnitude of the electric field E at the distance a = 0.5 m from the conductor along its perpendicular bisector equals 200 V/m.

2.21. Two long straight conductors are located parallel to each other and separated by a distance d = 16 cm. The conductors are uniformly charged with charges of opposite signs and equal linear densities $|\tau|$ = 150 μC/m. What is magnitude of the electric field E at the point equidistant from the both conductors by r = 10 cm?

2.22. Electric charge is uniformly distributed over a surface ($\sigma = 1$ nC/m^2) of an infinitely long thin-walled metal tube with a radius $R = 2$ cm. Determine magnitude of the electric field E at the points distant from the axis of the tube by $r_1 = 1$ cm, $r_2 = 3$ cm. Plot a graph of the dependence $E(r)$.

2.23. Two long thin-walled coaxial tubes with radii $R_1 = 2$ cm and $R_2 = 4$ cm carry electric charges uniformly distributed along their length with linear densities $\tau_1 = 1$ nCl/m and $\tau_2 = -0.5$ nCl/m. The space between the tubes is filled with ebonite. Determine magnitude of the electric field E at the points located at distances $r_1 = 1$ cm, $r_2 = 3$ cm, $r_3 = 5$ cm from the axis of the tubes. Plot a graph of the dependence $E(r)$.

2.24. Electric charge is uniformly distributed with a linear density $\tau = 3$ μC/m along a segment of a thin straight conductor of length $l = 10$ cm. Determine the electric field E at the point located on the axis of the conductor and distant from the proximal end of the segment by the distance equal to the length of that segment.

2.25. A thin rod of length $l = 10$ cm is charged with a linear density $\tau = 400$ nC/m. Find the electric field E at the point distant by $r = 8$ cm from the rod along the perpendicular to the rod, built from one of its ends.

2.26. A thin rod of length $l = 12$ cm is charged with a linear density $\tau = 200$ nC/m. Find the electric field E at the point located at a distance $r = 5$ cm from the rod along its perpendicular bisector.

2.27. Electric field is created by a thin uniformly charged rod, which is bent as three sides of a square (Fig. 2.7). The length a of the sides of the square is 20 cm. The linear charge density τ is 500 nC/m. Determine the electric field E at the point A.

2.28. Two thin straight rods of length $l_1 = 12$ cm and $l_2 = 16$ cm each are charged with equal linear densities $\tau = 400$ nC/m. The rods make a right angle. Find the electric field E at the point A (Fig. 2.8).

2.29. A very thin rod of length $2l$ is located in vacuum. The rod is charged with a linear density τ. Find magnitude of the electric field E as a function of the distance from the center of the rod for the points lying on the prolongation of its axis. Investigate the case $r >> l$.

2.30. An infinitely thin thread is charged uniformly with a linear density τ. Using the Gauss's law, find magnitude of the electric field E as a function of the distance from the thread.

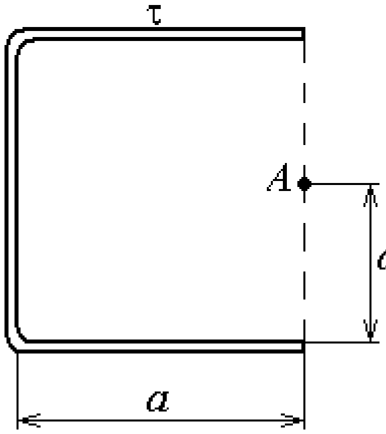


Figure 2.7

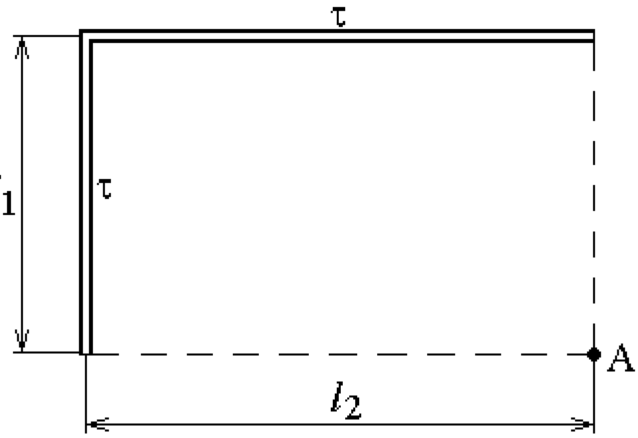


Figure 2.8

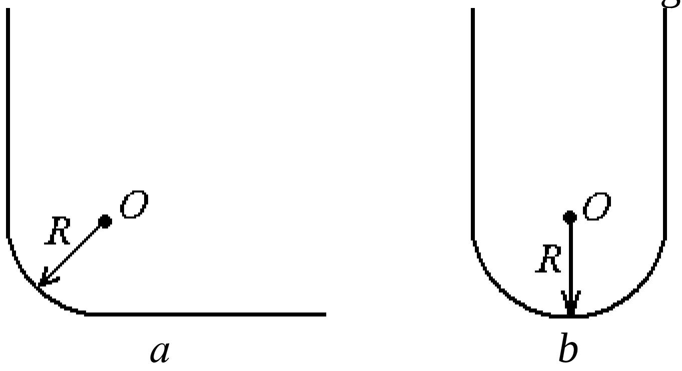


Figure 2.9

2.31. A thread is charged uniformly with a linear density τ and is bent as shown in Fig. 2.9, *a*, *b*. The curvature radius R is much smaller than the length of the thread. Using the result of the previous problem, find magnitude of the electric field E at the point O for the configurations (*a*) and (*b*).

2.32. Two long parallel threads are uniformly charged with a linear density $\tau = 0.5$ μC/m. The distance between the threads is $l =$ = 45 cm. Find the maximum value of the electric field in the plane of symmetry of such a system, which is located between the threads.

Electric field due to a charged plane

2.33. Electric field is created by two infinitely parallel plates, uniformly charged with surface densities $\sigma_1 = 2$ nC/m^2 and $\sigma_2 = -5$ nC/m^2. Determine the electric field E: a) between the plates; b) outside the plates. Plot a graph of the change in the electric field along a line perpendicular to the plates.

2.34. Two infinite parallel planes are uniformly charged with surface densities $\sigma_1 = 10$ nC/m^2 and $\sigma_2 = -30$ nC/m^2. Determine the force of interaction between the plates per unit area.

2.35. Two infinite planes carry equal charges distributed uniformly with surface density σ = 100 nC/m. The planes intersect at an angle α = 60^0. Find the electric field produced by the planes and draw a pattern of the electric field lines.

2.36. Two infinite plates are placed at a right angle to each other. Electric charges are uniformly distributed over the surface of the plates with surface densities $\sigma_1 = 1$ nC/m^2 and $\sigma_2 = 2$ nC/m^2. Determine the electric field created by the plates. Draw a pattern of the electric field lines.

2.37. Two identical rectangular parallel plates with sides a = 10 cm and b = 15 cm are located at a small distance (comparing to the linear dimensions of the plates) from each other. Charges Q_1 = = 50 nC and Q_2 = 150 nC are uniformly distributed over the surfaces of the plates. Determine the electric field between the plates.

2.38. Two round parallel plates of radius R = 10 cm are at a small distance (comparing to the radius) from each other. Electric charges of the same magnitude but opposite signs are given to the plates: $|Q_1| = |Q_2| = Q$. Determine the value of that charge Q if the plates are attracted to each other with the force F = 2 mN. Assume that the charges are distributed uniformly over the surface of the plates.

2.39. A parallel plate capacitor consists of two round plates of radius r, separated by a distance $2a$ ($a << r$). The plates have equal charges of opposite signs (+σ and –σ). The origin is in the center of the capacitor and the X axis is perpendicular to the plates passing through their centers. Investigate electric field at the points lying on the X axis. To do that, find: a) $E(x)$; b) $E(0)$, i.e. the field in the center of the capacitor; c) $E(a-0)$, $x = a - \delta$, $\delta \to 0$; d) $E(a+0)$, i.e. the field at the point with coordinate $x = a + \delta$, $\delta \to 0$; e) $E(x)$ for the case when $x >> r$. Neglect the plane of the plates.

2.40. An infinitely long cylindrical surface of circular cross-section is charged uniformly along its length with a surface density $\sigma = \sigma_0 \cos\varphi$, where φ is the polar angle of the cylindrical coordinate system where the axis OZ coincides with the axis of the considered surface. Find magnitude and direction of the electric field vector on the axis.

Electric field due to a charge distributed throughout a volume

2.41. A large flat plate of thickness $d = 1$ cm contains a charge uniformly distributed throughout a volume with a volume density $\rho = 100$ nC/m^3. Find the electric field E near the central part of the plate and outside the plate at a small distance from its surface.

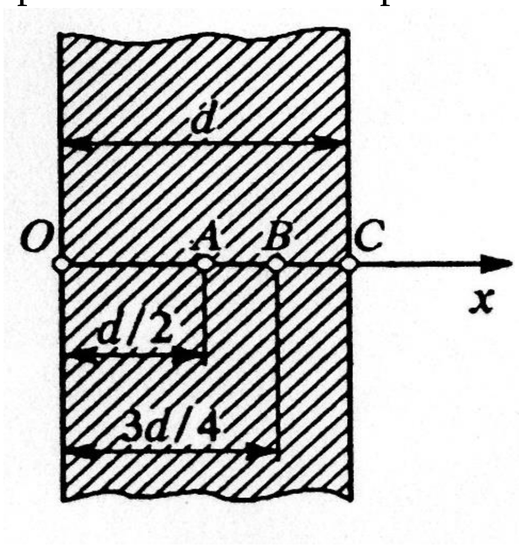


Figure 2.10

2.42. A sheet of glass of thickness $d = 2$ cm is uniformly charged with a volume density $\rho = 1$ nC/m^3. Determine the electric field E and the electric displacement D at the points A, B, C (Fig. 2.10). Plot a graph of the dependence $E(x)$ (the x-axis is perpendicular to the glass surface).

2.43. An ebonite solid sphere of radius $R = 5$ cm is uniformly charged with a volume density $\rho = 10$ nC/m^3. Determine the electric field E and the electric displacement D at the points: a) at a distance $r_1 = 3$ cm from the center of the sphere; b) on the surface of the sphere; c) at a distance $r_2 = 10$ cm from the center of the sphere. Plot the dependencies $E(r)$ and $D(r)$.

2.44. A glass sphere carries a uniformly distributed charge with a volume charge density ρ = 100 nC/m^3. The inner radius R_1 of the sphere is 5 cm, the outer radius R_2 is 10 cm. Calculate the electric field E and the electric displacement D at the points distant from the center of the sphere by: a) $r_1 = 3$ cm; b) $r_2 = 6$ cm; c) $r_3 = 12$ cm. Plot the dependencies $E(r)$ and $D(r)$.

2.45. A long paraffin cylinder of radius R = 2 cm is uniformly charged with a volume density ρ = 10 nC/m^3. Determine the electric field E and the electric displacement D at the points distant by a) $r_1 = 1$ cm; b) $r_2 = 3$ cm from the axis of the cylinder. The both points are equidistant from the ends of the cylinder. Plot the dependencies $E(r)$ and $D(r)$.

2.46. There is a spherical cavity containing no electric charges inside a sphere charged with a volume charge density ρ. The shift of the center of the cavity relative to the center of the sphere is determined by the vector $\vec{a}$. Find the electric field E inside the cavity. Consider the case $\vec{a}$ = 0.

2.47. A space is filled with electric charge of density $\rho = \rho_0 \exp(-\alpha r^3)$, where ρ_0 and α are constants. Find the electric field $\vec{E}$ as a function of $\vec{r}$. Investigate behavior of the field for large and small $\vec{r}$ (consider $\vec{r}$ is large for the condition $\alpha r^3 >> 1$, small for the condition $\alpha r^3 << 1$).

Force on an electric charge in an electric field

2.48. A thin thread is uniformly charged with a linear charge density τ = 2 μC/m. A point charge Q = 0.1 μC is located on the perpendicular bisector at a distance r = 1 cm from the thread, r is small comparing to the length of the thread. Determine the force F exerted on the charge Q.

2.49. A large metal plate is uniformly charged with a surface charge density σ = 10 nC/m^2. A point charge Q = 100 nC is located at a small distance from the plate. Find the force F exerted on the charge.

2.50. A point charge Q = 1 μC is located near a large uniformly charged plate opposite its middle. Determine the surface charge density σ of the plate if the force F = 0.06 N is exerted on the point charge.

2.51. A thin thread with uniform charge distribution along a length (τ = 0.4 nC/m) is located parallel to an infinite plate, which is uniformly charged with a surface density σ = 20 nC/m^2. Determine the force F exerted on the thread segment of length l = 1 m.

2.52. An infinitely long straight thread is uniformly charged with a linear charge density $\tau_1 = 10^3$ nC/m. A thin uniformly charged circle with a linear charge density $\tau_2 = 10$ nC/m shares the same axis with the thread. Determine the force F that stretches the circle. Interaction between individual parts of the circle is neglected.

2.53. Two parallel infinitely long straight threads are uniformly charged with linear densities $\tau_1 = 100$ nC/m and $\tau_2 = 200$ nC/m. Determine the force F of their interaction per unit length l = 1 m. The distance R between the threads is 10 cm.

2.54. Two identical round plates of area S = 100 cm^2 each are located parallel to each other. The charge Q_1 of the first plate equals +100 nC, the charge Q_2 of the second plate equals –100 nC. Determine the force F of mutual attraction of the plates for the cases when the distance between them is a) $r_1 = 2$ cm; b) $r_2 = 10$ m.

2.55. A metal sphere carries a charge $Q_1 = 100$ nC. There is a thin thread going along the electric field line of the sphere, beginning at the distance from the sphere’s surface which equals to the sphere’s radius. The length of the thread is equal to the radius of the sphere. A charge $Q_2 = 10$ nC is uniformly distributed along the length of the thread. Determine the force F acting on the thread if the radius of the sphere R is 10 cm.

2.56. A thin infinitely long thread carries a charge τ per unit length and is located parallel to a conductive plane. The distance between the thread and the plane is l. Find: a) magnitude of the force acting per unit length of the thread; b) distribution of the surface charge density $\sigma(x)$ over the plane (where x is the distance from the line on the plane, σ = max).

Flux of an electric field vector and an electric displacement vector

2.57. An infinite plane is uniformly charged with a surface charge density $\sigma = 1$ μC/m^2. A circle of radius $r = 10$ cm is placed parallel to the plane at a certain distance from it. Determine the flux Φ_E of the electric field vector passing through that circle.

2.58. A flat square plate with sides of length $a = 10$ cm is located at some distance from an infinite uniformly charged plane ($\sigma = 1$ μC/m^2). The plane of the plate makes an angle $\beta = 30^\circ$ with the electric field lines. Find the flux N_D of the electric displacement field through that plate.

2.59. A point charge $Q = 10$ nC is located in the center of a sphere of radius $R = 20$ cm. Determine the flux Φ_E of the electric field vector through a part of the spherical surface with the area $S = 20$ cm^2.

Topic 3. ELECTRIC POTENTIAL. ENERGY OF A SYSTEM OF ELECTRIC CHARGES

What a student should know

1. Potential of electrostatic field, units of the electric potential.
2. Potential of the field due to a point charge.
3. Potential of the field due to a charged sphere.
4. Potential energy of interaction in a system of point charges.
5. Work by forces of electrostatic field.
6. Superposition principle for electric fields.
7. Gauss's law.
8. Differentiation and integration method (**DI-method**).

Literature: [6, § 25.1 – 25.8]; [7, § 7.1 – 7.3]; [9, § 24.1 – 24.5]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: §25 No 3, 13, 16, 40, 42], examples 3.1 - 3.4.

Homework: see Table A.3 on p. 139.

3.1. Brief theoretical information

Potential of the electric field is a value equal to the ratio of the potential energy of a point positive charge placed at a given point of the field to the value of that charge:

$$\varphi = \frac{W}{q}.$$

Electric potential can be found as the work done by the electric field in carrying a unitary charge from that point to infinity without any acceleration:

$$\varphi = \frac{A}{q}.$$

The SI unit for electric potential is V (volt).

Potential is a scalar quantity. If at some point in space electric field is created by *n* charges with potentials φ_1, φ_2, φ_3, ... φ_n, then the total potential is found by the superposition principle:

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + ... + \varphi_n, \tag{3.1}$$

where $\varphi_i > 0$, if $q_i > 0$, and $\varphi_i < 0$ if $q_i < 0$.

Electric potential of the field due to a **point charge** *q* at the distance *r* from that charge:

$$\varphi = k\frac{q}{r},$$

where $k = \frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9$ m/F is the coefficient of proportionality.

Electric potential of the field due to a **metal sphere** of radius *R* and charge *q* at the distance *r* from the center of the sphere:

inside the sphere ($r > R$): $\varphi = k\frac{q}{\varepsilon_r R}$,

on the surface of the sphere ($r = R$): $\varphi = k\frac{q}{\varepsilon_r R}$,

outside the sphere ($r < R$): $\varphi = k\frac{q}{\varepsilon_r r}$.

Energy of interaction of a system of point charges is calculated using the principle of superposition of fields by the formula

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\varphi_i, \tag{3.2}$$

where φ_i is the potential of the field due to $n - 1$ charges (except for the *i*-th charge) at the point where the charge *q* is located.

3.2. Methodical guidelines

1. If a problem considers a point charge, then electric potential at some point of the field distant by r from that charge is determined by the formula

$$\varphi = k\frac{q}{r}.$$

2. Electric potential due to a system of charges is found by the superposition principle (formula 3.1). One should remember that the potential is an algebraic scalar quantity, so the calculations must take into account the signs of the individual charges of the system, because they determine the signs of the potentials of the corresponding field components.

3. For an arbitrary charge distribution, one can use the **DI-method**. It is especially effective for determining electric potential due to charges distributed linearly or over a surface.

4. In the case of volume charge distribution, the **DI-method** almost loses its effectiveness due to the complexity of integral expressions. In the cases where the charge system has a certain symmetry, the potential can be found by application of the Gauss's law.

5. Potential energy W of interaction of a system of point charges is determined by the work that the system can perform if the charges move away from each other to infinity. It can be found by formula (3.2).

6. Potential energy of repulsion of charges of the same sign is positive and increases if the charges move towards each other. Potential energy of attraction of charges of opposite signs is negative and increases to zero if one of the charges moves away from the other to a long distance ($r \to \infty$).

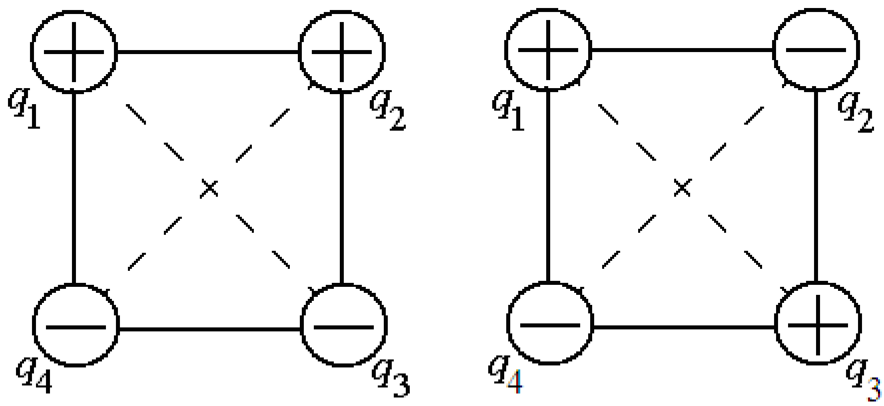
3.2.1. Potential energy and potential of the electric field due to a point charge

Example 3.1

Determine the potential energy W of a system of four point charges located at the vertices of a square with side a = 10 cm. The charges have the same values q = 10 nC, but two of them are negative. Consider two possible cases of arrangement of the charges.

Given:
$q_1 = q_2 = -q_3 = -q_4 = q = 10$ nC $= 1 \cdot 10^{-8}$ C
$a = 10$ cm $= 10^{-1}$ m
W – ?

I. Physical model



a *b*

Figure 3.1

II. Mathematical model

1. Potential energy of interaction of a system of point charges (q_1, q_2, ... , q_n) is equal to the sum of the interaction energies of separate charge pairs:

$$W_{ij} = q_i \cdot \varphi_j = \sum_{i \neq j} k \frac{q_i q_j}{r_{ij}}, \qquad \textbf{(3.3)}$$

where r_{ij} is the distance between the charges q_i and q_j in the separate pair.

2. According to formula (3.3) in the system of four charges (q_1, q_2, q_3, q_4) the potential energy of interaction is equal to:

$$W = W_{12} + W_{13} + W_{14} + W_{23} + W_{24} + W_{34}.$$

3. All the charges have the same value. The distances between them are, respectively:

$$a_{13} = a_{24} = a\sqrt{2},\; a_{14} = a_{12} = a_{34} = a_{23} = a.$$

4. In the first case (Fig. 3.1, *a*), the potential energy of the charge system is equal to:

$$W = kq^2\left(\frac{1}{a} - \frac{1}{a} + \frac{1}{a} - \frac{1}{a} - \frac{1}{a\sqrt{2}} - \frac{1}{a\sqrt{2}}\right) =$$

$$= -kq^2\left(\frac{2}{a\sqrt{2}}\right).$$

Here we take into account that the potential energy is an algebraic quantity, i.e., its sign depends on the sign of charges (see guidelines on p. 58, no. 6).

5. In the second case (Fig. 3.1, *b*), the potential energy of the system is:

$$W = kq^2\left(-\frac{1}{a} - \frac{1}{a} - \frac{1}{a} - \frac{1}{a} + \frac{1}{a\sqrt{2}} + \frac{1}{a\sqrt{2}}\right) =$$

$$= -kq^2\left(\frac{4}{a} - \frac{2}{a\sqrt{2}}\right) = -2kq^2\,\frac{\left(2\sqrt{2}-1\right)}{a\sqrt{2}}.$$

III. Numerical calculations:

$$\text{a) } W = -\frac{9\cdot 10^9 (10^{-8})^2 \cdot \sqrt{2}}{10^{-1}} = -12.7\cdot 10^{-6}\,(\text{J});$$

$$\text{б) } W = -\frac{9\cdot 10^9 \left(10^{-8}\right)^2 \cdot \left(2\sqrt{2}-1\right)}{10^{-1}\sqrt{2}} = -23.1\cdot 10^{-6}\,(\text{J}).$$

Answer: a) $W = -12.7\,\mu\text{J}$; b) W = – 23.1 μJ.

3.2.2. Electric potential of the field due to charges distributed along a line

Example 3.2

Electric charge with a linear density τ = 10 nC/m is uniformly distributed along a segment of a thin straight conductor. Calculate the potential φ created by that charge at the point on the axis of the conductor, which is distant from the nearest end of the segment by the distance equal to the length of that segment.

Given:

$\tau = 10^{-8}$ C/m

φ_A – ?

I. Physical model

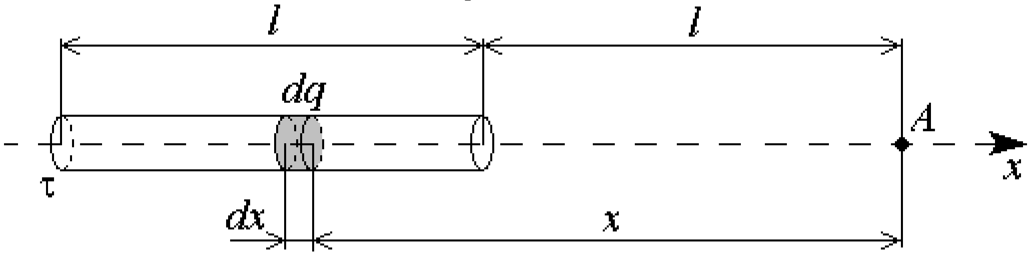


Figure 3.2

II. Mathematical model

Let's use the **DI-method** (see p. 8).

1. Consider an element dx of the segment of the conductor which carries an elementary charge $dq = \tau dx$ (Fig. 3.2), where τ is the linear charge density for the charge element located at the distance x from the point A.

2. The charge dq creates electric field, the potential $d\varphi$ of which at the point A can be calculated by the formula

$$d\varphi_A = k\frac{\tau dx}{x}.$$

3. Electric potential created by all the elements dx of the segment at the point A is determined by the superposition principle, and the integration variable x varies within the limits from l to $2l$, that is:

$$\varphi_A = \int_l^{2l} d\varphi_A = \int_l^{2l} \frac{k\tau dx}{x} = k\tau \ln\frac{2l}{l} = k\tau \ln 2 .$$

III. Numerical calculations:

$$\varphi_A = 9\cdot 10^9 \cdot 10^{-8} \cdot \ln 2 = 62.4\,(\text{V}).$$

Answer: $\varphi_A = 62.4$ V.

3.2.3. Electric potential of the field due to charges distributed over a surface

Example 3.3

Find the potential at the edge of a thin disk of radius R = 20 cm, which carries uniformly distributed electric charge with a surface density σ *= 0.25 μC/m².*

Given:
R = 0.2 m
σ = 0.25·10^{-6} C/m^2
φ – ?

I. Physical model

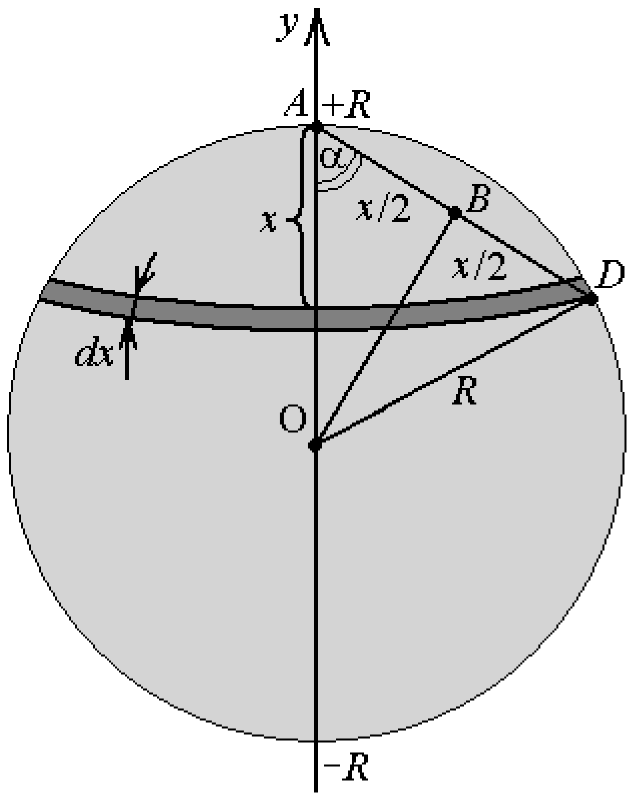


Figure 3.3

II. Mathematical model

Let's use the **DI-method**.

1. Mark the point *A* on the disk (Fig. 3.3) and divide the disk into a system of infinitely thin circular ribbons with centers in that point. Consider one of the ribbons of radius *x* and thickness *dx*.

2. Make some geometric and trigonometric transformations (Fig. 3.3). Divide the side *AD* in half. Then, from the right triangle *OAB* we obtain:

$$\cos\alpha = \frac{x/2}{R} = \frac{x}{2R};$$

hence, we have that α = arccos ($x/2R$), the length of the ribbon is $l = 2x\alpha = 2x\cdot\arccos(x/2R)$, its infinitesimal area is $dS = ldx$.

3. The infinitesimal charge of the ribbon $dq = \sigma dS$ (σ is the surface charge density) produces electric field, and its potential $d\varphi$ at the point *A* can be calculated by the formula

$$d\varphi = \frac{\sigma dS}{4\pi\varepsilon_0 x}. \qquad \textbf{(3.4)}$$

4. Find the potential at the edge of the thin disk at the point *A* by integrating expression (3.14) within the limits from $+R$ to $-R$:

$$\varphi = \int d\varphi = \frac{\sigma}{4\pi\varepsilon_0}\int_{-R}^{+R}\arccos\frac{x}{2R}dx =$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(x\arccos\frac{x}{2R} - \sqrt{(2R)^2 - x^2}\right)\Bigg|_{-R}^{+R} = \frac{\sigma R}{\pi\varepsilon_0}.$$

III. Numerical calculations:

$$\varphi = \frac{0.25\cdot10^{-6}0.2}{3.14\cdot8.85\cdot10^{-12}} = 1.8\cdot10^{3}(\text{V}).$$

Answer: $\varphi = 1.8$ kV.

3.2.4. Electric potential of the field due to charges distributed throughout a volume

Example 3.4

A charge $q = 1\ \mu C$ is uniformly distributed throughout a volume of a sphere of radius $r = 10$ mm. Find the potential in the center of the sphere.

Given:
$q = 10^{-6}$ C
$r = 10^{-2}$ m
φ – ?

I. Physical model

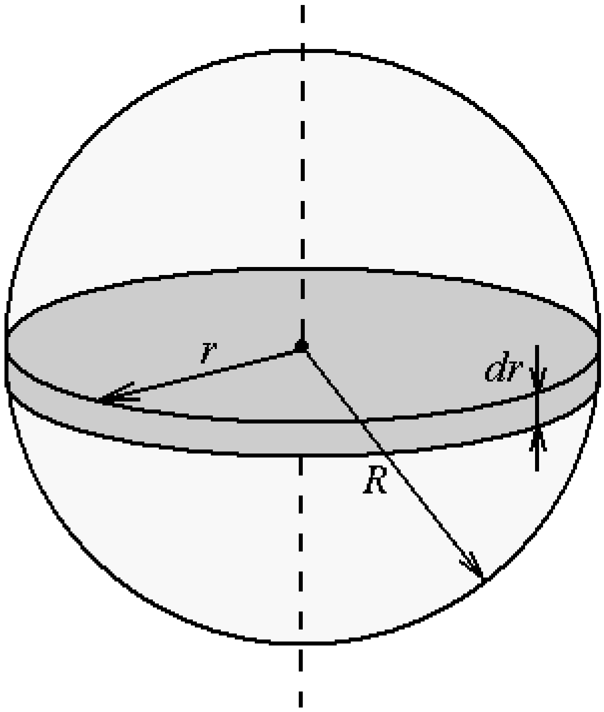


Figure 3.4

II. Mathematical model

Let's use the **DI-method** (see p. 8).

1. Divide the sphere into infinitesimal layers of radii r and thickness dr. Each layer carries a charge $dq = \rho dV$, where ρ is the volume charge density equal to:

$$\rho = \frac{q}{V},$$

then:

$$dq = q\frac{dV}{V} = q\frac{4\pi r^2 dr}{4\pi R^3/3} = q\frac{3r^2 dr}{R^3},$$

where dV is the volume of the layer; V is the volume of the sphere.

2. All the charges dq are at the same distance r from the center of the sphere and produce the following potential in the center:

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}\cdot 3rdr. \qquad \textbf{(3.5)}$$

3. Integrating expression (3.5) within the limits from 0 to R, we obtain the potential in the center of the sphere:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{3q}{R^3}\int_0^R rdr = \frac{1}{4\pi\varepsilon_0}\frac{3q}{R^3}\frac{R^2}{2} = \frac{1}{4\pi\varepsilon_0}\frac{3q}{2R}.$$

III. Numerical calculations:

$$\varphi = 9\cdot 10^9\,\frac{3\cdot 10^{-6}}{2\cdot 10^{-2}} = 1.35\cdot 10^6 (\text{V}).$$

Answer: φ = 1.35 MV.

3.3. Problems for independent work

Potential energy and electric potential of the field due to point charges

3.1. A point charge Q = 10 nC, located at some point in the field, has a potential energy Π = 10 μJ. Find the potential φ at that point of the field.

3.2. A work A = 4 μJ is done by external forces in moving a charge Q = 20 nC from one point in the field to another. Determine the work A_1 by the field forces and the potential difference $\Delta\varphi$ between these two points.

3.3. Electric field is created by a positive point charge Q_1 = 6 nC. A positive charge Q_2 is moved from the point A in the field to the point B (Fig. 3.5). What is the change in the potential energy $\Delta\Pi$ per unit of the moved charge if r_1 = 20 cm and r_2 = 50 cm?

3.4. Electric field is created by a positive point charge Q_1 = 50 nC. Without using the concept of potential, determine the work A by external forces in moving a point charge Q_2 = –2 nC from the point C to the point B (Fig. 3.6), if r_1 = 10 cm, r_2 = 20 cm. Also, determine the change $\Delta\Pi$ in the potential energy of the system of charges.

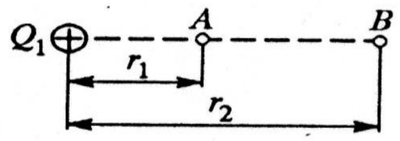


Figure 3.5

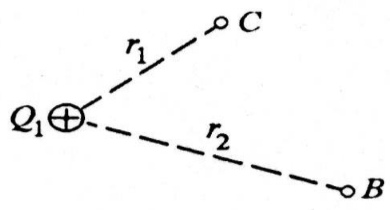


Figure 3.6

3.5. Electric field is produced by a point charge Q = 1 nC. Determine the potential φ of the field at the point distant from the charge by r = 20 cm.

3.6. Determine the potential φ of the electric field at the point which is distant from the charges Q_1 = – 0.2 μC and Q_2 = 0.5 μC by r_1 = 15 cm and r_2 = 25 cm, respectively. Also determine the minimum and the maximum distance between the charges for which this solution is possible.

3.7. Charges Q_1 = 1 μC and Q_2 = –1μC are separated by a distance d = 10 cm. Determine the electric field E and the electric potential φ at the point distant by r = 10 cm from the first charge and lying on the line passing through the first charge perpendicular to the direction from Q_1 to Q_2.

3.8. Determine the potential energy Π of a system of two point charges Q_1 = 100 nC and Q_2 = 10 nC located at a distance d = 10 cm from each other.

3.9. Find the potential energy of a system of three point charges Q_1 = 10 nC, Q_2 = 20 nC and Q_3 = – 30 nC located at the vertices of an equilateral triangle with a side a = 10 cm.

3.10. What is the potential energy Π of a system of four identical point charges Q = 10 nC located at the vertices of a square with a side a = 10 cm?

3.11. Electric field is created by two point charges $+2Q$ and $-Q$, located at a distance d = 12 cm from each other. Determine the geometric location of the points on the plane for which the electric potential is zero (write the equation of the line of zero potential).

Electric potential of the field due to linearly distributed charges

3.12. A charge Q = 1 nC is uniformly distributed along a thin rod of length l = 10 cm. Determine the potential φ of the electric field at the point lying on the axis of the rod at a distance a = 20 cm from its nearest end.

3.13. Electric charge is uniformly distributed along a thin circle of radius R = 10 cm with a linear charge density τ = 10 nC/m. Determine the potential φ at the point lying on the axis of the circle at a distance a = 5 cm from its center.

3.14. Electric charge is uniformly distributed along a length of an infinitely long thin straight thread with a linear charge density τ = 0.01 μC/m. Determine the potential difference Δφ of the two points of the field distant from the thread by r_1 = 2 cm and r_2 = 4 cm.

3.15. Thin rods make a square with a side a. The rods are charged with a linear density τ = 1.33 nC/m. Find the potential φ in the center of the square.

Electric potential of the field due to charges distributed over a surface

3.16. A metal ball of diameter of d = 2 cm is negatively charged to the potential φ = 150 V. How many electrons are there on the surface of the ball?

3.17. One hundred identical droplets of mercury are charged to the potential φ = 20 V each and then merge into one large drop. What is the potential φ_1 of the resultant drop?

3.18. Electric charge is uniformly distributed over an infinite plane with a surface charge density $\sigma = 10$ nC/m^2. Determine the potential difference $\Delta\varphi$ of the two points if one of them is located on the plane and another is distant from the plane by $d = 10$ cm.

3.19. Two infinite parallel planes are at a distance $d = 0.5$ cm from each other. Charges with surface densities $\sigma_1 = 0.2$ μC/m^2 and $\sigma_1 = -0.3$ μC/m^2 are uniformly distributed over a surface of the planes. Determine the potential difference U between the planes.

3.20. A charge $Q = 1$ nC is uniformly distributed over a surface of a thin round plate. The radius R of the plate is 5 cm. Determine the potential φ of the electric field at the two points: a) in the center of the plate; b) at the point lying on the axis perpendicular to the surface of the plate and distant from its center by $a = 5$ cm.

3.21. There are two concentric metal spheres, the radii of which are $R_1 = 3$ cm and $R_2 = 6$ cm. The space between the spheres is filled with paraffin. The charge Q_1 of the inner sphere is –1 nC, the charge Q_2 of the outer sphere is 2 nC. Find the potential φ of the electric field at a distance: a) $r_1 = 1$ cm; b) $r_2 = 5$ cm; c) $r_3 = 9$ cm.

3.22. Determine the potential φ to which you can charge an isolated metal ball of radius $R = 10$ cm, if the air breakdown occurs at the electric field $E = 3$ MV/m.

Electric potential of the field due to charges distributed throughout a volume

3.23. A flat glass plate of thickness $d = 2$ cm is uniformly charged with a volume density $\rho = 10$ nC/m^3. Find the potential difference $\Delta\varphi$ between the point on the surface of the plate and the point inside the plate. Assume that the size of the plate significantly exceeds its thickness.

3.24. An ebonite ball is hollow inside and charged uniformly with a volume density $\rho = 2$ nC/m^3. The inner radius of the ball is $R_1 = 3$ cm, the outer is $R_2 = 6$ cm. Find the potential φ of the electric field at the points: 1) on the outer surface of the ball; 2) on the inner surface of the ball; 3) in the center of the ball.

3.25. A solid paraffin ball of radius $R = 10$ cm is uniformly charged with a volume charge density $\rho = 1$ nC/m^3. Find the potential φ of the electric field in the center of the ball and on its surface. Plot the dependence $\varphi(r)$.

3.26. Potential of the electric field in some region of space depends only on the coordinate x according to the law $\varphi = -ax^3 + b$, where a and b are constant. Find the volume charge distribution $\rho(x)$.

Topic 4. POTENTIAL GRADIENT. WORK IN MOVING AN ELECTRIC CHARGE. MOTION OF CHARGED PARTICLES IN ELECTRIC FIELD

What a student should know

1. Relation between potential gradient and electric field.
2. Work in moving an electric charge.
3. Potential energy of a charged particle.
4. Superposition principle for electric fields.
5. Gauss's law.
6. Newton's second law.
7. Law of conservation of energy and momentum.
8. Differentiation and integration method (**DI-method**).

Literature: [6, § 25.3 – 25.4, 23.7]; [7, §7.4 – 7.6]; [9, § 24.6 – 24.8, 22.6]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: § 25 No 17, 23, 35; § 23 No 46], examples 4.1– 4.6.

Homework: see Table A.3 on p. 139.

4.1. Brief theoretical information

Relation between electric potential and electric field:

$$\vec{E} = -\mathrm{grad}\,\varphi. \tag{4.1}$$

In the case of electric field of spherical symmetry,

$$\vec{E} = -\frac{\partial\varphi}{\partial r}\frac{\vec{r}}{r} = -\frac{\partial\varphi}{\partial r}\vec{e}_r, \tag{4.2}$$

or in the scalar form:

$$E_x = -\frac{\partial\varphi}{\partial x}. \tag{4.3}$$

And in the case of a uniform field when the electric field vector is constant at every point in space, the relation is:

$$E = \frac{\varphi_1 - \varphi_2}{d},$$

where φ_1 and φ_2 are the potentials of the points of two equipotential surfaces; d is the distance between these surfaces along the electric field line.

Potential difference and electric field are related by

$$\varphi_1 - \varphi_2 = \int_1^2 \vec{E}d\vec{r} = \int_1^2 E_n dr. \qquad \textbf{(4.4)}$$

Gauss's law in the differential form:

$$div\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}, \quad div\vec{E} = \frac{\rho}{\varepsilon_0},$$

where ρ is the volume charge density.

Work done by the electric field in carrying a point charge q' from one point of the field with potential φ_1 to another point with potential φ_2 equals

$$A = q'(\varphi_1 - \varphi_2), \text{ or } A = q'\int \vec{E}d\vec{l}, \qquad \textbf{(4.5)}$$

for the uniform field:

$$A = q'El\cos\alpha,$$

where l is the displacement, α is the angle between the vector $\vec{E}$ and the direction of the displacement.

Work done by external forces is equal in absolute value to the work done by the field forces and opposite in sign:

$$A_{ext.f} = -A_{field.f}.$$

Potential energy of a charged particle in the electric field:

$$\Pi = Q \cdot \varphi.$$

Law of conservation of energy and momentum:

$$\begin{aligned} &E = \Pi + T = const, \\ &\sum \vec{p}_i = const, \end{aligned} \qquad \textbf{(4.6)}$$

where T is the kinetic energy of the body; Π is the potential energy.

4.2. Methodical guidelines

1. General method for determining potential difference is based on formula (4.4), which relates electric field and electric potential difference between two points in the field. It is important that integral in (4.4) may be calculated along any line joining these two points.

2. If spatial distribution of electric potential in an inhomogeneous field is known, then formulas (4.1) – (4.3) allow to find the electric field vector. The problem is simplified in the case of symmetric fields, when direction of the vector $\vec{E}$ is given. In such a case, it is enough to take the derivative of the potential with respect to the coordinate in the given direction.

3. To determine the work by forces of electric field in moving an electric charge, it is necessary to make an equation based on the laws of conservation and conversion of energy. In the case of interaction of charged bodies and redistribution of charges that occurs, the equations are made in accordance with the law of conservation of charge. The resulting system of equations is solved relative to the desired value.

4. The work done by electric field in carrying a point charge q is determined by formula (4.5). The electric field potentials φ_1 and φ_2 can be determined using the **DI-method** or the Gauss's law.

5. One should remember that the field created by an electric charge is potential, which means that the work by the field does not depend on the trajectory followed in the field, but depends on the initial and final positions.

6. In many problems considering motion of a particle, it is important to choose a correct reference frame depending on the conditions given in the problem. Particular attention should be paid to the properties of the reference frame where motion of the particle is considered: is the reference frame closed? If the system is closed, the law of conservation of energy and momentum (4.6) is valid.

7. In order to solve problems, it may be necessary to write the equation of motion of a particle – the Newton's second law. One should remember that the electric force is $\vec{F} = q\vec{E}$, where q is an algebraic quantity.

8. During motion of a particle, a force $\vec{F}$ always (except when $\vec{v} \perp \vec{E}$) does work that replaces the kinetic energy of the particle.

9. When solving problems on particle collisions, it is convenient to solve the problem in the system of the center of mass. The total momentum of a system of particles in the C-system is always zero, while the momentums of the both particles in such a system are the same in magnitude and opposite in direction. Kinetic energy of the both particles can be expressed in terms of the reduced mass and the relative velocity of the particles.

4.2.1. Potential gradient and its relation with electric field

Example 4.1

Determine the potential difference between two metal balls of radius r_0 = 0.5 cm each, separated by a distance r = 1 m from each other, if the charge of the first ball is q_1 = 1.5 nC, and the charge of the second ball is q_2 = –1.5 nC.

Given:
$r_0 = 0.005$ m
$r = 1$ m
$q_1 = 1.5 \cdot 10^{-9}$ C
$q_2 = -1.5 \cdot 10^{-9}$ C
$\varphi_A - \varphi_B$ – ?

I. Physical model

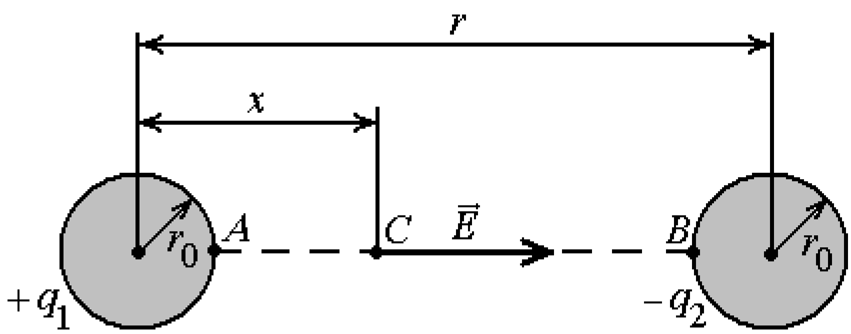


Figure 4.1

1. Since $r \gg r_0$, mutual attraction of the balls can be neglected when considering the forces of mutual repulsion of the charges of the same sign within each ball. That is, we take that distribution of charges over the surface of the balls is uniform.

2. The potential difference is determined by formula (4.4). To use that formula, first we must determine the electric field E at any point

in the space between the balls. To do that, we use the superposition principle for electric fields (see formula 2.3).

3. Let's choose the trajectory of integration along the line *AB* (Fig. 4.1). The electric field vectors due to the both balls at all the points of the line are directed from *A* to *B* (from positive charge to negative).

II. Mathematical model

1. The net electric field at the point *C*, located at the distance *x* from the center of the left ball, is equal to:

$$\vec{E}_c = \vec{E}_1 + \vec{E}_2,$$

or in projections onto the *OX* axis:

$$E = E_1 + E_2,$$

where E_1 and E_2 can be found by formula (2.2):

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

Then we have:

$$E_C = \frac{1}{4\pi\varepsilon_0}\left(\frac{|q_1|}{x^2} + \frac{|q_2|}{(r-x)^2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{x^2} + \frac{1}{(r-x)^2}\right),$$

where $q = |q_1| = |q_2|$ is absolute value of each charge.

2. By formula (4.4) we determine the potential difference:

$$\varphi_A - \varphi_B = \int_{r_0}^{r-r_0} E dx = \frac{q}{4\pi\varepsilon_0}\int_{r_0}^{r-r_0}\left(\frac{1}{x^2} - \frac{1}{(r-x)^2}\right)dx.$$

After integrating and making some simplifications, we get:

$$\varphi_A - \varphi_B = \frac{1}{4\pi\varepsilon_0}\frac{2q(r-2r_0)}{r_0(r-r_0)}.$$

3. Given the ratio $r \gg r_0$, we have:

$$\varphi_A - \varphi_B = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r_0}.$$

III. Numerical calculations:

$$\varphi_A - \varphi_B = 9\cdot 10^9\,\frac{2\cdot 1.5\cdot 10^{-9}}{5\cdot 10^{-3}} = 5.4\cdot 10^3 (\text{V}).$$

Answer: $\varphi_A - \varphi_B = 5.4$ kV.

Example 4.2

Two coaxial rings, each of radius R are made of thin wire. The rings are located at a small distance l from each other (l << R) and have charges q and –q. Find the electric potential and the electric field on the axis of the system as a function of the coordinate x. Draw approximate graphs of the obtained dependences in one figure. Investigate the obtained functions if $|x| >> R$.

Given:
R, a
$a << R$
$q, -q$
φ – ?
E_x – ?

I. Physical model

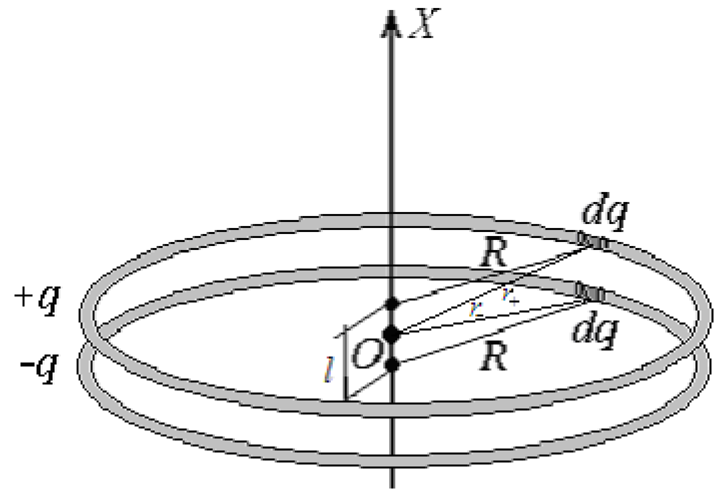


Figure 4.2

II. Mathematical model

According to the superposition principle, the potential created by two charged rings is equal to the sum of the potentials created by each charged ring separately:

$$\varphi = \varphi_1 + \varphi_2. \qquad \textbf{(4.7)}$$

Let's use the **DI-method**.

1. Divide the rings into infinitesimal elements dx with charges dq, which can be taken as point charges. Then the potential produced by the charge dq at the distance r_+ is:

$$d\varphi_1 = \frac{dq}{4\pi\varepsilon_0 r_+}, \qquad \textbf{(4.8)}$$

and the potential produced by the charge dq at the distance r_- is:

$$d\varphi_2 = \frac{dq}{4\pi\varepsilon_0 r_-}, \qquad \textbf{(4.9)}$$

where r_+ and r_- are the distances from the rings' elements to the point O, which do not change if we consider one element of the ring or another.

From Fig. 4.2 we have: $r_+ = \sqrt{R^2 + \left(x - \frac{l}{2}\right)^2}, r_- = \sqrt{R^2 + \left(x + \frac{l}{2}\right)^2}$.

2. Integrate expressions (4.8) and (4.9):

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0 \sqrt{R^2 + \left(x - \frac{l}{2}\right)^2}} \int\limits_{(q)} dq = \frac{q}{4\pi\varepsilon_0 \sqrt{R^2 + \left(x - \frac{l}{2}\right)^2}}, \qquad \textbf{(4.10)}$$

because regardless of the nature of the charge distribution $\int\limits_{(q)} dq = q$.

The potential φ_2 is respectively equal to:

$$\varphi_2 = -\frac{q}{4\pi\varepsilon_0 \sqrt{R^2 + \left(x + \frac{l}{2}\right)^2}}. \qquad \textbf{(4.11)}$$

3. Substituting expressions (4.10) and (4.11) into formula (4.7), we obtain:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{R^2 + \left(x - \frac{l}{2}\right)^2}} - \frac{1}{\sqrt{R^2 + \left(x + \frac{l}{2}\right)^2}}\right) = \frac{q}{4\pi\varepsilon_0} \times$$

$$\times \left(\frac{2xl}{\sqrt{R^2 + \left(x - \frac{l}{2}\right)^2} \cdot \sqrt{R^2 + \left(x + \frac{l}{2}\right)^2} \cdot \left(\sqrt{R^2 + \left(x - \frac{l}{2}\right)^2} + \sqrt{R^2 + \left(x + \frac{l}{2}\right)^2}\right)}\right)$$

In the last expression we ignore the component $l/2$ which is small comparing to R and obtain the final formula:

$$\varphi = \frac{ql}{4\pi\varepsilon_0} \frac{x}{\left(x^2 + R^2\right)^{3/2}}. \qquad \textbf{(4.12)}$$

4. Use formula (4.3) and expression (4.12) to determine the electric field on the axis of the system:

$$E_x = -\frac{\partial\varphi}{\partial x} = -\frac{ql}{4\pi\varepsilon_0} \frac{d}{dx}\left(\frac{x}{\left(x^2 + R^2\right)^{3/2}}\right) =$$

$$= -\frac{ql}{4\pi\varepsilon_0}\left(\frac{\left(x^2 + R^2\right)^{3/2}}{\left(x^2 + R^2\right)^3} - \frac{3}{2}\frac{\left(x^2 + R^2\right)^{1/2} \cdot 2x^2}{\left(x^2 + R^2\right)^3}\right) = -\frac{ql}{4\pi\varepsilon_0}\left(\frac{2x^2 - R^2}{\left(x^2 + R^2\right)^{5/2}}\right),$$

where E_x is the projection of the vector $\vec{E}$ onto the X axis.

5. For $|x| >> R$ we can neglect the component R^2 comparing to x^2, then the potential $\varphi \approx \frac{ql}{4\pi\varepsilon_0 x^2}$ (for large $|x|$ the potential decreases proportional to the square of the distance from the rings) and the electric field $E \approx \frac{ql}{4\pi\varepsilon_0 x^3}$ (for large $|x|$ the field decreases as the cube of the distance from the rings).

6. Draw approximate graphs of $E(x)$ and $\varphi(x)$ (Fig. 4.3).

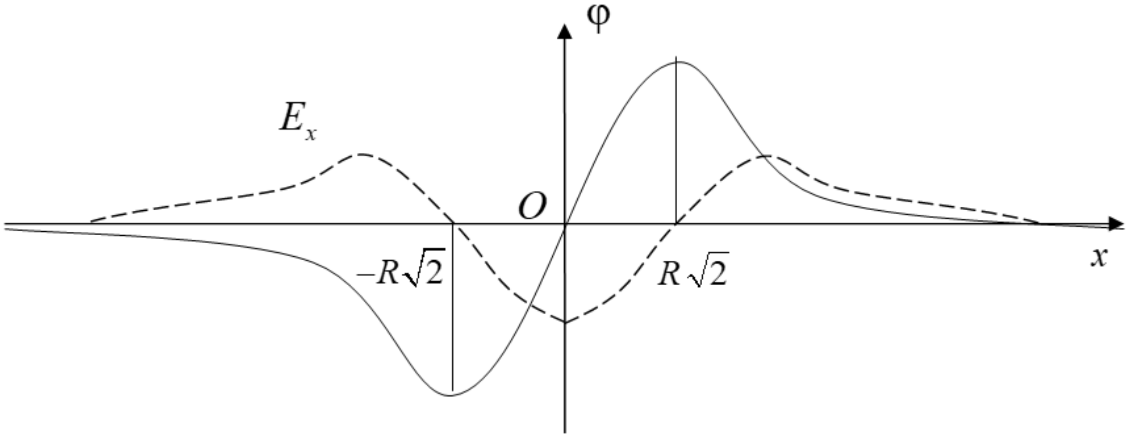


Figure 4.3

Answer: $E_x = \frac{ql}{4\pi\varepsilon_0}\frac{\left(R^2 - 2x^2\right)}{\left(x^2 + R^2\right)^{5/2}}$; $\varphi = \frac{qlx}{4\pi\varepsilon_0\left(x^2 + R^2\right)^{3/2}}$.

The corresponding graphs are shown in Fig. 4.3.

Example 4.3.

Electric potential of the field inside a charged sphere depends only on the distance from its center as φ = $ar^2 + b$*, where a and b are constants. Find the volume charge distribution* ρ(r) *inside the sphere.*

Given:

$\varphi = ar^2 + b$

a, b

$\rho(r)$ – ?

I. Physical model

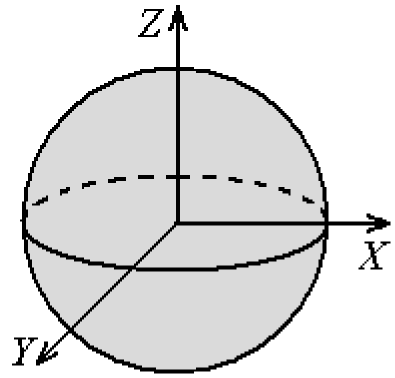


Figure 4.4

II. Mathematical model

1. Let's use the differential form of the Gauss's law:

$$\operatorname{div}\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}.$$

As far as $\vec{E}$ = –gradφ, we can write:

$$div\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right] = \frac{\rho}{\varepsilon_0},$$

therefore,

$$\rho = -\varepsilon_0\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right). \qquad \textbf{(4.13)}$$

2. As given, $\varphi = ar^2 + b = a\left(x^2 + y^2 + z^2\right) + b.$ Substitute this expression into equation (4.13) and take the partial derivative of the second order. Therefore,

$$\rho = -\varepsilon_0 \left(\frac{\partial^2}{\partial x^2}\left(a\left(x^2+y^2+z^2\right)+b\right) + \frac{\partial^2}{\partial y^2}\left(a\left(x^2+y^2+z^2\right)+b\right) + \right.$$

$$\left. + \frac{\partial^2}{\partial z^2}\left(a\left(x^2+y^2+z^2\right)+b\right)\right) = -6a\varepsilon_0.$$

The negative sign means that the sphere is uniformly charged with the negative charge.

Answer: $\rho(r) = -6a\varepsilon_0$.

4.2.2. Work in moving an electric charge in electric field

Example 4.4

Electric charge is uniformly distributed along an infinitely long straight thread (τ = 0.1 μC/m). Determine the work A_{12} by the field forces to move a charge q' = 50 nC from the point 1 to the point 2.

Given:
$\tau = 0.1 \cdot 10^{-6}$ C/m
$q' = 5 \cdot 10^{-8}$ C
$\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m

A_{12} – ?

I. Physical model

1. According to the law of conservation of energy, the work by the field forces to move a charge from one point in the field to another is equal to:

$$A_{12} = \Delta\Pi = q'(\varphi_1 - \varphi_2),$$

where q' is the moved charge, φ_1 and φ_2 are the potentials produced by the charged thread at the points 1 and 2, respectively.

2. The potential difference is determined by formula (4.4). Configuration of the charges allows us to assume that the field has axial symmetry. Thus, using the Gauss's law, first we find the electric field created by the charged thread.

3. Draw the auxiliary Gaussian surfaces with radii r_1 and r_2 (Fig. 4.5). For each surface, the Gauss's law has the form:

$$\oint_{S_{1,2}} \vec{E} d\vec{S} = \frac{\sum Q_i}{\varepsilon_0}. \qquad \textbf{(4.14)}$$

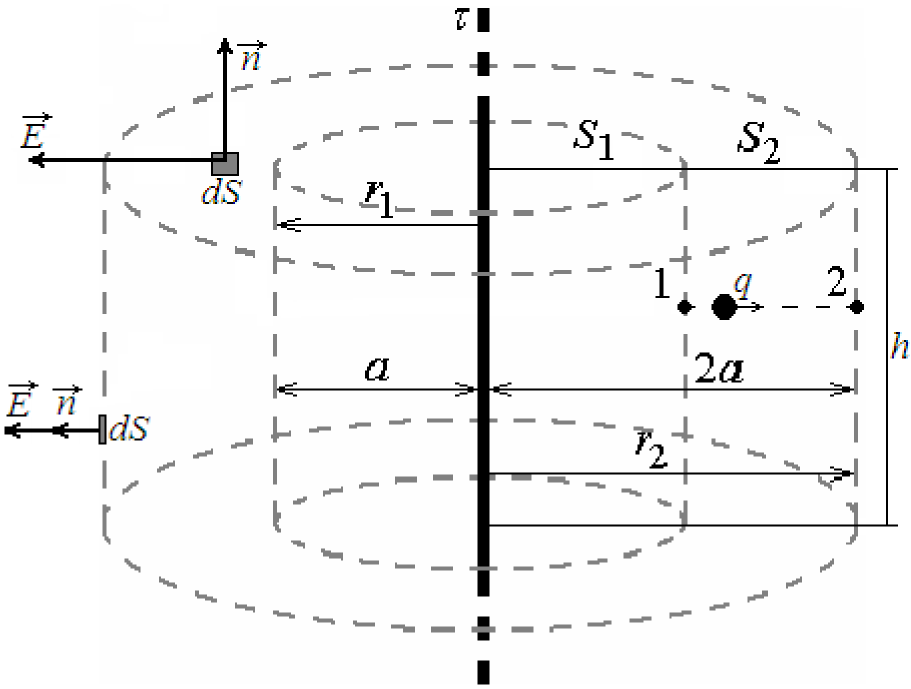


Figure 4.5

II. Mathematical model

1. Similarly to the example (2.4), we find the electric field $\vec{E}$ on the Gaussian surfaces. To do that, we transform the left side of the Gauss's law:

$$\oint \vec{E} d\vec{S} = \int_{S_{1,2lateral}} E_n dS + 2 \int_{S_{1,2ends}} E_n dS ,$$

where E_n is the projection of the electric field vector onto the normal to the Gaussian surfaces.

Consider the *ends of cylinders* of the Gaussian surfaces (Fig. 2.5). For them, the projection of the electric field vector onto the normal is zero ($E_n = 0$), then

$$\int_{S_{1,2ends}} E_n dS = 0 \text{, since } E_n = 0 \text{ (Fig. 4.5), } dS \neq 0.$$

All points of the *lateral surface* are in the same conditions with respect to the charge, which allows us to consider E_n as a constant value (E_n = const), then

$$\int_{S_{1,2lat.}} E_n dS = E_n \int_{S_{1,2lat.}} dS = E_n S_{1,2lat.} = E_n \cdot 2\pi r \cdot h , \qquad \textbf{(4.15)}$$

where *r* and *h* are the radius and the height of the auxiliary surfaces.

Substitute the right side of equation (4.15) into equation (4.14)

$$E_n 2\pi r h = \frac{\sum Q_i}{\varepsilon_0},$$

or

$$E_n \cdot 2\pi r \cdot h = \frac{\tau h}{\varepsilon_0},$$

where τ is the linear charge density, and *h* ~ *l*.

Then the electric field due to the infinitely long uniformly charged thread is:

$$E_n = E_r = \left|\vec{E}\right| = \frac{\tau}{2\pi\varepsilon_0 r}.$$

2. Use formula (4.4) to find the potential difference in the field between the points 1 and 2 (the integration limits are from *a* to 2*a*):

$$\varphi_1 - \varphi_2 = \int_a^{2a} E dr = \frac{\tau}{2\pi\varepsilon_0}\int_a^{2a}\frac{dr}{r} = \frac{\tau}{2\pi\varepsilon_0}\ln\Big|_a^{2a} = \frac{\tau}{2\pi\varepsilon_0}\ln 2.$$

3. The work done by the field to move the charge q' from the point 1 to the point 2 is:

$$A_{12} = q'(\varphi_1 - \varphi_2) = q'\frac{\tau}{2\pi\varepsilon_0}\ln 2.$$

III. Numerical calculations:

$$A_{12} = \frac{5\cdot 10^{-8}\cdot 0.1\cdot 10^{-6}}{2\pi\cdot 1\cdot 8.85\cdot 10^{-12}}\ln 2 = 62.4(\mu\text{J}).$$

Answer: $A_{12} = 62.4$ μJ.

4.2.3. Motion of charged particles in an electric field

Example 4.5

An electron (with initial speed v_0 = 0) flies out of the point 1 on the surface of an infinitely long negatively charged cylinder (τ = 20 nC/m). Determine the kinetic energy T of the electron at the *point 2 located at the distance of 9R from the surface of the cylinder, if R is its radius.*

Given:
R
$L = 9R$
$\tau = 2\cdot10^{-8}$ C/m
$v_0 = 0$

T_2 – ?

I. Physical model

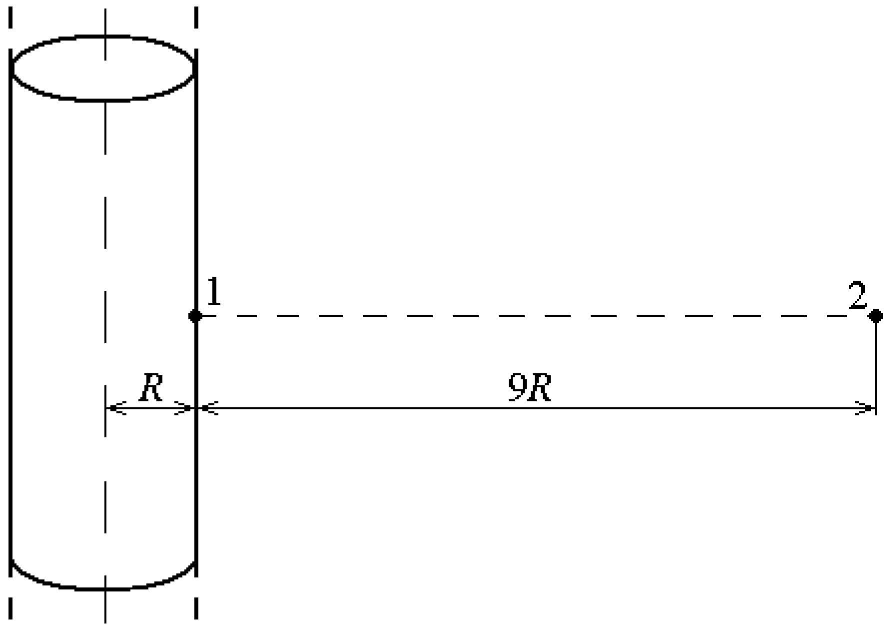


Figure 4.6

II. Mathematical model

1. The infinitely long negatively charged cylinder with a linear charge density τ creates electric field around it. That field is inhomogeneous but symmetrical about the axis of the cylinder, and its magnitude

$$E = E(r) = -\frac{1}{4\pi\varepsilon_0}\frac{2\tau}{\varepsilon_r r}. \qquad \textbf{(4.16)}$$

The field lines are radially (perpendicularly) approaching the axis of the cylinder.

2. Since the electron and the cylinder are charged with the same signs, the electron is repelled from the cylinder by the force:

$$F = -eE,$$

where e is the elementary charge ($e = 1.6\cdot10^{-19}$ C).

3. As given, the initial speed of the electron is zero ($v_0 = 0$), so its kinetic energy T_1 at the point 1 is also zero. Under the action of the force F, it starts to move along the field line in a rectilinear trajectory.

4. The kinetic energy T_2 of the electron at the point 2 is found by formula (4.5), and we use the work - kinetic energy relation:

$$T_2 = A = e\int_L E_l dl = e\int_1^2 E(r)\,dr.$$

Taking into account (4.16) and the fact that the trajectory of the electron is a straight line, we integrate within the limits from R to $R+9R$:

$$T_2 = e\frac{2\tau}{4\pi\varepsilon_0\varepsilon_r}\int_R^{(R+9R)}\frac{dr}{r} = \frac{e\tau}{2\pi\varepsilon_0\varepsilon_r}\ln r\Big|_R^{10R} = \frac{e\tau}{2\pi\varepsilon_0\varepsilon_r}\ln 10,$$

where $\varepsilon_r = 1$ (we assume that the electron is flying in vacuum).

III. Numerical calculations:

$$T_2 = \frac{1.602\cdot 10^{-19}\cdot 20\cdot 10^{-9}}{2\pi\cdot 8.85\cdot 10^{-12}}\cdot \ln 10 \approx 1.327\cdot 10^{-16}\text{ (J)} \approx 828.18\text{ (eV)}.$$

Answer: T_2 = 828.18 eV.

Example 4.6

A proton approaches an α - particle. The speed v_1 of the proton in a laboratory (stationary) reference frame at a sufficiently large distance from the α - particle is equal to 300 km/s, and the speed v_2 of the α - particle is considered to be zero. Determine the minimum distance r_{min}, at which the proton can approach the α - particle. The charge of the α - particle is equal to two elementary positive charges, and its mass m_2 can be considered four times greater than the mass m_1 of the proton.

Given:

$v_1 = 3\cdot 10^5$ m/s

$v_2 = 0,\ q_\alpha = 2e$

$m_2 = 4m_1$

$e = 1.6\cdot 10^{-19}$ C

$m_p = 1.6\cdot 10^{-27}$ kg

$r_{\min}$ – ?

I. Physical model

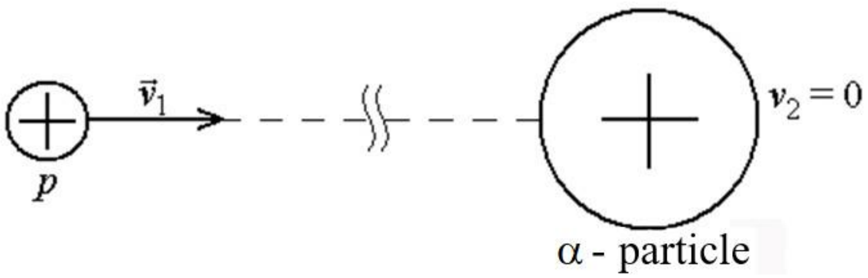


Figure 4.7

II. Mathematical model

1. The proton and the α - particle are positively charged, then the repulsive forces will act between them. The proton will move uniformly slowing down with constant acceleration, while the α - particle will move uniformly speeding up with constant acceleration (because we consider that it is at rest at the initial moment of time).

2. Let's take the origin of the coordinate system in the center of mass of the system of the two particles (C-system*).

To determine the minimum distance that the proton can approach the α - particle, we apply the law of conservation of energy, according to which the total mechanical energy $\tilde{E}$ of an isolated system does not change, that is

$$\tilde{E} = \tilde{T} + \tilde{\Pi},$$

where $\tilde{T}$ is the sum of kinetic energies of the both particles relative to the center of mass; $\tilde{\Pi}$ is the potential energy of the system of charges.

3. As given, at the initial moment of time the proton is quite far from the α - particle, so the potential energy can be neglected ($\tilde{\Pi} = 0$). Therefore, for the initial moment, the total energy is equal to the kinetic energy of the particles, that is $\tilde{E} = \tilde{T}$.

At the final moment, when the particles get as close as possible, the velocity and the kinetic energy become zero, and the total energy equals to the potential energy: $\tilde{E}' = \tilde{\Pi}'$. Therefore, according to the law of conservation of energy:

$$\tilde{T} = \tilde{\Pi}'.$$

The kinetic energy $\tilde{T}$ in the C-system has the form:

$$\tilde{T} = \frac{\mu v_{12}^2}{2},$$

where $\mu = m_1 m_2 / \left(m_1 + m_2\right)$ is the reduced mass; $\vec{v}_{12} = \left|\vec{v}_1 - \vec{v}_2\right|$ is the relative velocity (velocity of one particle relative to another).

---

*This problem considers collision of particles, so, for convenience, we will further denote all the values related to the system after the "collision" (meaning the moment of time that corresponds to the maximum interaction between the particles) with a dash; and the values in the C-system – with an above tilde (~) sign.

Substituting the expression for the potential energy in vacuum, we obtain the relation to find the desired distance:

$$\frac{\mu\nu_{12}^2}{2}=\frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{\min}} \quad\Rightarrow\quad r_{\min}=\frac{1}{4\pi\varepsilon_0}\frac{2q_1q_2}{\mu\nu_{12}^2}=\frac{1}{4\pi\varepsilon_0}\frac{2\left(m_1+m_2\right)q_1q_2}{m_1m_2\left|\nu_1-\nu_2\right|^2},$$

where q_1, q_2 are the charges of the proton and the α-particle, respectively.

4. Given that the charge of the α-particle is equal to two elementary positive charges, and its mass m_2 can be considered four times greater than the mass m_1 of the proton, we obtain:

$$r_{\min}=\frac{5}{4\pi\varepsilon_0}\frac{e^2}{m_p\nu_1^2}.$$

III. Numerical calculations:

$$r_{\min}=\frac{5}{4\pi 8.85\cdot10^{-12}}\frac{1.602^2\cdot10^{-38}}{1.672\cdot10^{-27}\cdot9\cdot10^{10}}\approx7.67\cdot10^{-12}(\text{m}).$$

Answer: $r_{\min}$ = 7.67 pm.

4.3. Problems for independent work

Potential gradient and its relation with electric field vector

4.1. An infinite plane is uniformly charged with a surface density σ = 4 nC/m^2. Determine the value and direction of the potential gradient of the electric field produced by this plane.

4.2. The magnitude of a uniform electric field E at some point in space is equal to 600 V/m. Determine the potential difference U between that point and the other point lying on the line that makes an angle α = 60^0 with the direction of the electric field vector. The distance Δr between the points is 2 mm.

4.3. The magnitude of a uniform electric field E at some point in space is equal to 120 V/m. Determine the potential difference U between that point and the other point that lies on the same electric field line and is distant from the first point by $\Delta r = 1$ mm.

4.4. Electric field is created by a positive point charge. Electric potential φ of the field at the point distant from the charge by r = 12 cm is equal to 24 V. Determine the value and direction of the potential gradient at that point.

4.5. An infinite straight thin thread is uniformly charged with a linear charge density τ = 1 nC/m. Find the potential gradient at the point at a distance r = 10 cm from the thread and indicate its direction.

4.6. A solid dielectric sphere ($\varepsilon_r = 3$) of radius R = 10 cm is charged with a volume charge density ρ = 50 nC/m^3. The magnitude of electric field inside and on the surface of such a sphere is determined by the formula $E = \dfrac{\rho}{3\varepsilon_0\varepsilon_r} r$, where r is the distance from the center of the sphere to the point at which the field is measured. Determine the potential difference between the center of the sphere and the points lying on its surface.

4.7. A charge q = 1 μC is uniformly distributed over a round thin plate of radius r = 0.1 m. Taking the axis of the plate for the axis Y, find: a) φ, E for the points lying on the axis as a function of y; investigate the obtained expressions for $y \ll r$; b) φ, E at the point y_1 = 100 mm.

4.8. A charge with a density ρ = ρ (r) is distributed throughout a region V. Write an expression for the electric potential φ and the electric field E at the point given by the radius vector $\vec{r}\,'$.

4.9. Determine the electric field, the potential of which has the form of φ = $\vec{a} \cdot \vec{r}$, where $\vec{a}$ is the constant vector, $\vec{r}$ is the radius vector of a point in the field.

4.10. Determine the electric field, the potential of which depends on the coordinates x, y by the law: a) $\varphi = a(x^2 - y^2)$; b) $\varphi = axy$, where a is constant. Draw an approximate pattern of these fields using the vector $\vec{E}$ (in the xy plane).

4.11. Find the potentials of the following electrostatic fields:
a) $\vec{E} = a(y\vec{i} + x\vec{j})$; b) $\vec{E} = 2axy\vec{i} + a(x^2 - y^2)\vec{j}$; c) $\vec{E} = ay\vec{i} + (ax + bz)\vec{j} + by\vec{k}$, where a and b are constants; $\vec{i}$, $\vec{j}$, $\vec{k}$ are the unit-vectors of axes x, y, z.

Work on moving charges in an electric field

4.12. Point charges $Q_1 = 1$ μC and $Q_2 = 0.1$ μC are located at a distance $r_1 = 10$ cm from each other. What is the work A done by the field forces, if the second charge, repelling from the first, moves away from it by a distance: a) $r_1 = 10$ м; b) $r_2 = \infty$?

4.13. A thin rod is bent in a semicircle. The rod is charged with a linear density τ = 133 nC/m. What is the work A by external forces required to carry a charge Q = 6.7 nC from the center of the semicircle to infinity?

4.14. A thin rod is bent in a circle of radius R =10 cm. It is charged with a linear density τ = 300 nC/m. What is the work A by external forces required to carry a charge Q = 5 nC from the center of the circle to the point located on its axis at the distance r = 20 cm from its center?

4.15. Two infinite planes, uniformly charged with a surface charge density σ = 0.2 nC/m^2, intersect at an angle α = 60^0. Draw a pattern of the equipotential surface and determine the work by the field forces in carrying a charge Q = 10 nC from the point A to the point B (Fig. 4.8).

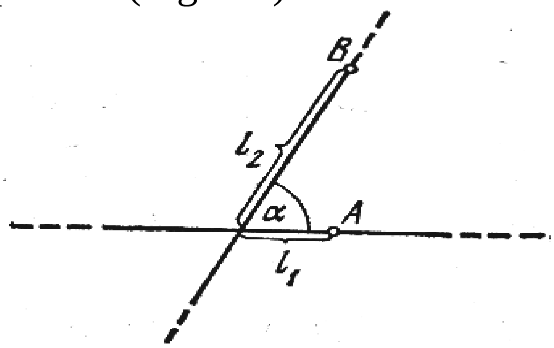


Figure 4.8

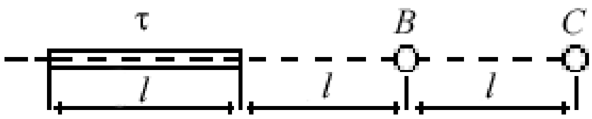


Figure 4.9

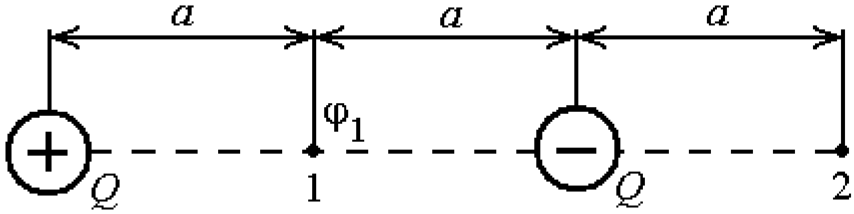


Figure 4.10

4.16. A charge is uniformly distributed along a segment of a straight conductor with a linear charge density $\tau = 1$ μC/m. Determine the work A by the field forces to move a charge q=1 nC from the point B to the point C (Fig. 4.9).

4.17. Electric field is created by two identical positive point charges Q. Find the work $A_{1,2}$ by the field forces to move a charge $Q_1 = 10$ nC from the point 1 with the potential $\varphi_1 = 300$ V to the point 2 (Fig. 4.10).

4.18. Determine the work $A_{1,2}$ in carrying a charge $Q_1 = 50$ nC from the point 1 to the point 2 (Fig. 4.11) in the field created by two charges of equal absolute value $|Q| = 1$ μC. The distance $a = 0.1$ m.

4.19. Electric field is created by a charge uniformly distributed along a ring ($\tau = 1$ μC/m). Determine the work $A_{1,2}$ by the field forces to move a charge $Q = 10$ nC from the point 1 (in the center of the ring) to the point 2 on the perpendicular to the plane of the ring (Fig. 4.12).

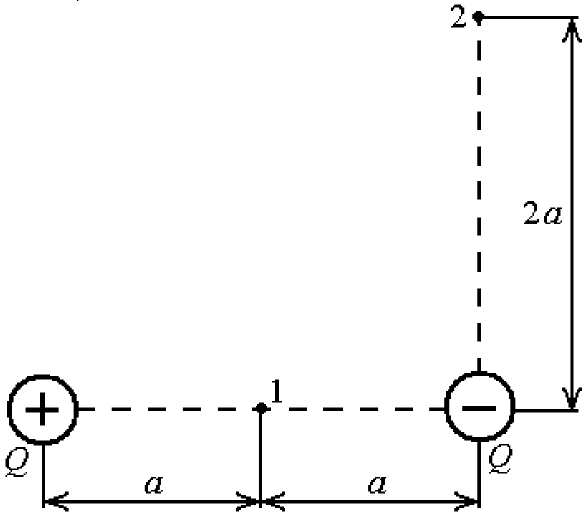


Figure 4.11

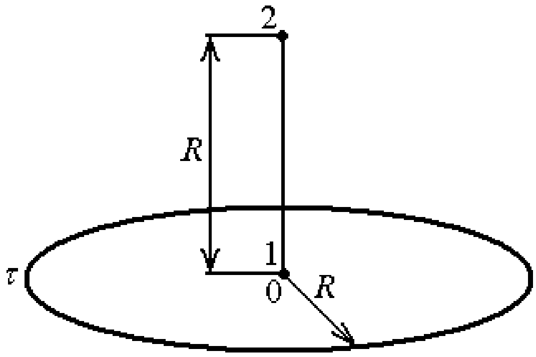


Figure 4.12

Motion of charged particles in electric field

4.20. A proton with initial speed $v_0 = 100$ km/s enters a uniform electric field ($E = 300$ V/cm) so that its velocity vector coincided with the direction of the field lines. What path l must the proton travel in the direction of the field lines to double its speed?

4.21. An infinite plane is negatively charged with a surface charge density σ = 35.4 nC/m^2. An electron flies in the direction of the electric field line produced by the plane. Determine the minimum distance l_{min} the electron can approach the plane, if it has the kinetic energy T = 80 eV at the distance l_0 = 5 cm.

4.22. An electron flies horizontally with the speed of v_0 = 1.6 Mm/s and enters a uniform electric directed vertically upwards and having the magnitude of E = 90 V/cm. What will the magnitude and direction of the electron's velocity be in time 1 ns?

4.23. A proton moves along a field line of a uniform electric field. At the point of the field with the potential φ_1 the proton has the speed of v_1 = 0.1 Mm/s. Determine the potential φ_2 of the point at which the speed of the proton increases 2 times. The ratio of the charge of the proton to its mass is e/m = 96 MC/kg.

4.24. An electron with a speed of v_0 = 1 Mm/s flies into a uniform electric field of magnitude E = 1 kV/m. Determine the distance l traveled by the electron to the point where its speed v_1 is half the initial.

4.25. What is the minimum speed v_{min} that a proton must have in order to reach the surface of a metal ball charged to the potential φ = 400 V (Fig. 4.13)?

4.26. An electron moves along the field line of a uniform electric field. At some point in the field with a potential φ_1 = 100 V, the electron has a speed of v_1 = 6 Mm/s. Determine the potential φ_2 of the field point at which the speed v_2 of the electron will be equal to 0.5v_1.

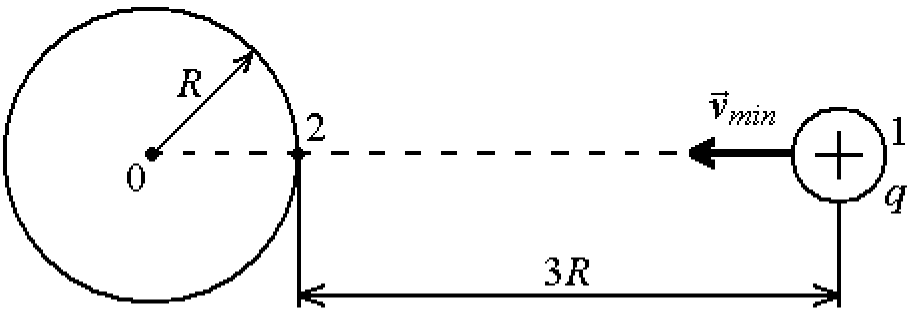


Figure 4.13

4.27. An electron with an initial speed v_0 = 3 Mm/s flies into a uniform electric field of magnitude E = 150 V/m. The initial velocity vector is perpendicular to the electric field lines. Find: 1) the force F acting on the electron; 2) the acceleration a acquired by the electron; 3) the speed v of the electron in time $t = 0.1$ μs.

4.28. An electron enters the space between the plates of a parallel plate capacitor with the velocity of magnitude $v = 10$ Mm/s directed parallel to the plates. How close will the electron approach the positively charged plate during its motion inside the capacitor, if the distance d between the plates is 16 mm, the potential difference across the plates is $U = 30$ V and the length l of the plates is 6 mm? The field is considered uniform.

4.29. An electron enters a parallel plate capacitor with the velocity of magnitude $v_0 = 10$ Mm/s directed parallel to the plates. At the moment of time when it exits the capacitor, the direction of the electron's velocity makes an angle $\alpha = 35^0$ with the initial velocity direction. Determine the potential difference across the plates (consider the field to be uniform) if the length l of the plates is 10 cm and the distance d between them is 2 cm.

4.30. An electron enters a parallel plate capacitor being at the same distance from each plate and having a velocity of magnitude $v_0 = 10$ Mm/s directed parallel to the plates. The distance d between the plates is 2 cm, the length l of each plate is 10 cm. What is the smallest potential difference U that must be applied across the plates so that the electron does not fly out of the capacitor?

4.31. A positively charged particle, the charge of which is equal to the elementary charge e, is accelerated through the potential difference $U = 60$ kV and approaches the nucleus of a lithium atom, the charge of which is equal to three elementary charges. At what smallest distance $r_{\min}$ can the particle approach the nucleus? The initial distance between the particle and the nucleus can be considered as infinitely large, and the mass of the particle is small compared to the mass of the nucleus.

4.32. Two electrons separated by a large distance start to approach each other with a relative initial speed $v = 10$ Mm/s. Determine the minimum distance $r_{\min}$ at which they can approach each other.

4.33. Two charged particles of the same sign with charges Q_1 and Q_2 approach each other from a large distance. The velocity vectors $\vec{v}_1$ and $\vec{v}_2$ of the particle lie on the same line. Determine the minimum distance r_{min} at which these particles can approach each other if their masses are m_1 and m_2, respectively. Consider two cases: 1) $m_1 = m_2$ and 2) $m_2 >> m_1$.

4.34. The mass ratio of two charged particles is equal to $k = m_1 / m_2$. The particles are at a distance r_0 from each other. What kinetic energy T_1 will the particle of mass m_1 have if it moves away from the other particle under the repulsive force by a distance $r >> r_0$. Consider three cases: 1) $k = 1$; 2) $k = 0$; 3) $k \to \infty$. Take the charges of the particles equal to Q_1 and Q_2. Initial velocities of the particles can be neglected.

Topic 5. ELECTRIC DIPOLE

What a student should know

1. Dipole, point dipole.
2. Electric dipole moment
3. Electric field and electric potential due to a dipole.
4. Energy of the dipole field.
5. Work by electric field forces.
6. Concept of internal forces of electric field.
7. Superposition principle for electric fields.
8. Relation between electric field and electric potential.
9. Electric field due to a system of charges at large distances.
10. Fundamental law of dynamics for rotational motion.

Literature: [6, § 23.4, 26.6]; [7, § 5.7]; [9, § 22.3, 22.7]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: § 26 No 49, 51], examples 5.1– 5.5.

Homework: see Table A.4 on p. 139.

5.1. Brief theoretical information

Electric dipole is a system consisting of two point charges, identical in magnitude and opposite in sign, which are separated by a fixed distance from each other.

Electric dipole moment:

$$\vec{p} = q\vec{l}, \qquad \textbf{(5.1)}$$

where q is the value of the dipole charges; $\vec{l}$ is the vector (arm), directed from the negative charge to the positive.

If the arm l of the dipole is much smaller than the distance r from the center of the dipole to the point of observation ($l << r$), the dipole is considered as a **point dipole**.

Electric field due to a point dipole with electric moment *p*, measured at the point with coordinates (*r*, α):

$$E = \frac{p}{4\pi\varepsilon_0\varepsilon r^3}\sqrt{1+3\cos^2\alpha},$$

where *r* is the distance from the observation point to the center of the dipole, α is the angle between the vector of electric dipole moment and the direction to the observation point (Fig. 5.1).

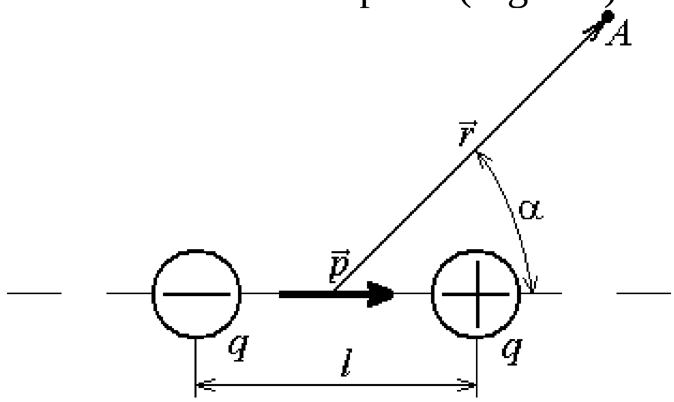


Figure 5.1

If α = 0, then

$$E = \frac{p}{2\pi\varepsilon_0\varepsilon r^3};$$

if α = π/2, then

$$E = \frac{p}{4\pi\varepsilon_0\varepsilon r^3}.$$

Electric potential due to a point dipole with electric dipole moment *p*, measured at the point with coordinates (*r*, α):

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\alpha}{r^2}.$$

If α = 0, then

$$\varphi = \frac{p}{4\pi\varepsilon_0\varepsilon r^2};$$

if α = π/2, then $\varphi = 0$.

Potential energy of a system of two charges (a dipole) in an external field is equal to the sum of the energies of each of the charges separately.

$$W_p = q\varphi_+ - q\varphi_- = q(\varphi_+ - \varphi_-), \qquad \textbf{(5.2)}$$

where φ_+ and φ_- are the potentials of the external field at the points where the charges $+q$ and $-q$ are located.

Electric dipole moment of a system of N charges is determined by the formula

$$\vec{p} = \sum_{i=1}^{N} q_i \vec{r}_i, \qquad \textbf{(5.3)}$$

where $\vec{r}_i$ determines the position of the i-th charge in the system. The signs of the charges must be taken into account in this formula.

Work by external field forces on an electric dipole:

$$A = -\Delta W = W_1 - W_2, \qquad \textbf{(5.4)}$$

where ΔW is the change in the potential energy of the dipole as the consequence of the action of forces, W_1 and W_2 are the initial and final value of the energy in the process of action of forces.

Torque on a dipole with an electric dipole moment $\vec{p}$ in a uniform electric field $\vec{E}$:

$$\vec{M} = \left[\vec{p} \times \vec{E}\right], \; M = pE\sin\alpha,$$

where α is the angle between the directions of the vectors $\vec{p}$ and $\vec{E}$.

In the case of an inhomogeneous electric field which is symmetrical about the OX axis, the force on the dipole is

$$F_x = p\frac{\partial E}{\partial x}\cos\alpha,$$

where $\partial E/\partial x$ is the degree of the field inhomogeneity in the OX direction.

5.2. Methodical guidelines

1. When solving problems on this topic, it is necessary to pay attention to the dipole under consideration: small size of the dipole allows us to make assumptions that it is a point dipole. That simplifies solution of the problem and obtained results.

2. If the observation point A lies on the prolongation of the dipole axis, or is at any arbitrary position in space, then the electric field $\vec{E}$ and the electric potential φ are determined by the superposition principle (see formulas (2.3), (3.1)).

3. Since the electric field due to a dipole has axial symmetry, the field pattern in any plane passing through the dipole axis does not change and the electric field vector $\vec{E}$ lies in that plane.

4. To calculate the electric field due to a dipole, there is no need to know q and l separately; it is enough to know their product, that is the electric dipole moment, according to formula (5.1).

5. The electric dipole moment of a multipole, i.e., of a system of N charges, is determined by formula (5.3), where $\vec{r}_i$ determines the position of the i-th charge in the system: it shows the direction and distance from some origin within the charge system to the i-th charge. The sign of the charge must be taken into account in that formula.

5.2.1. Electric field and electric potential due to a dipole. Electric dipole moment

Example 5.1

Determine the potential φ and the magnitude E of the electric field due to a dipole as functions of r and θ (r is the distance from the center of the dipole, θ is the angle between the axis of the dipole and the direction from the center of the dipole to a given point). The dipole moment is equal to p.

Given:

p

$\varphi\ (r,\theta),\ E\ (r,\theta)$ – ?

I. Physical model

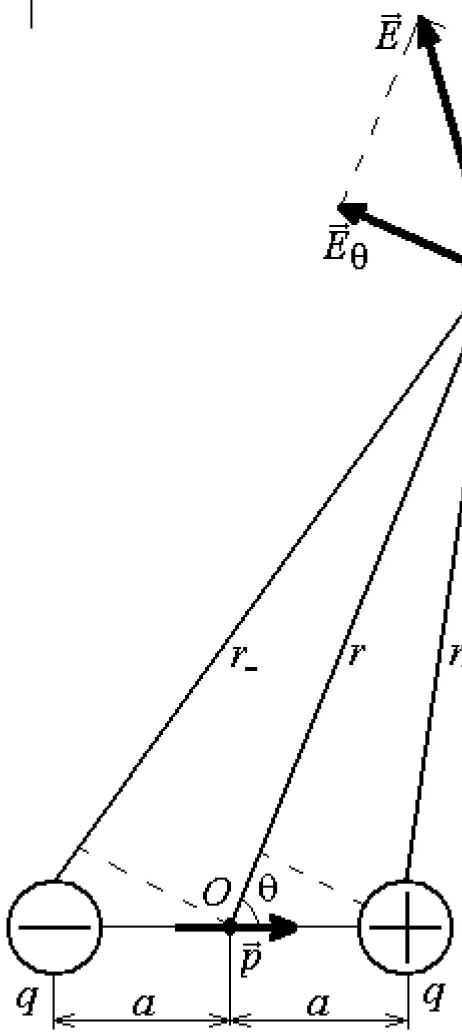


Figure 5.2

II. Mathematical model

1. According to the superposition principle, the total potential is equal to the sum of potentials created by the charges $+q$ at the distance r_+ and –q at the distance r_- (Fig. 5.2):

$$\varphi(r_+, r_-) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q(r_- - r_+)}{r_+ r_-}. \qquad \textbf{(5.5)}$$

2. According to Fig. 5.2, the distance between the charges is $l = 2a$, where a is the distance from the center of the dipole to the charges.

3. Assuming that the dipole is a point dipole ($r >> l$), we obtain:

$$r_+ r_- = r^2,\ r_+ = r - a\cos\theta,\ r_- = r + a\cos\theta, \qquad \textbf{(5.6)}$$

that is

$$r_- - r_+ = 2a\cos\theta = l\cos\theta. \qquad \textbf{(5.7)}$$

Substituting equations (5.6) and (5.7) into expression (5.5), we obtain:

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{ql\cos\theta}{r^2},$$

or according to formula (5.1):

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}. \qquad \textbf{(5.8)}$$

4. To determine the electric field due to the dipole, we use formula (4.3), which gives relation between the electric potential and electric field. Let’s calculate projections of the vector $\vec{E}$ onto two mutually perpendicular directions* E_r and E_θ (Fig. 5.2). These projections determine the vector $\vec{E}$ with respect to the change in the variables r and θ, respectively, i.e., E_r is projection onto the radial direction which characterizes change in the value of r, and E_θ is projection of the vector $\vec{E}$ onto the tangential direction which characterizes change in the angle θ. The tangential direction is determined by the value $rd\theta$ of displacement of the end of the radius-vector r when it turns through the angle $d\theta$. Thus, as a result of differentiation we have:

$$E_\theta(r,\theta) = -\frac{\partial\varphi}{r\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3},$$

$$E_r(r,\theta) = -\frac{\partial\varphi}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}.$$

5. With help of the obtained components of the vector $\vec{E}$, it is easy to find the magnitude of the electric field produced by the dipole:

---

*It should be noted that these are the directions of displacement of the end of the vector $\vec{r}$ which defines the observation point.

$$E=\sqrt{E_r^2+E_\theta^2}=\sqrt{\left(\frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}\right)^2+\left(\frac{1}{4\pi\varepsilon_0}\frac{2p\sin\theta}{r^3}\right)^2}=$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{4\cos^2\theta+\sin^2\theta}=\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta+1}.$$

Answer: $\varphi(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$; $E(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{3\cos^2\theta+1}.$

Example 5.2

A point electric dipole with moment p is located in an external electric field of magnitude E_0, so that the vectors $\vec{p}$ and $\vec{E}_0$ are collinear. It happens that one of the equipotential surfaces enclosing the dipole is a sphere. Find its radius.

Given:
p, E_0
R – ?

I. Physical model

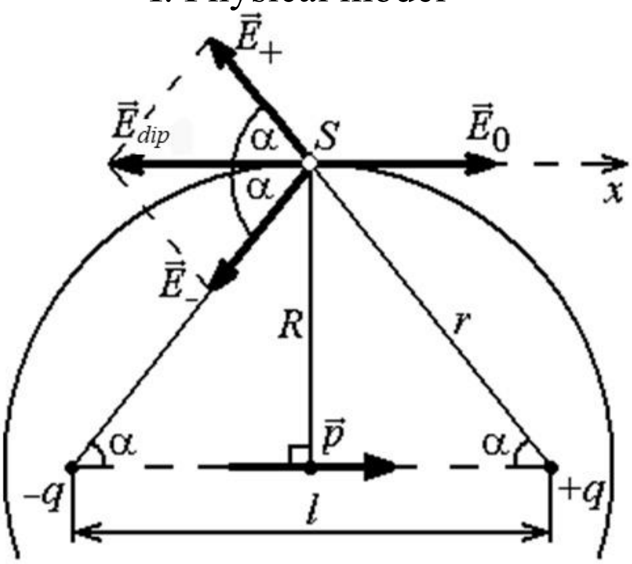


Figure 5.3

II. Mathematical model

1. As given, the equipotential surface* is a sphere. According to the superposition principle, the electric field at any point on that

*The **equipotential surface** is the locus of points in space having the same electric potential, i.e., at any point on the equipotential surface the electric potential is a constant value.

sphere is the vector sum of the external electric field and the electric field created by the dipole:

$$\vec{E} = \vec{E}_0 + \vec{E}_{dip},$$

but this vector sum at any point on the sphere must have zero projection onto the tangent to this sphere.

2. According to the requirements of symmetry of the dipole field with respect to the plane passing through the dipole's center perpendicular to the vector of the electric dipole moment (the perpendicular bisector plane), the center of the dipole must coincide with the center of the equipotential sphere.

3. Consider the point S on the sphere, which belongs also to the perpendicular bisector plane of the electric dipole moment vector (Fig. 5.3). At that point, the electric field $\vec{E}_{dip}$ due to the dipole, as shown in Fig. 5.3, is directed opposite to the vector $\vec{E}_0$ of the external electric field. In addition, the vector $\vec{E}_{dip}$ of the electric field due to the dipole is the geometric sum of the vectors $\vec{E}_+$ and $\vec{E}_-$ of the fields created by the positive and the negative charge of the dipole separately.

4. According to Fig. 5.3, let's project all the vectors onto the axis tangent to the sphere at the point S:

$$E_0 = E_{dip} = \left(E_+ + E_-\right)\cos\alpha. \qquad \textbf{(5.9)}$$

The distances from the charges of the dipole to the point S are the same:

$$r = \sqrt{R^2 + l^2/4},$$

where R is the radius of the sphere; l is the distance between the charges of the dipole.

Magnitudes $|E_+|$ and $|E_-|$ are equal, therefore, substituting the known formula (2.3) into equation (5.9) we obtain:

$$E_0 = \frac{1}{4\pi\varepsilon_0}\frac{2q}{R^2 + \frac{l^2}{4}}\cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{2q}{R^2 + \frac{l^2}{4}}\frac{l}{2\sqrt{R^2 + \frac{l^2}{4}}}, \qquad \textbf{(5.10)}$$

because according to Fig. 5.3 $\cos\alpha = \dfrac{l}{2\sqrt{R^2 + \dfrac{l^2}{4}}}$.

5. As given, we have the point dipole, i.e., $R >> l$, therefore, in the denominator of the obtained expression (5.10) the term $l^2/4$ can be neglected in comparison with R^2, while in the numerator the product ql equals p.

Finally: $E_0 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{R^3} \Rightarrow R = \sqrt[3]{\dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{E_0}}$.

Answer: $R = \sqrt[3]{\dfrac{1}{4\pi\varepsilon_0}\dfrac{p}{E_0}}$.

Example 5.3

A point charge $q = -2\cdot 10^{-10}$ C is located on the prolongation of the axis of the dipole with the electric dipole moment $p_e = 1.5\cdot 10^{-10}$ C. The distance between the point charge and the center of the dipole is $r = 10$ cm (the charge is located closer to the positive pole of the dipole). What work is required to move that charge to the symmetrically located point to the other side of the dipole? The dipole arm $l << r$.

Given:
$q = -2\cdot 10^{-10}$ C
$r = 0.1$ m
$p_e = 1.5\cdot 10^{-10}$ C·m

$A_{ext.f}$ – ?

I. Physical model

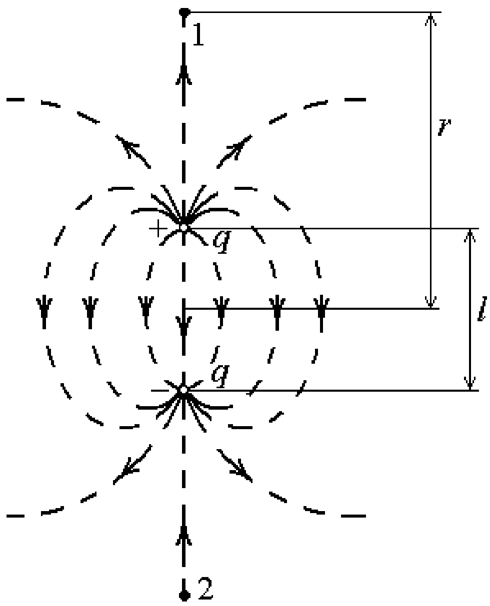


Figure 5.4

II. Mathematical model

1. The work by external forces is equal in magnitude and opposite in sign to the work by the field forces:

$$A_{ext.f} = -A_{field.f} = -q\left(\varphi_1 - \varphi_2\right), \qquad \textbf{(5.11)}$$

where φ_1 and φ_2 are the potentials of the initial and final points.

2. The field is created by the two point charges $+q$ and $-q$ of the dipole, i.e. the potentials of the points 1 and 2 must be found according to the superposition principle:

$$\varphi_1 = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r-\frac{l}{2}} - \frac{1}{r+\frac{l}{2}}\right),\ \varphi_2 = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r+\frac{l}{2}} - \frac{1}{r-\frac{l}{2}}\right), \qquad \textbf{(5.12)}$$

where, according to Fig. 5.4: $(r - l/2)$ is the distance from the point 1 to the positive pole; $(r + l/2)$ is the distance from the point 1 to the negative pole.

3. Reduce to a common denominator each of the expressions (5.21):

$$\varphi_1 = \frac{q}{4\pi\varepsilon_0}\frac{l}{r^2\left(1-\frac{l^2}{4r^2}\right)},\ \varphi_2 = -\frac{q}{4\pi\varepsilon_0}\frac{l}{r^2\left(1-\frac{l^2}{4r^2}\right)}. \qquad \textbf{(5.13)}$$

As we know, $ql = p$. If $r >> l$ then the term $l^2/(4r^2)$ can be neglected, and (5.13) takes the form:

$$\varphi_1 = \frac{p_e}{4\pi\varepsilon_0 r^2};\ \varphi_2 = -\frac{p_e}{4\pi\varepsilon_0 r^2}. \qquad \textbf{(5.14)}$$

4. Substituting equation (5.14) into equation (5.11), we obtain:

$$A_{з.с} = -q\left(\frac{p_e}{4\pi\varepsilon_0 r^2} + \frac{p_e}{4\pi\varepsilon_0 r^2}\right) = -\frac{2qp_e}{4\pi\varepsilon_0 r^2}.$$

III. Numerical calculations:

$$A_{ext.f} = -\frac{2\left(-2\cdot10^{-10}\right)\cdot1.5\cdot10^{-10}}{9\cdot10^9\cdot\left(0.1\right)^2} = 5.4\cdot10^{-8}\,(\text{J}).$$

Answer: $A_{ext.f.}$ = 54 nJ.

5.2.2. Dipole in an external electric field

Example 5.4

A uniform electric field of magnitude E = 300 kV/m is applied perpendicular to the arm of a dipole with the electric moment p = 12 pC·m. Under the action of the field forces, the dipole starts to rotate about the axis passing through its center. Find the angular speed ω of the dipole at the moment when it passes the equilibrium position. The moment of inertia J of the dipole about its perpendicular bisector axis is 2·10 $^{-9}$ kg·m^2.

Given:
$p = 1.2 \cdot 10^{-11}$ C·m
$E = 3 \cdot 10^5$ V/m
$J = 2 \cdot 10^{-9}$ kg·m^2
ω – ?

I. Physical model

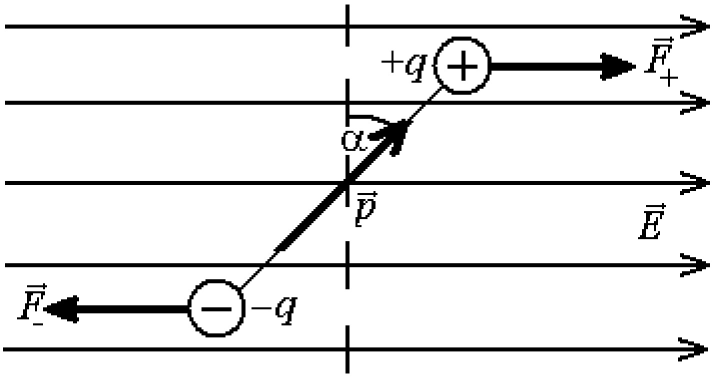


Figure 5.5

II. Mathematical model

1. Fundamental law of dynamics for rotational motion is:

$$\frac{dL}{dt} = \frac{d(J\omega)}{dt} = M_{ext},$$

for the case under consideration:

$$J\frac{d\omega}{dt} = M_{ext},$$

then

$$\omega(t) = \frac{1}{J}\int_0^t M_{ext}dt, \qquad \textbf{(5.15)}$$

where M_{ext} is the net torque due to external forces acting on the dipole.

2. Let the position of the dipole in the external field be as shown in Fig. 5.5: the forces of the same magnitude, but opposite direction are exerted on the dipole (i.e. on the charges +*q* and –*q* of the dipole) in the external electric field. Directions of these forces are parallel to the external field lines.

3. According to the superposition principle:

$$\vec{F} = \vec{F}_+ + \vec{F}_-,$$

since these forces have the same magnitude, i.e., $|F_+| = |F_-| = qE$, then the net force

$$F = 2qE. \qquad \textbf{(5.16)}$$

4. The torque due to the net force exerted on the dipole, is by definition:

$$M = 2qEl\cos\alpha,$$

where $l\cdot\cos\alpha$ is the arm (Fig. 5.5); l is the distance from the negative charge $-q$ to the positive charge $+q$.

According to formula (5.1), expression (5.16) takes the form

$$M = 2pE\cos\alpha. \qquad \textbf{(5.17)}$$

5. Calculating the angular speed ω from integration with respect to time, we perform integration with respect to the angle α, that is

$$\omega = \frac{d\alpha}{dt} \Rightarrow dt = \frac{d\alpha}{\omega}. \qquad \textbf{(5.18)}$$

The equilibrium position of the dipole is the position when the net torque on the dipole is zero ($M = 0 \Rightarrow \alpha = \pi / 2$).

Therefore, we substitute equations (5.17) and (5.18) into expression (5.15), and integrate within the limits from 0 to π / 2:

$$\omega = \frac{1}{J}\int_0^{\frac{\pi}{2}} 2pE\cos\alpha\frac{d\alpha}{\omega} = \frac{2pE}{J\omega}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{2pE}{J\omega}. \qquad \textbf{(5.19)}$$

From expression (5.19), we obtain the angular speed ω of the dipole:

$$\omega = \sqrt{\frac{2pE}{J}}.$$

III. Numerical calculations:

$$\omega = \sqrt{\frac{2\cdot 1.2\cdot 10^{-11}\cdot 3\cdot 10^{5}}{2\cdot 10^{-11}}} = 6 \text{ (rad/s)}.$$

Answer: ω = 6 rad/s.

Example 5.5

What work must be done against the electric field forces to move a dipole with the electric moment p from the position 1, where the electric field equals to E_1, to the position 2 with the electric field of magnitude E_2 (Fig. 5.6) and to turn it through the angle 90°?

Given:
p, E_1, E_2
$\alpha = 90°$
A – ?

I. Physical model

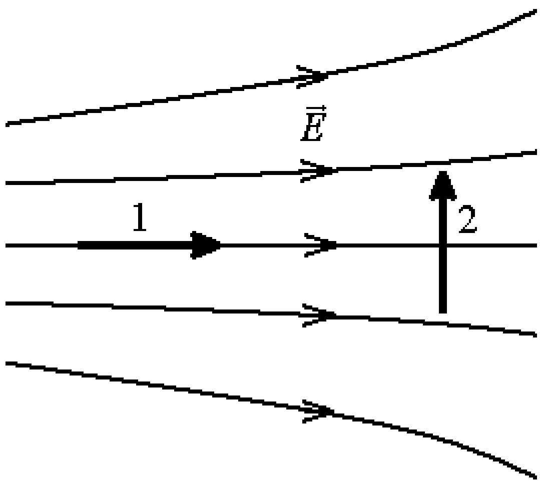


Figure 5.6

II. Mathematical model

1. Consider motion of the dipole: let it be carried in the field, and then rotated (or vice versa – we'll show that the work depends only on the initial and final position of the dipole, and not on the path of transition between these positions). The work in this motion is the sum of works in moving the dipole in the field and in rotating the dipole. Let's calculate these works separately.

2. Use formula (5.2), (5.4) and the known formula for the potential difference:

$$\varphi_a - \varphi_b = \int_a^b \vec{E}d\vec{l}.$$

In our case, the direction $\vec{l}$ coincides with the direction of the electric dipole moment. Having chosen the center of the dipole as the origin (we have a right to do that because the dipole is a neutral system of charges), we denote coordinates of the positive and the negative charge as $+l/2$ and $-l/2$, where l is the distance between the charges $+q$ and $-q$.

Since the distance between the charges of the dipole is much smaller than all other distances in the problem (we have the point dipole), the external electric field does not change within the dipole. Therefore, the potential energy of the dipole in the position 1 is:

$$W_1 = q \int_{-l/2}^{l/2} E_1 dl = -qlE_1.$$

And in the position 2 it is:

$$W_2 = q \int_{-l/2}^{l/2} E_2 dl = -qlE_2.$$

Then the work in moving the dipole between these positions is:

$$A_{12} = W_2 - W_1 = ql\left(E_1 - E_2\right).$$

3. Let's calculate the energy of the dipole when the vector of the electric moment is perpendicular to the vector of the external electric field (the dipole is turned through the angle $\alpha = 90°$). Fig. 5.6 shows that the external field is symmetrical about the axis passing through the center of the dipole perpendicular to the vector of its electric moment. That means that electric potential at the points equidistant from the axis of symmetry (at the points of location of charges $+q$ and $-q$ of the dipole) is the same. Thus, the potential energy of the dipole in this position is zero.

Therefore, the work in rotating the dipole at the point 2 through the angle $\alpha = 90°$ is:

$$A_{90^\circ} = W_{90^\circ} - W_2 = 0 - \left(-qlE_2\right) = qlE_2.$$

4. Thus, we have proven that no matter how complex is the motion, the work required for it is the same, i.e., if the dipole is carried from the position 1 to the position 2 and rotated, the work is

$$A = A_{12} + A_{90^\circ} = ql\left(E_1 - E_2\right) - \left(-qlE_2\right) = qlE_1,$$

if, vice versa, it is rotated at the position 1 and so moved to the position 2, then the work in rotating the dipole with energy W_1 is:

$$A = W_{90^\circ} - W_1 = 0 - \left(-qlE_1\right) = qlE_1. \qquad \textbf{(5.20)}$$

The work in moving the dipole is zero.

5. Since the electric dipole moment $p = ql$, the formula (5.20) finally takes the form:

$$A = pE_1.$$

Answer: $A = pE_1$.

5.3. Problems for independent work

Electric field due to a dipole

5.1. The distance between the charges $q = \pm$ 3.2 nC of the dipole is 12 cm. Find the magnitude E and the potential φ of the field produced by the dipole at the point distant by r = 8 cm both from the first and the second charge.

5.2. A dipole with an electric moment p = 0.12 nC·m is formed by two point charges $Q = \pm$ 1 nC. Find the magnitude E and the potential φ of the electric field at the points A and B (Fig. 5.7), which are at the distance r = 8 cm from the center of the dipole.

5.3. Determine the magnitude E and the potential φ of the electric field produced by a point dipole with an electric moment p = 4 pC·m at the distance r = 10 cm from the center of the dipole, in the direction making angle $\alpha = 60°$ with the vector of the electric dipole moment.

5.4. A point dipole with an electric moment p = 1 pC·m rotates uniformly with a frequency $\nu = 10^3$ s^{-1} about its perpendicular bisector axis. Derive the law of change of the electric potential as a function of time at some point distant by r = 1 cm from the center of the dipole and lying in the plane of rotation of the dipole. Assume that at the initial moment of time the potential φ_0 of that point is zero. Plot a graph of the dependence $\varphi(t)$.

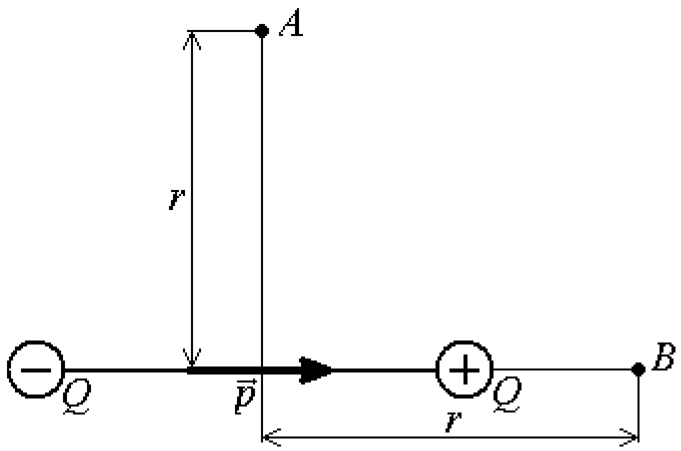


Figure 5.7

5.5. Two point dipoles with electric moments p_1 = 1 pC·m and p_2 = 4 pC·m are located at a distance r = 2 cm from each other. Find the force of their interaction if the axes of the dipoles lie on the same line.

5.6. Two point dipoles with electric moments p_1 = 20 pC·m and p_2 = 50 pC·m are located at a distance r = 10 cm from each other so that their axes lie on the same line. Calculate the potential energy of the dipoles corresponding to their stable equilibrium.

5.7. What is the property of the electric dipole moment $\vec{p}$ of a neutral system of charges?

5.8. What is the work A required to rotate a dipole with an electric dipole moment $\vec{p}$ from the position along the field to the position against the field?

5.9. What is the electric moment $\vec{p}$ of: a) a quadrupole, b) an octupole?

Dipole in an external electric field

5.10. A dipole with an electric moment p = 100 pC·m is attached to an elastic thread (see Fig. 5.8). When an electric field of magnitude E = 3 kV/m is applied in the direction perpendicular to the dipole arm, the dipole turns through an angle α = 30°. Determine the torsion constant C of the thread. (The torsion constant is the value equal to the torque that causes torsion of the thread by 1 rad).

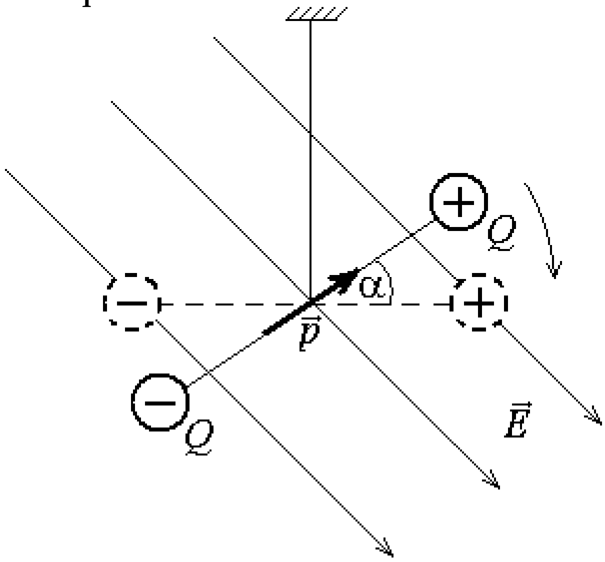


Figure 5.8

5.11. A dipole with an electric moment p = 100 pC·m is located in a uniform electric field of magnitude E = 50 V/m. The vector of the electric dipole moment makes an angle α = 60° with the direction of the field lines. What is the potential energy W_p of the dipole?

Hint: For zero potential energy take the energy corresponding to the position of the dipole when the vector of the electric dipole moment is perpendicular to the field lines.

5.12. A dipole with an electric moment p = 100 pC·m is freely placed into a uniform electric field of magnitude E = 150 V/m. Calculate the work A required to rotate the dipole through the angle α = 180°.

5.13. A point dipole with an electric moment p = 100 pC·m is freely placed into a uniform electric field of magnitude E = 9 V/m. The dipole is turned through a small angle and left on its own. Determine the natural frequency ν of the dipole oscillations in the electric field. The moment of inertia J of the dipole about the axis passing through its center is $4 \cdot 10^{-12}$ kg·m^2.

5.14. A dipole with an electric moment p = 20 pC·m is located in an inhomogeneous electric field. The degree of the field inhomogeneity is characterized by the value $dE/dx = 1$ MV/m^2 taken in the direction of the dipole axis. Calculate the force F_x exerted on the dipole in this direction.

5.15. A point dipole with an electric moment $p = 5$ pC·m is freely placed in the field produced by a point charge $Q = 100$ nC at the distance $r = 10$ cm from that charge. For that point in space, determine the value $|dE/dr|$ which characterizes the degree of the field inhomogeneity in the direction of the field line, and the force F exerted on the dipole.

5.16. Find the force F of interaction of two water molecules separated by a distance $r = 1 \cdot 10^{-3}$ m. The electric dipole moment of the water molecule is $p = 0.62 \cdot 10^{-29}$ C·m. The dipole moments of the molecules are considered to be aligned with the line joining the centers of the molecules.

5.17. A dipole with an electric dipole moment $\vec{p}$ is located at a distance r from a long uniformly charged thread with a linear charge density τ. Find the force $\vec{F}$ acting on the dipole if the vector $\vec{p}$ is oriented: a) along the thread; b) along the radius-vector $\vec{r}$; c) perpendicular to the thread and to the radius-vector $\vec{r}$.

5.18. A point dipole with an electric dipole moment $\vec{p}$ is located at a distance d from a conductive plane. Find the force acting on the dipole if the vector $\vec{p}$ is perpendicular to the plane.

Topic 6. CAPACITANCE, CAPACITORS. ENERGY STORED IN A CHARGED CONDUCTOR. ENERGY STORED IN ELECTRIC FIELD

What a student should know

1. Electrical capacitance of an isolated conductor, units of the capacitance.
2. Capacitance of a parallel-plate, spherical, cylindrical capacitor.
3. Capacitance of an isolated conductive sphere.
4. Capacitance for capacitors in parallel and series combination.
5. Energy stored in the field of a charged capacitor.
6. Energy density stored in the electric field.
7. Work by electric field forces.
8. Force exerted on a conductor or dielectric in an electric field.
9. Gauss's law.
10. DI-method.
11. Rules for junctions and loops in electric circuits.

Literature: [6, § 26.1 – 26.5]; [7, § 8.1 – 8.5]; [9, § 25]; brief theoretical information.

Tasks that determine normative level of knowledge and skills: [6: § 26 No 4, 19, 23, 53], examples 6.1, 6.2.

Homework: see Table A.5 on p. 140.

6.1. Brief theoretical information

Electrical capacitance of an isolated conductor or capacitor:

$$C = \frac{\Delta Q}{\Delta \varphi},$$

where ΔQ is the charge transferred to the conductor (capacitor); $\Delta\varphi$ is the change in potential caused by this charge.

Or,

$$C = \frac{Q}{\varphi},$$

where Q is the charge transferred to the conductor, φ is the potential of the conductor.

Capacitance of a capacitor:

$$C = \frac{Q}{\varphi_1 - \varphi_2}, \qquad \textbf{(6.1)}$$

where $\varphi_1 - \varphi_2$ is the potential difference across the plates of the capacitor.

Capacitance of a parallel-plate capacitor:

$$C = \frac{\varepsilon_r \varepsilon_0 S}{d},$$

where S is the area of the plates (of one plate); d is the distance between the plates; ε_r is the relative permittivity of the dielectric material filling the space between the plates.

Capacitance of an isolated conductive sphere of radius R, which is located in an infinite medium with relative permittivity ε_r:

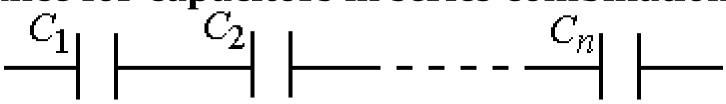
$$C = 4\pi\varepsilon_0\varepsilon_r R.$$

If the sphere is filled with dielectric material, its capacitance does not change.

Electrical capacitance of a spherical capacitor (two concentric spheres of radii R_1 and R_2, the space between them is filled with a dielectric material of relative permittivity ε_r):

$$C = \frac{4\pi\varepsilon_r\varepsilon_0 R_1 R_2}{R_2 - R_1}.$$

Capacitance for capacitors in series combination:



a) in general case:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots + \frac{1}{C_n} = \sum_{i=1}^{n} \frac{1}{C_i}, \qquad \textbf{(6.2)}$$

where n is the number of capacitors in combination;

b) for the case of two capacitors,

$$C = \frac{C_1 C_2}{C_1 + C_2};$$

c) for the case of n identical capacitors of capacitance C_1 each,

$$C = \frac{C_1}{n}.$$

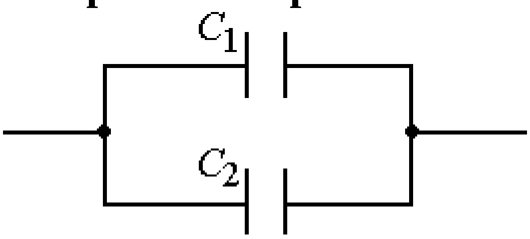
Charge of the entire combination:

$$Q = Q_1 = Q_2 = Q_3 = \ldots = Q_n.$$

Voltage across the entire combination:

$$U = U_1 + U_2 + U_3 + \ldots + U_n.$$

Capacitance for capacitors in parallel combination:



a) in general case:

$$C = C_1 + C_2 + \ldots + C_n;$$

$$C = \sum_{i=1}^{n} C_i,$$

where n is the number of capacitors in combination;

b) for the case of two capacitors,

$$C = C_1 + C_2;$$

c) for the case of n identical capacitors of capacitance C_1 each,

$$C = nC_1.$$

Charge of the entire combination:

$$Q = Q_1 + Q_2 + \ldots + Q_n.$$

Voltage across the entire combination:

$$U = U_1 = U_2 = \ldots = U_n.$$

Energy stored in electric field that occupies a volume V and is characterized by an energy density w is determined by the expression

$$W = \int_V w dV. \qquad \textbf{(6.3)}$$

Energy density stored in electric field is determined by the values that characterize the electric field:

$$w = \frac{\varepsilon_r \varepsilon_0 E^2}{2} = \frac{1}{2} ED, \qquad \textbf{(6.4)}$$

where D is the electric displacement field; E is the electric field.

Energy stored in a charged capacitor:

$$W = \frac{qU}{2} = \frac{CU^2}{2} = \frac{q^2}{2C}. \qquad \textbf{(6.5)}$$

Work by electric field forces:

$$A = -\Delta W = W_1 - W_2, \qquad \textbf{(6.6)}$$

where ΔW is the change in the field energy.

Force exerted on a conductor or dielectric in electric field in the x direction at a constant charge q:

$$F_x = -\frac{\partial W}{\partial x}.$$

6.2. Methodical guidelines

1. Capacitance depends only on the shape, size and material of a conductor and does not depend on whether the conductor has cavities or not.

2. If a parallel plate capacitor is connected to a power supply, charged and then disconnected, then in the case of change of the capacitance C of the capacitor by increasing (or reducing) the distance d between its plates, or by introducing (or removing) a dielectric material between the plates, the charge on the capacitor does not change.

3. A mixed combination of capacitors, which consists of connection of groups of capacitors in series and in parallel, is the easiest to calculate. After replacing each group by the equivalent capacitance, the circuit is simplified until it becomes possible to find the total capacitance of the system. It should be noted that in the case of such a replacement, the charges and voltages do not change. It is

possible to simplify a circuit by finding the points of equal potentials (these are the points that have symmetry). Points of the circuit have the same potential if they are connected directly by a conductor, the resistance of which is assumed to be zero in problems. By connecting and disconnecting such points, you can reduce complex combination to combination of capacitors in series and in parallel.

4. To calculate an electrical circuit consisting of capacitors and constant voltage sources, if the circuit cannot be decomposed into groups of series and parallel combinations, the following two rules should be applied:

- *the junction* (*node) rule* is a consequence of the law of conservation of electric charge: if plates of several capacitors are connected to one node which is not connected to a power supply, then the algebraic sum of the charges on these plates is zero, i.e.

$$\sum q = 0; \qquad \textbf{(6.7)}$$

- *the loop rule* is a consequence of the law of conservation of energy: the algebraic sum of potential differences across every capacitor and power supply that occurs while traveling around any closed loop in the circuit is zero, i.e.

$$\sum U = 0. \qquad \textbf{(6.8)}$$

Direction of traveling around the loop is chosen arbitrary: clockwise or counterclockwise.

5. To solve problems on the topic “Energy stored in a charged capacitor”, the equation of energy balance is usually used for external action on capacitors associated with the change in their capacitances. Change in the capacitance of a system may be accompanied by motion of charges, that is, by electric current flow. It is always believed that arbitrary motion of charges due to changes in the capacitance of the system is so slow that the loss of energy for Joule heating, which is proportional to the square of the current, can be neglected.

6. If it is necessary to find the force of attraction of the plates of the capacitor, one should keep in mind that the electric field produced by one plate is twice less than the field between the plates of the capacitor.

7. In order to increase the capacitance of a capacitor (energy stored in it) it is necessary to do work equal to the change in the energy stored in the capacitor (6.6).

8. When calculating the total electric energy of charged non-point bodies, it should be assumed that it consists not only of the energy of interaction, but also of the energy consumed for appearance of the charge on each body (their own energy) (6.3).

6.2.1. Complex combination of capacitors

Example 6.1

A system of capacitors (Fig. 6.1) is charged to the potential difference U_0 = 200 V, and then disconnected from the power supply. How will the energy stored in the system change if the switch K is closed? The capacitances $C_1 = C_2 = C_3 = C_5 = 1\ \mu F$, $C_4 = 0.5\ \mu F$.

Given:

U_0 = 200 V

$C_1 = C_2 = C_3 = C_5 = 10^{-6}$ F

$C_4 = 0.5\cdot 10^{-6}$ F

ΔW – ?

I. Physical model

After disconnecting the system from the battery, its charge, which corresponds to the sum of the charges of all the plates connected to one of the battery terminals, remains unchanged regardless of the position of the switch *K*. However, if the switch is closed, the combination pattern of the capacitors changes, which brings change in the capacitance of the system. Therefore, according to formula (6.5) let's determine the change in the energy stored in the system:

$$\Delta W = W - W_0 = \frac{q^2}{2C} - \frac{q^2}{2C_0} = \frac{q^2}{2}\frac{C_0 - C}{C_0 C},$$

where C_0, W_0 and C, W are the capacitances and energies stored in the system before and after closing the switch.

Since the charge on the system $q = C_0U_0$, then

$$\Delta W = \frac{C_0^2U_0^2}{2}\frac{C_0 - C}{C_0C} = \frac{C_0U_0^2}{2}\frac{C_0 - C}{C}.$$

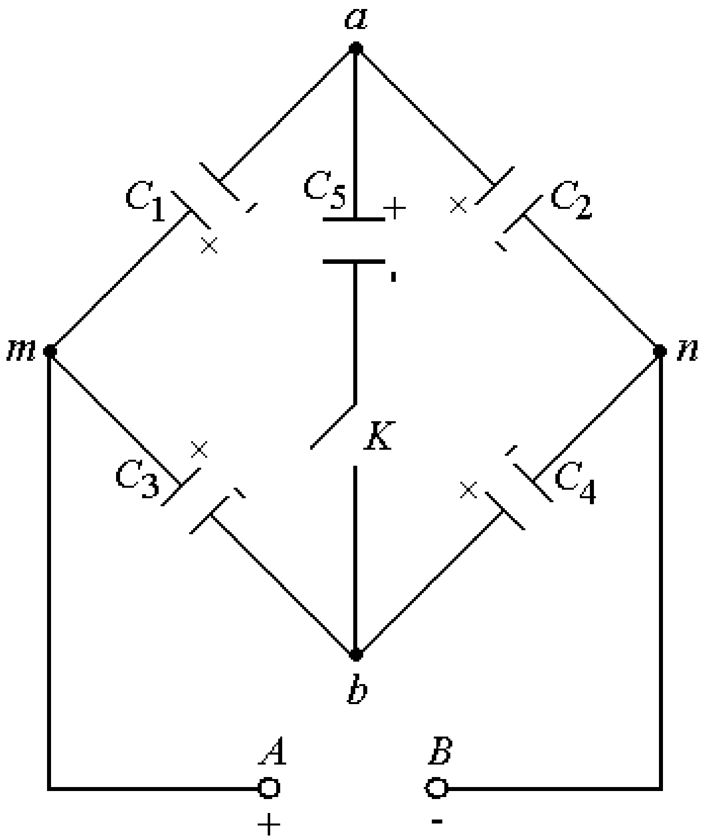


Figure 6.1

II. Mathematical model

To find the capacitance C_0 (before closing the switch) consider the system in Fig. 6.1. That system is a parallel combination of two branches, while each branch is a series combination of two capacitors. Using formula (6.2), we obtain

$$C_0 = \frac{C_1C_2}{C_1 + C_2} + \frac{C_3C_4}{C_3 + C_4}.$$

2. Let’s determine the capacitance C (after closing the switch). From formula (6.1), the charge is proportional to the voltage U:

$$q = f\left(C_1, C_2, C_3, C_4, C_5\right)U, \qquad \textbf{(6.9)}$$

where U is the terminal voltage of the battery in the case of the closed switch, $f\left(C_1, C_2, C_3, C_4, C_5\right) = C$.

3. Let's calculate the charge of the system adding the charges of capacitors according to rule (6.7):

$$q = q_1 + q_3. \qquad \textbf{(6.10)}$$

To determine the charges q_1 and q_3 let's put the signs of charges on the plates of all capacitors depending on the chosen signs of the battery terminals (Fig. 6.1). By rule (6.7) for the nodes a and b, we can write:

$$-q_1 + q_2 + q_5 = 0; \qquad \textbf{(6.11)}$$

$$-q_3 + q_4 - q_5 = 0. \qquad \textbf{(6.12)}$$

4. Equations (6.10) – (6.12) have six unknown quantities. We use rule (6.8). Choose the travelling direction around the loops, for example, clockwise. To avoid errors in the signs, keep in mind the following: if the potential across the section (1 – 2) of the loop decreases in the travelling direction, then the potential difference $\varphi_1 - \varphi_2$ is positive, otherwise it is negative. Therefore, taking into account relation (6.1), for the loops *mabm*, *anba* and A*man*BA we obtain:

$$\frac{q_1}{C_1} + \frac{q_5}{C_5} - \frac{q_3}{C_3} = 0; \qquad \textbf{(6.13)}$$

$$\frac{q_2}{C_2} - \frac{q_4}{C_4} - \frac{q_5}{C_5} = 0; \qquad \textbf{(6.14)}$$

$$\frac{q_1}{C_1} + \frac{q_2}{C_2} - U = 0. \qquad \textbf{(6.15)}$$

5. Solving the system of equations (6.10) – (6.15), containing six unknown quantities q, q_1, q_2, q_3, q_4, q_5 with respect to q, we obtain

$$q = \left(\frac{11}{13}\right)U = 0.85U. \qquad \textbf{(6.16)}$$

6. Comparing formulas (6.9) and (6.13), we can find the capacitance after closing the switch: C = 0.85 μF.

III. Numerical calculations:

$$C_0 = \frac{\left(10^{-6}\right)^2}{2\cdot 10^{-6}} + \frac{0.5\left(10^{-6}\right)^2}{1.5\cdot 10^{-6}} = 0.5\cdot 10^{-6} + 0.33\cdot 10^{-6} = 0.83\cdot 10^{-6}\,(\text{F});$$

$$C = 0.85\cdot 10^{-6}\,(\text{F});$$

$$\Delta W = \frac{0.83\cdot 10^{-6}\cdot 4\cdot 10^{4}}{2}\cdot\frac{0.83\cdot 10^{-6} - 0.85\cdot 10^{-6}}{0.85\cdot 10^{-6}} = -3.9\cdot 10^{-4}\,(\text{J}).$$

Answer: ΔW = – 0.39 mJ, the negative sign in the answer shows that in the case of closing the switch the energy of the system decreases, while the charge remains constant.

6.2.2. Energy stored in a charged conductor. Energy stored in electric field

Example 6.2

A charge Q is uniformly distributed throughout a volume of a sphere of radius R. Assuming the relative permittivity ε_r = 1, find the electrical energy stored in the sphere.

Given:

Q, R

$W - ?$

I. Physical model

1. The intrinsic electrical energy stored in the sphere is equal to:

$$W = \int_V w_e dV, \tag{6.17}$$

where w_e is the volume energy density.

2. To determine the volume energy density, we use formula (6.4).

3. We find the unknown electric field using the Gauss's law, taking into account the spherical symmetry:

$$\oint_{S_{1,2}} \vec{E}d\vec{S} = \frac{\sum Q_i}{\varepsilon_0}, \tag{6.18}$$

where dS is the area element of the auxiliary surface, which has the shape of a sphere.

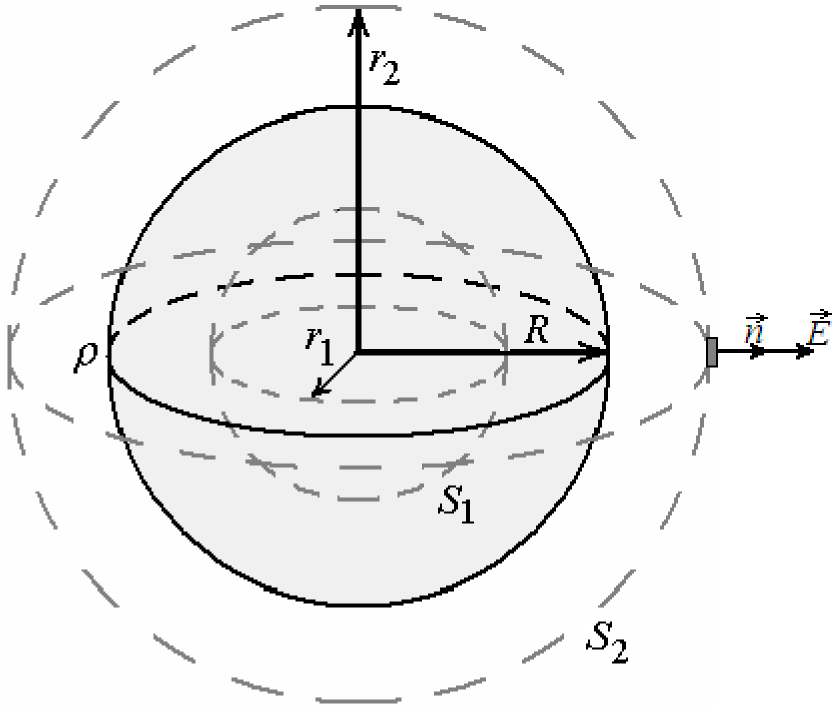


Figure 6.2

II. Mathematical model

1. To calculate the electric field inside and outside the sphere of radius R, draw the auxiliary surfaces S_1 and S_2 (Fig. 6.2). At all points of these surfaces, the angle between $\vec{E}$ and $d\vec{S}$ is zero (for the positive charge) and $\vec{E}$ = const, so

$$\oint \vec{E}d\vec{S} = \int_{S_{1,2}} E_n dS = E_n \int_{S_{1,2}} dS = E_n 4\pi r^2, \qquad \textbf{(6.19)}$$

where r is the radius of the auxiliary surface.

If $r < R$, then the sum of the charges enclosed by the surface S_1 is equal to

$$\sum Q = \frac{4}{3}\rho\pi r^3,$$

where $\rho = 3Q/4\pi R^3$ is the volume density of the charge.

Therefore,

$$\sum Q = \frac{Qr^3}{R^3}. \qquad \textbf{(6.20)}$$

Substituting equations (6.19) and (6.20) into formula (6.18), we obtain expression for the electric field inside the volume of the sphere:

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3}.$$

If $r > R$, then the sum of the charges enclosed by the surface S_2 is equal to

$$\sum Q = Q. \tag{6.21}$$

Substituting equations (6.21) and (6.19) into expression (6.18), we find the electric field outside the sphere:

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}. \tag{6.22}$$

2. Substitute (6.21) and (6.22) into formula (6.4). In this case, the volume energy density is also a function of the distance r:

$$w_e = \frac{Q^2 r^2}{32\pi^2\varepsilon_0 R^6},\ (r < R),$$
$$w_e = \frac{Q^2}{32\pi^2\varepsilon_0 r^4},\ (r > R). \tag{6.23}$$

Since the dependence $w_e\,(r)$ is not the same for the regions of space inside and outside the charge Q, we calculate the integral in the right part of (6.17) as the sum of two integrals:

$$W = \int_{V_1} w_e dV + \int_{V_2} w_e dV, \tag{6.24}$$

where V_1 is the volume of space occupied by the charge Q; V_2 is the volume of the rest of space; dV is the infinitesimal volume in the form of a thin layer of the sphere with thickness dr (within such a volume the values E and w_e are constant): $dV = 4\pi r^2 dr$.

4. Substitute expressions (6.23) into equation (6.24) taking into account that within the volume V_1 the variable r varies from 0 to R, and within V_2 it varies from R to ∞. Finally, we have:

$$W = \frac{Q^2}{8\pi\varepsilon_0}\left(\int_0^R r^4 dr + \int_R^\infty \frac{dr}{r^2}\right) = \frac{3Q^2}{20\pi\varepsilon_0 R}.$$

Answer: $W = \dfrac{3Q^2}{20\pi\varepsilon_0 R}$.

6.3. Problems for independent work

Capacitance of capacitors

6.1. Electric charge is uniformly distributed over the plates of a parallel plate capacitor with a surface density σ = 0.2 μC/m^2. The distance d between the plates is 1 mm. How much will the potential difference across the capacitor change if the distance between the plates increases to 3 mm?

6.2. A parallel plate capacitor is charged to the potential difference U = 600 V. There are two layers of dielectric between the plates: the glass layer of thickness $d_1 = 7$ mm and the ebonite layer of thickness $d_2 = 3$ mm. The area S of each plate of the capacitor is 200 cm^2. Find: a) the capacitance C of the capacitor; b) the electric displacement D, the electric field E and the potential drop $\Delta\varphi$ across each layer.

6.3. A paraffin slab of thickness d = 1 cm is inserted between the plates of a parallel plate air capacitor fitting exactly the space between the plates. By what value should the distance between the plates be increased to obtain the capacitance equal to that the initial air capacitor has had?

6.4. The distance d between the plates of a parallel plate capacitor is 1.33 mm, the area S of the plates is 20 cm^2. There are two layers of dielectric in the space between the plates of the capacitor: the mica of thickness $d_1 = 0.7$ mm and the ebonite of thickness $d_2 = 0.3$ mm. Determine the capacitance C of such a capacitor.

6.5. The capacitance C of a parallel plate capacitor is 1.5 μF. The distance d between the plates is 5 mm. What will the capacitance of the capacitor be, if one puts a slab of ebonite of thickness $d_1 = 3$ mm on the bottom plate?

6.6. There is a tight-fitting glass slab between the plates of a parallel plate capacitor. The capacitor is charged to the potential difference $U_1 = 100$ V. What will the potential difference U_2 be if the glass slab is removed from the capacitor?

6.7. Two concentric metal spheres with radii $R_1 = 2$ cm and $R_2 = 2.1$ cm make a spherical capacitor. Determine its capacitance C if the space between the spheres is filled with paraffin.

6.8. A capacitor consists of two concentric spheres. The radius R_1 of the inner sphere is 10 cm, the radius of the outer is $R_2 = 10.2$ cm. The space between the spheres is filled with paraffin. A charge Q = 5 μC is given to the inner sphere. Determine the potential difference U between the spheres.

6.9. A KD type capacitor is made of low-frequency ceramics in the form of a disk with electrodes applied to its both sides. The M750 capacitor has the capacitance C = 6800 pF, the disk of diameter D = 13.5 mm and thickness d = 0.6 mm. Determine the relative permittivity of its dielectric ceramics.

6.10. A KT type capacitor has a dielectric made of capacitor ceramics in the form of a tube of mean diameter D and length l. Determine the relative permittivity for the capacitor KT-2, if its capacitance C = 30 pF, diameter D = 3.5 mm, length l = 7 mm, and the tube thickness d = 0.3 mm.

6.11. A coaxial radio frequency cable (RK-75-4-12) consists of a central wire, a concentric cylindrical sheath (screen) and a polyethylene insulation between them. Find the capacitance per unit length of such a cable (μF/m) if the wire diameter is d = 1.2 mm and the screen diameter is D = 4.6 mm.

Combinations of capacitors

6.12. Two capacitors with capacitances $C_1 = 3$ μF and $C_2 = 6$ μF are connected to each other and connected to the battery ε = 120 V. Determine the charges Q_1 and Q_2 on the capacitors and the potential differences U_1 and U_2 across their plates, if the capacitors are connected: a) in parallel; b) in series.

6.13. A capacitor of capacitance $C_1 = 0.6$ μF is charged to the potential difference $U_1 = 300$ V. Then it is connected to another capacitor of capacitance $C_2 = 0.4$ μF, which is charged to the potential difference $U_2 = 150$ V. Find the charge ΔQ that will flow from the plate of the first capacitor to the second capacitor.

6.14. Three identical parallel plate capacitors are connected in series. The capacitance C of such a capacitor bank is 89 pF. The area S of each plate is 100 cm^2. The dielectric material is glass. What is the thickness d of the dielectric?

6.15. A capacitor of capacitance $C_1 = 0.2$ μF is charged to the potential difference $U_1 = 320$ V. Then it is connected to another capacitor charged to the potential difference $U_2 = 450$ V, and the voltage across the first capacitor changes to 400 V. Determine the capacitance of the second capacitor.

6.16. Capacitors with capacitances $C_1 = 10$ nF, $C_2 = 40$ nF, $C_3 = 2$ nF and $C_4 = 30$ nF are connected as shown in Fig. 6.3. Determine the capacitance C of the capacitor bank.

6.17. Capacitors are connected as shown in Fig. 6.4. Capacitances of the capacitors are: $C_1 = 0.2$ μF, $C_2 = 0.6$ μF, $C_3 = 0.3$ μF, $C_4 = 0.5$ μF. Determine the capacitance C of the capacitor bank.

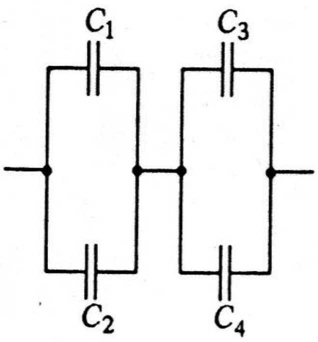


Figure 6.3

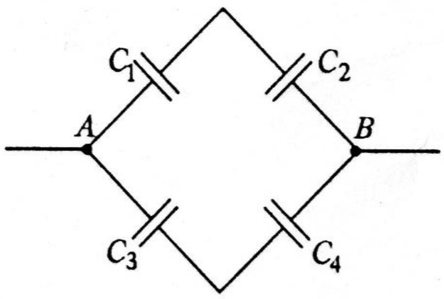


Figure 6.4

6.18. An air capacitor is charged to the potential difference $U = 600$ V and disconnected from the power supply, and then connected in parallel with an uncharged capacitor of the same shape and size, but with a dielectric (porcelain). Determine the relative permittivity ε_r of the porcelain, if after connecting the second capacitor the potential difference decreased to $U_1 = 100$ V.

6.19. Capacitors of capacitances $C_1 = 2$ μF, $C_2 = 2$ μF, $C_3 = 3$ μF, $C_4 = 1$ μF are connected as shown in Fig. 6.5. The potential difference across the plates of the fourth capacitor is $U_4 = 100$ V. Find the charges and the potential differences across the plates of each capacitor, as well as the total charge and the total potential difference across the capacitor bank.

6.20. Five different capacitors are connected according to the scheme (Fig. 6.6). Determine the capacitance C_4 at which the capacitance of the entire system does not depend on the value of capacitance C_5. Take $C_1 = 8$ pF, $C_2 = 12$ pF, $C_3 = 6$ pF.

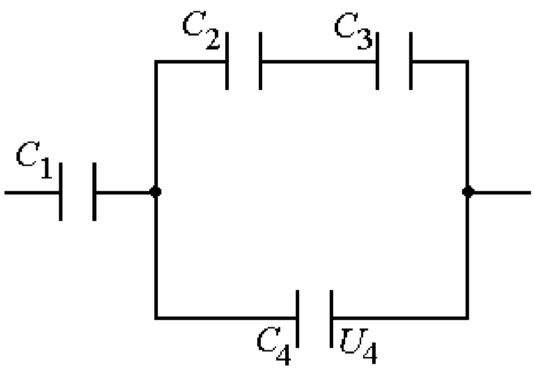


Figure 6.5

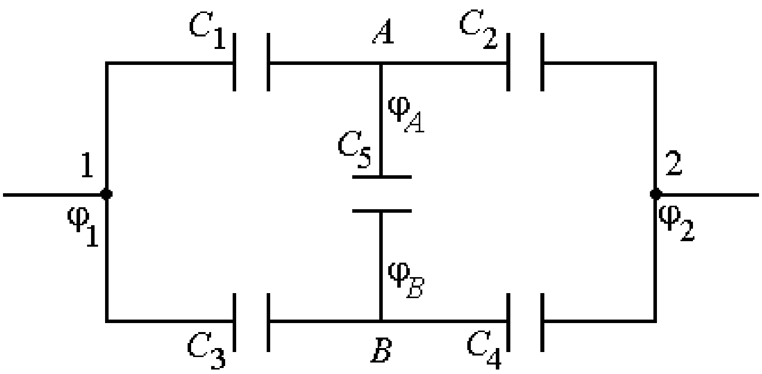


Figure 6.6

Energy stored in the field of a parallel plate capacitor

6.21. The distance d between the plates of a parallel plate capacitor is 2 cm, the potential difference across the plates is $U = 6$ kV. The charge Q on each plate is equal to 10 nC. Calculate the field energy W stored in the capacitor and the force F of mutual attraction of the plates.

6.22. What amount of heat Q will be released during the discharge of a parallel plate capacitor, if the potential difference across the plates is 15 kV, the distance d is 1 mm, the dielectric is mica, the area of each plate is 300 cm^2?

6.23. The force F of attraction between the plates of a parallel plate air capacitor is 50 mN. The area S of each plate is 200 cm^2. Find the energy density w stored in the field of the capacitor.

6.24. A parallel plate air capacitor consists of two round plates of radius r = 10 cm each. The distance d_1 between the plates is 1 cm. The capacitor is charged to the potential difference U = 1.2 kV and then disconnected from the power supply. What work A in distancing the plates from each other is required to increase the distance between them to $d_2 = 3.5$ cm?

6.25. A parallel plate air capacitor of capacitance C = 1.11 nF is charged to the potential difference U = 300 V. After disconnecting from the power supply, the distance between the plates of the capacitor is increased 5 times. Determine: 1) the potential difference U' across the plates after their distancing; 2) the work A by external forces required to distance the plates.

6.26. A capacitor of capacitance $C_1 = 666$ pF is charged to the potential difference U = 1.5 kV and disconnected from the power supply. Then a second, uncharged capacitor of capacitance $C_2 = 444$ pF is connected in parallel to the first capacitor. Determine the energy spent on the formation of the spark that has occurred at the connection of the capacitors.

6.27. Capacitors with capacitances $C_1 = 1$ μF, $C_2 = 2$ μF, $C_3 = 3$ μF are connected in a circuit with voltage U = 1.1 kV.

Determine the energy stored in each capacitor in the cases of 1) series and 2) parallel combination.

6.28. The capacitance C of a parallel plate capacitor is 111 pF, the dielectric is ceramics. The capacitor is charged to the potential difference $U = 600$ V and disconnected from the power supply. What work A is required to remove the dielectric from the capacitor? Neglect friction.

6.29. The space between the plates of a parallel plate capacitor is filled with a dielectric (ceramics) of volume $V = 100\ \text{cm}^3$. The surface charge density σ over the capacitor plates is $8.85\ \text{nC/m}^2$. Find the work A required to remove the dielectric from the capacitor. Neglect friction of the dielectric against the capacitor plates.

6.30. An ebonite slab of thickness $d = 2$ mm and area $S = 300\ \text{cm}^2$ is placed into a uniform electric field of magnitude $E = 1$ kV/m so that the field lines are perpendicular to the slab's flat surface. Find: 1) the density σ' of bound charges on the surface of the slab; 2) the energy W stored in the electric field concentrated within the slab.

6.31. The slab from the previous problem is moved to the region of space without external electric field. Neglecting the decrease of the field in the dielectric over time, find the energy W stored in the electric field within the slab.

6.32. Determine the work required to increase by $\Delta x = 0.2$ mm the distance x between the plates of a parallel plate air capacitor with charges $\pm q = 0.2\ \mu\text{C}$ on its plates. The area of each plate is $S = 400\ \text{cm}^2$.

6.33. The space between the plates of a parallel plate capacitor is filled with a dielectric material of relative permittivity ε_r. What happens with the energy density w stored in the field between the plates, if the capacitor: a) is connected to a power supply; b) is disconnected from the power supply?

6.34. A parallel plate capacitor with the distance between the plates $d = 1$ mm is immersed into the water in a horizontal position, and the water fills it completely. After that, the capacitor is connected to a DC power supply $U = 500$ V. Find the increase in the water pressure inside the capacitor.

6.35. A parallel plate capacitor is placed so that one of its plates is above the surface of the liquid, the other is below the surface. The relative permittivity of the liquid is ε_r, its density is ρ. What height will the liquid level in the capacitor rise after charging its plates with the surface density σ?

Energy stored in a charged sphere

6.36. A metal sphere with capacitance C = 10 pF is charged to the potential φ = 3 kV. Find the energy W stored in the field within a spherical layer bounded by that sphere and a spherical surface concentric to it with the radius three times larger than the radius of the sphere.

6.37. Electric field is created by a charged (Q = 0.1 μC) sphere of radius R = 10 cm. What energy W is stored in the volume bounded by that sphere and a spherical surface concentric to it with the radius twice as large as the radius of the sphere?

6.38. A metal sphere of radius R_1 = 6 cm contains a charge Q. A spherical surface concentric to the sphere divides space into two parts (the inner part is limited and the outer part is infinite) so that the electric field energies stored in the both parts are equal. Find the radius R_2 of the spherical surface.

6.39. A solid paraffin ball of radius R = 10 cm is charged uniformly throughout a volume with the volume charge density ρ = 10 nC/m^3. Find the energy W_1 stored in the electric field within the ball and the energy W_2 stored in the field outside the ball.

6.40. An ebonite ball is uniformly charged throughout a volume. How many times does the energy stored in the electrostatic field outside the ball exceed the energy stored in the field concentrated within the ball?

6.41. A charge q = 10^{-10} C is uniformly distributed over the surface of a sphere with radius r = 1 cm. The relative permittivity of the medium surrounding the sphere is ε_r = 1. a) Calculate the energy W stored in the field due to the sphere; b) What part η of that energy is stored within the region bounded by the spere and an imaginary concentric spherical surface of radius $R = 1$ m? c) What is the radius of the spherical surface enclosing half of the energy?

6.42. Initially, a charge $q = 10^{-10}$ C is distributed uniformly throughout the volume of a sphere with radius $r = 1$ cm. Then, as a result of mutual repulsion, the charges pass to the surface of the sphere. What work A is done on the charges by the electric forces ($\varepsilon_r = 1$)?

6.43. A point charge $q = 3$ μC is located in the center of a spherical layer of a homogeneous and isotropic dielectric ($\varepsilon_r = 3$). The inner radius of the layer is $a = 250$ mm, the outer is $b = 500$ mm. Find the energy W stored within the dielectric.

6.44. A system consists of two concentric metal shells with radii R_1, R_2 and corresponding charges q_1 and q_2. Find the intrinsic energy W_1 and W_2 stored in the field of each shell, the interaction energy of the shells W_{12} and the total electrical energy W of the system.

6.45. A spherical shell of radius R_1, carrying a uniformly distributed charge q, is expanded to radius R_2. Find the work done by the electric forces.

6.46. A long cylindrical dielectric layer with relative permittivity ε_r is introduced into the cylindrical capacitor filling almost the entire space between the plates. The average radius of the plates is R, the distance between them is d, and $d << R$. The plates of the capacitor are connected to a DC power supply U. Find the magnitude of the electric force that draws the dielectric into the capacitor.

ANSWERS

1.1. F = 9 GN. **1.2.** Q = 50.1 nC. **1.3.** ε_r = 2. **1.4.** Q = 86.7 fC. **1.5.** ν = 219 km/s, $n = 6.59\cdot10^{14}$ s^{-1}. **1.6.** F = 54 mN. **1.7.** Q_1 = 0.14 μC, Q_2 = 20 nC. **1.8.** Q_1 = 0.09 μC, Q_2 = – 0.01 μC. **1.9.** Between the charges at the distance x = 40 cm from the charge 4Q, positive. **1.10.** The point is located at the distance l_1 = 20 cm from Q_1, $Q_3 = -8\cdot10^{-8}$ C, unstable. **1.11.** Q_1 = = –0.577 nC, no. **1.12.** Q_1 = –0.287 nC. **1.13.** $m_p = 1.86\cdot10^{-9}$ kg. **1.14.** $\vec{F} = \dfrac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N_1}\sum_{k=1}^{N_2}\dfrac{q_i q_k'}{\left(\left|\vec{r}_i - \vec{r}_k'\right|\right)^3}\left(\vec{r}_i - \vec{r}_k'\right)$. **1.15.** For electrons: $F_e/F_g \approx 4.2\cdot10^{42}$, for protons: $F_e/F_g \approx 1.24\cdot10^{36}$, $q/m = 0.86\cdot10^{-10}$ C/kg. **1.16.** $F = 1.35\cdot10^{-8}$ N. **1.17.** $\dfrac{dq}{dt} = \dfrac{3}{2}a\sqrt{\dfrac{2\pi\varepsilon_0 mg}{l}}$. **1.18.** $\vec{r}_3 = \dfrac{\vec{r}_1\sqrt{q_2} + \vec{r}_2\sqrt{q_1}}{\sqrt{q_1} + \sqrt{q_2}}$, $q_3 = -\dfrac{q_1 q_2}{\left(\sqrt{q_1} + \sqrt{q_2}\right)^2}$. **1.19.** F = 1.5 mN. **1.20.** F = 4.5 mN. **1.21.** F = 3.6 mN. **1.22.** F = 1.27 μN. **1.23.** F = 9 mN. **1.24.** F = 4.03 mN. **1.25.** 1) F_1 = 0.16 mN, 2) F_2 = 2.25 μN. **1.26.** F = 35 μN. **1.27**. $\vec{F} = k\int_V\int_{V'}\dfrac{\rho(\vec{r})\rho(\vec{r}')(\vec{r} - \vec{r}')}{\left|\vec{r} - \vec{r}'\right|^3}dVdV'$. **1.29.** ΔF = 50 N.

2.1. E = 2.99 kV/m, E = 607 V/m. **2.2.** E = 280 V/m. **2.3.** x_1 = 6 cm, x_2 = 12 cm. **2.4.** At the distance $d_1 = d\left(\sqrt{2} + 1\right)$ from the negative charge **2.5.** E = 34 kV/m. **2.6.** a) E = 0, b) E = 0. **2.7.** $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\dfrac{q_i}{\left(\vec{r} - \vec{r}_i\right)^2}$. **2.8.** $\vec{E} = 2.7\vec{i} - 3.6\vec{j}$; E = 4.5 kV/m. **2.9.** $E = \dfrac{ql}{\pi\sqrt{2}\varepsilon_0\left(l^2 + x^2\right)^{3/2}}$. **2.10.** 0, E_2 = 900 V/m, E_3 = 400 V/m, graph at Fig. 1. **2.11.** 0, E_2 = 1.1 kV/m, E_3 = 200 V/m, graph at Fig. 2. **2.12.** E = 2.71 kV/m.

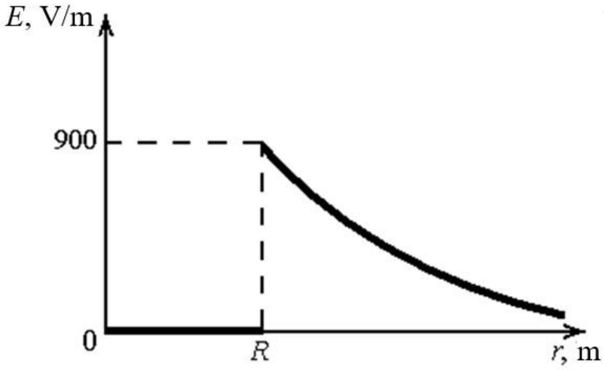


Figure 1

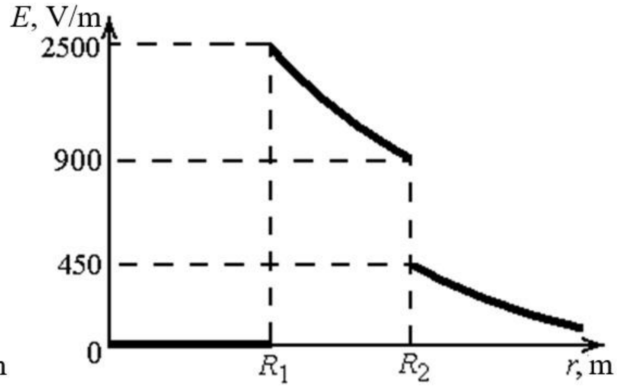


Figure 2

2.13. a) $E = \dfrac{kxq}{\left(\sqrt{x^2+R^2}\right)^3}$, b) $x_m = \pm\, 42.8$ mm, $E_m = 1.93 \cdot 10^4$ V/m.

2.14. $E = 0.1$ kV/m. **2.15.** $E = k\dfrac{lq}{\left(l^2+R^2\right)^{3/2}}$, $E_m = \dfrac{q}{6\sqrt{3}\pi\varepsilon_0 R^2}$ if $l = \dfrac{R}{\sqrt{2}}$.

2.16. $E = \dfrac{3kqR^2}{x^4}$. **2.17.** a) $E = \dfrac{\lambda_0}{4\varepsilon_0 R}$, b) $E = \dfrac{\lambda_0 R^2}{4\varepsilon_0\left(x^2+R^2\right)^{3/2}}$, if $x >> R$ $E = \dfrac{\lambda_0 R^2}{4\varepsilon_0 x^3}$. **2.18.** $\vec{E} = -\dfrac{\vec{a}R}{3\varepsilon_0}$. **2.19.** E = 64.3 kV/m. **2.20.** τ = 5.55 nC/m. **2.21.** E = 43.2 MV/m. **2.22.** 0, E_2 = 75.5 V/m, graph at Fig. 3. **2.23.** 0, E_2 = 200 V/m, E_3 = 180 V/m, graph at Fig. 4. **2.24.** E = 135 kV/m.

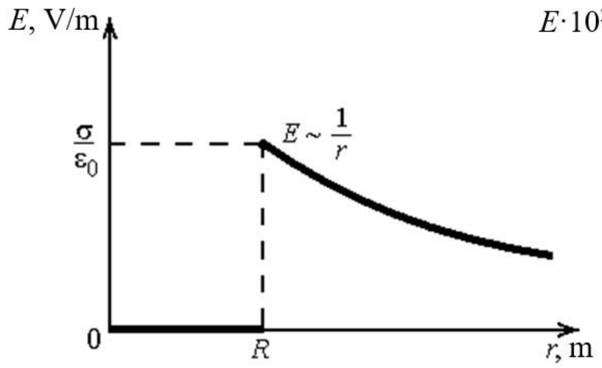


Figure 3

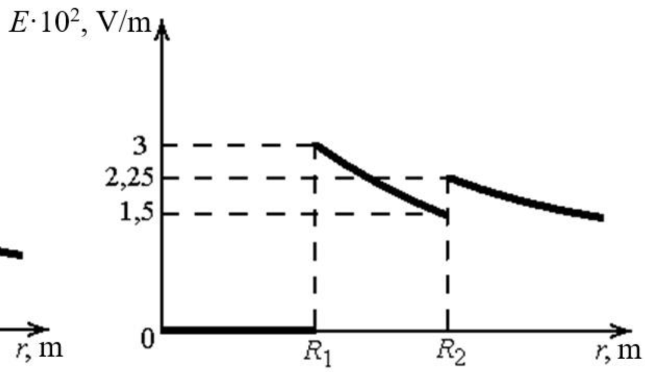


Figure 4

2.25. E = 35.6 kV/m. **2.26.** E = 55.7 kV/m. **2.27.** E = 60.2 kV/m. **2.28.** E = 38 kV/m. **2.29.** $E(r) = \dfrac{\tau}{4\pi\varepsilon_0}\dfrac{2l}{r^2 - l^2}$; if $r >> l$, $E(r) = \dfrac{\tau}{4\pi\varepsilon_0}\dfrac{l}{r^2}$.

2.30. $E(r) = \dfrac{\tau}{2\pi\varepsilon_0 r}$. **2.31.** a) $E = \dfrac{\tau\sqrt{2}}{4\pi\varepsilon_0 R}$, b) 0. **2.32.** $E_{\max} = \dfrac{\tau}{\pi\varepsilon_0 l} = 40\,\text{kV/m}$. **2.33**. 1) E = 396 V/m, 2) E = 170 V/m, graph at Fig. 5. **2.34.** P = 17 μPa. **2.37.** E = 377 kV/m. **2.38.** $|Q|$ = 33.3 nC. **2.39.** a) $E(0) \approx \dfrac{\sigma}{\varepsilon_0}\left(1 - \dfrac{a}{r}\right)$, b) $E(a-0) \approx \dfrac{\sigma}{\varepsilon_0}\left(1 - \dfrac{a}{r}\right)$, c) $E(a+0) \approx -\dfrac{\sigma}{\varepsilon_0}\dfrac{a}{r}$, d) $E(x \gg r) = -k\dfrac{q4a}{x^3}$. **2.40.** $E = \dfrac{\sigma_0}{2\varepsilon_0}$, direction of the vector $\vec{E}$ corresponds to the angle φ = π. **2.41.** E = 56.5 V/m.

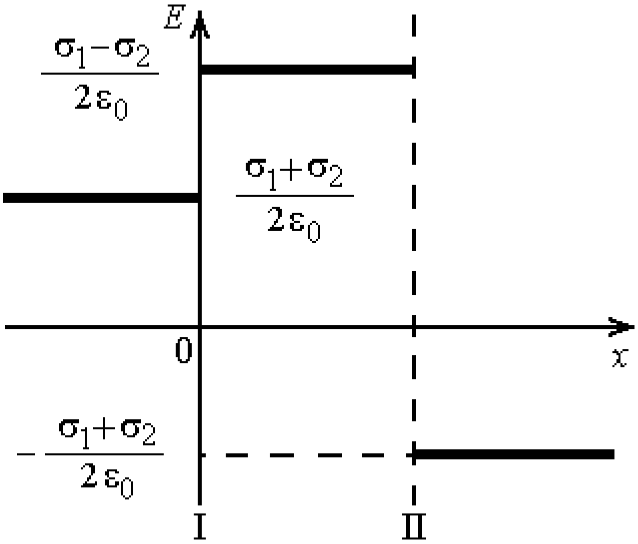


Figure 5

2.42. E_A = 0, D_A = 0, E_B = 80.8 V/m, D_B = 5 nC/m^2, E'_C = 162 V/m, E''_C = 1.13 kV/m, D_C = 10 nC/m^2, graph at Fig. 6. **2.43.** 1) E_1 = 3.78 V/m, D_1 = 0.1 nC/m^2, 2) E'_2 = 6.28 V/m ($r >> R$), E''_2 = 18.8 V/m, 3) E_3 = 4.72 V/m, D_3 = 41.7 nC/m^2, graph at Fig. 7. **2.44.** 1) E_1 = 0, D_1 = 0, 2) E_2 = 13.6 V/m, D_2 = 843 pC/m^2, 3) E_3 = 229 V/m, D_3 = 2.02 nC/m^2, graph at Fig. 8. **2.45.** 1) E_1 = 2.83 V/m, D_1 = 50 pC/m^2, 2) E_2 = 7.55 V/m, D_2 = 66.7 pC/m^2, graph at Fig. 9. **2.46.** $\vec{E}_{tot} = \dfrac{\rho\vec{a}}{3\varepsilon_0}$, the field inside the cavity is uniform. **2.47.** $\vec{E}(r) = \dfrac{\rho_0}{2\varepsilon_0}\dfrac{\vec{r}}{r}$. **2.48.** F = 0.36 N. **2.49.** F = 56.5 μN. **2.50.** σ = = 1.06 μC/m^2. **2.51.** F/l = 452 nN/m. **2.52.** F = 1.13 mN. **2.53.** F/l = = 3.6 mN/m. **2.54.** 1) F = 56.5 mN, 2) F = 0.9 μN. **2.55.** F = 150 μN.

2.56. $F = \dfrac{\tau^2}{4\pi\varepsilon_0 l}$, $\sigma(x) = \dfrac{\tau l}{\pi\left(l^2 + x^2\right)}$. **2.57.** Φ_E = 1.78 kVm. **2.58.** ψ = 2.5 nC. **2.59.** Φ_E = 4.5 V·m. **2.60.** Φ_E = 2.7 V·m.

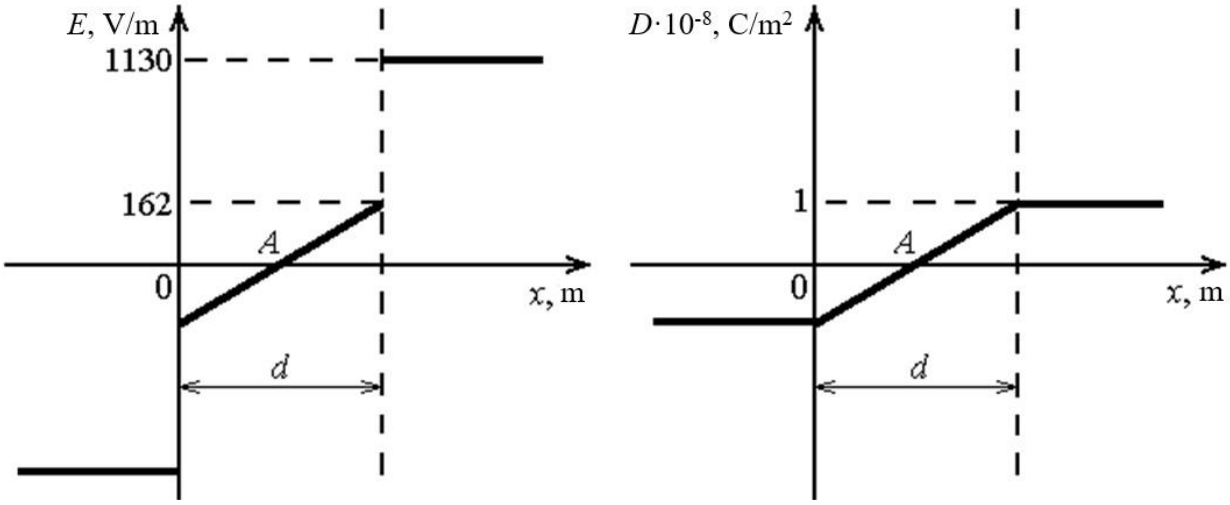


Figure 6

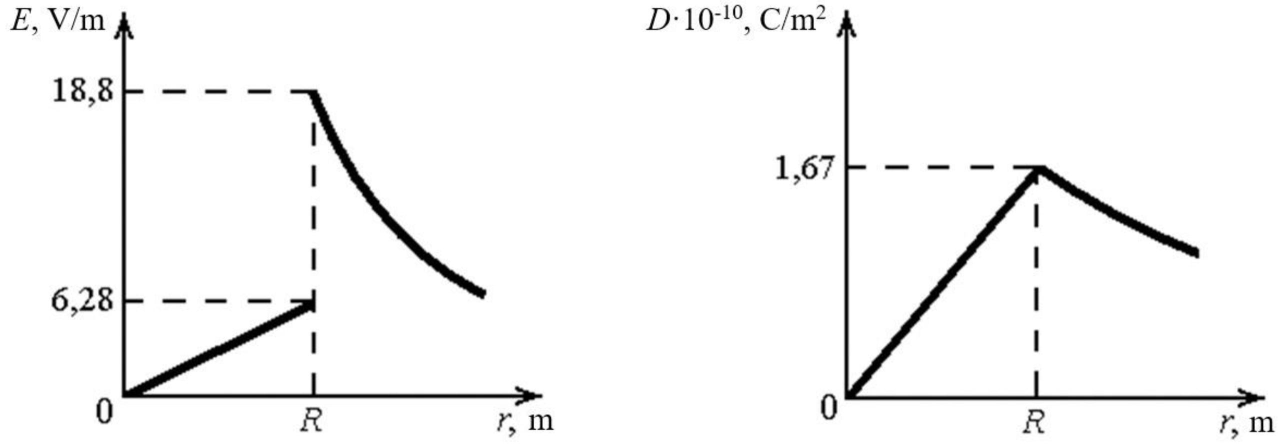


Figure 7

3.1. φ = 1 kV. **3.2.** A_1 = – 4 μJ, $\Delta\varphi$ = 200 V. **3.3.** $\Delta\Pi/Q_2$ = – 162 J/C. **3.4.** A = 4.5 μJ. **3.5.** φ = 45 V. **3.6.** φ = 6 kV, d_{min} = 10 cm, d_{max} = 40 cm. **3.7.** E = 664 kV/m, φ = 26.4 kV. **3.8.** Π = 90 μJ. **3.9.** Π = – 63 μJ. **3.10.** Π = 48.8 μJ. **3.11.** $(x - 10)^2 + y^2 = 64$. **3.12.** φ = 36.5 V. **3.13.** φ = 505 V. **3.14.** $\Delta\varphi = \dfrac{\tau}{2\pi\varepsilon_0}\ln\dfrac{r_2}{r_1} = 125$ V. **3.15.** φ = 33.6 V. **3.16.** N = $1.04\cdot10^9$. **3.17.** φ = 432 V. **3.18.** $\Delta\varphi$ = 56.6 V. **3.19.** U = 141 V. **3.20.** 1) φ = 360 V, 2) φ = 149 V. **3.21.** 1) φ = 75 V, 2) φ = 135 V, 3) φ = 100 V. **3.22.** φ = = $E\cdot R$ = 300 kV. **3.23.** $\Delta\varphi$ = 8.07 V. **3.24.** φ_1 = 238 V, $\varphi_2 = \varphi_3$ = 116 V. **3.25.** φ_1 = 472 V, φ_2 = 377 V, graph at Fig. 10. **3.26.** $\rho = 6\varepsilon_0 ax$.

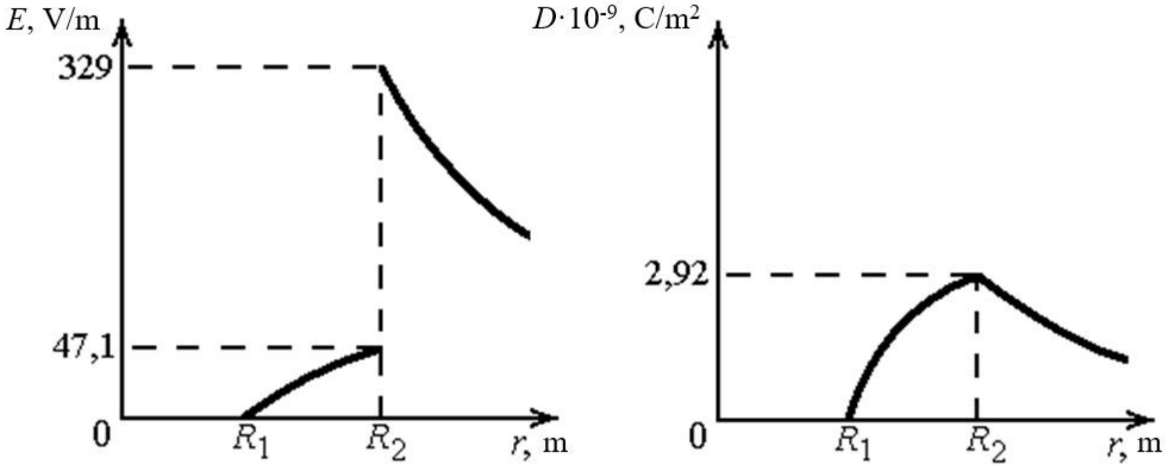


Figure 8

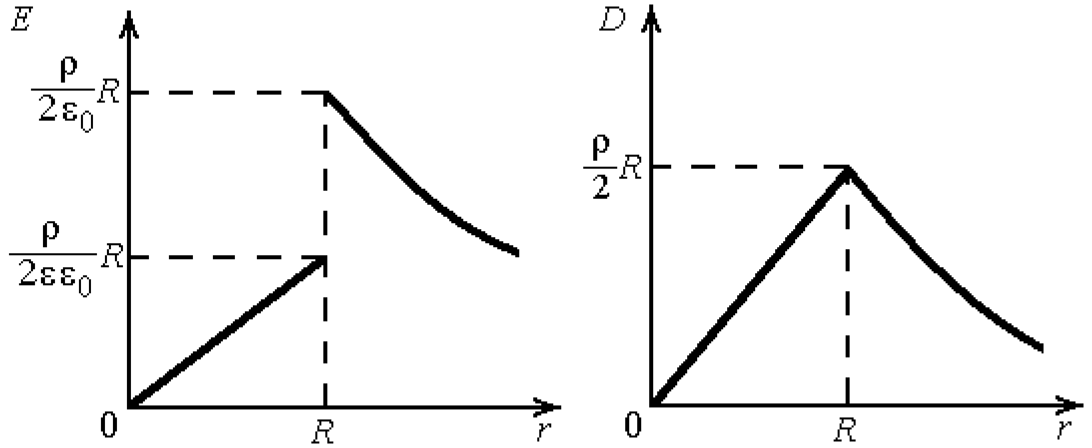


Figure 9

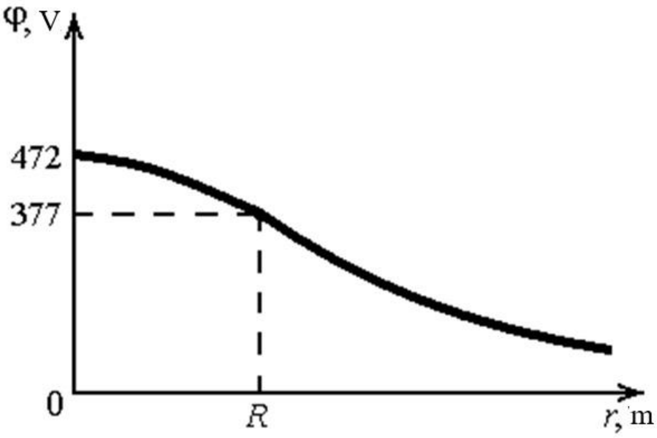


Figure 10

4.1. $\left|\text{grad}\,\varphi\right| = -\vec{E}$; $\left|\text{grad}\,\varphi\right| = \vec{E} = 226\,\text{V/m}$, gradient is directed perpendicular to the plane. **4.2.** U = 0.6 V. **4.3.** U = 0.12 V. **4.4.** $\left|\text{grad}\,\varphi\right| = \dfrac{\varphi}{r} =$

$= 200$ V/m, gradient is directed towards the charge. **4.5.** $\left|\operatorname{grad}\varphi\right| = \dfrac{\tau}{2\pi\varepsilon_0 r} =$ $= 180$ V/m, gradient is directed towards the thread along the field line. **4.6.** $\Delta\varphi = \dfrac{\rho R^2}{6\varepsilon_0\varepsilon} = 3.14$ V. **4.7.** a) $\varphi = \dfrac{q}{2\varepsilon_0\pi r^2}\left(\sqrt{r^2+x^2} - x\right)$, $E_x = \dfrac{q}{2\varepsilon_0\pi r^2}\left(1 - \dfrac{x}{\sqrt{r^2+x^2}}\right)$. At $|x| >> r$ the field due to the point charge: $\varphi = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|x|}$, $E_x = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{x^2}\dfrac{x}{|x|}$; b) $\varphi = 75$ kV, $E_x = 0.53$ MB/m. **4.8.** $\varphi = \dfrac{1}{4\pi\varepsilon_0}\int_V \dfrac{\rho(\vec{r})dV}{|\vec{r}' - \vec{r}|}$, $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\int_V \dfrac{\rho(\vec{r})(\vec{r}' - \vec{r})dV}{|\vec{r}' - \vec{r}|^3}$. **4.9.** $\vec{E} = -\vec{a}$, the field is uniform. **4.10.** a) $\vec{E} = -2a\,(x\vec{i} - y\vec{j})$, b) $\vec{E} = -a\,(y\vec{i} - x\vec{j})$. **4.11.** a) $\varphi =$ $= -axy$ + const, b) $\varphi = ay\,((y^2/3) - x^2)$ + const, c) $\varphi = -axy - byz$ + const. **4.12.** $A_1 = 8.91$ mJ, $A_2 = 9$ mJ. **4.13.** $A = Q\tau / 4\varepsilon_0 = 25.2$ μJ. **4.14.** $A = 47$ μJ. **4.16.** $A = 2.62$ μJ. **4.17.** $A_{1,2} = (Q_1\varphi_1)/3 = 1$ μJ. **4.18.** $A_{1,2} = 659$ μJ. **4.19.** $A_{1,2} =$ $= 165$ μJ. **4.20.** $l = 5.19$ mm. **4.21.** $l = 1$ cm. **4.22.** $\upsilon = 2.24$ Mm/c, makes an angle 45° with the initial direction. **4.23.** $\varphi_2 = 289$ V (m and e are the mass and the charge of the proton). **4.24.** $l = 2.13$ mm. **4.25.** $\upsilon_{min} = 0.24$ Mm/c (e/m is the specific charge of the electron). **4.26.** $\varphi_2 = 23.3$ V (m is the mass of the electron). **4.27.** $F = 2.4\cdot10^{-17}$ N, $a = 2.75\cdot10^{13}$ m/s^2, $\upsilon = 4.07$ Mm/s. **4.28.** $l_{min} =$ $= 5.9$ mm. **4.29.** $\Delta\varphi = 79.6$ V. **4.30.** $U = 22.5$ V. **4.31.** $r_{min} = 72$ fm. **4.32.** $r_{min} = \dfrac{e^2}{\pi\varepsilon_0 m\upsilon^2} = 10.1$ pm (m is the mass of the electron).

4.33. $r_{min} = \dfrac{Q_1Q_2(1 + m_1/m_2)}{2\pi\varepsilon_0 m_1(\upsilon_1 + \upsilon_2)^2}$; $r_1 = \dfrac{Q_1Q_2}{\pi\varepsilon_0 m_1(\upsilon_1 + \upsilon_2)^2}$; $r_2 = \dfrac{Q_1Q_2}{2\pi\varepsilon_0 m_1(\upsilon_1 + \upsilon_2)^2}$.

4.34. $T_1 = \dfrac{Q_1Q_2}{4\pi\varepsilon_0 r_0(1+k)}$; 1) $T_1 = \dfrac{Q_1Q_2}{8\pi\varepsilon_0 r_0}$; 2) $T_1 = \dfrac{Q_1Q_2}{4\pi\varepsilon_0 r_0}$; 3) $T_1 = 0$.

5.1. $E = 6.75$ kV/m; $\varphi = 0$. **5.2.** $\varphi_A = 0$, $\varphi_B = 385$ B, $E_A = 1.08$ kV/m, $E_B = 22$ kV/m. **5.3.** $\varphi = 1.8$ V, $E = 47.6$ V/m. **5.4.** $\varphi\,(t) = 90\cos\,(6.28\cdot10^3\cdot t +$ $+ \pi/2)$. **5.5.** $F = 1.35$ μN. **5.6.** $W_p = -18$ nJ. **5.7.** The electric dipole moment $\vec{p}$ of the neutral charge system does not depend on the choice of coordinate system, which determines this system. **5.8.** $A = 2pE$. **5.9.** a) $\vec{p} = 0$, b) $\vec{p} = 0$. **5.10.** $C = \dfrac{pE\sin\alpha}{\alpha} = 286$ nN·m/rad. **5.11.** $W_p = -500$ μJ. **5.12.** $A = 30$ μJ.

5.13. ν = 239 Hz. **5.14.** $F = p\dfrac{dE}{dx} = 0.2\,\text{mN}$. **5.15.** $\left|\dfrac{dE}{dx}\right| = 1.8\,\text{MB/м}^2$, F = 9 μN.
5.16. F = 2.1·10^{-16} N. **5.17.** a) $\vec{F} = 0$, b) $\vec{F} = -\dfrac{\lambda\vec{p}}{2\pi\varepsilon_0 r^2}$, c) $\vec{F} = \dfrac{\lambda\vec{p}}{2\pi\varepsilon_0 r^2}$.
5.18. $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{3p^2}{8d^4}$, the direction is from the dipole to the conductive plane.

6.1. ΔU = 22.6 V. **6.2.** C = 88.5 pF, 2) $D_1 = D_2$ = 2.66 μC/m^2, E_1 = = 42.8 kV/m, E_2 = 100 kV/m, $\Delta\varphi_1 = \Delta\varphi_1$ = 300 V. **6.3.** Δd = 0.5 cm. **6.4.** C = 35.4 pF. **6.5.** C = 2.5 μF. **6.6.** U = 700 V. **6.8.** U = 4.41 kV. **6.10.** ε_r = 13.2. **6.11.** 100 pF/m. **6.12.** 1) Q_1 = 360 μC, Q_2 = 720 μC, $U_1 = U_2$ = = 120 V; 2) $Q_1 = Q_2$ = 240 μC, U_1 = 80 V, U_2 = 40 V. **6.13.** ΔQ = 36 μC. **6.14.** d = 2.32 mm. **6.15.** $C_2 = \dfrac{U - U_1}{U_2 - U_1}C_1 = 0.32$ μF. **6.18.** ε_r = 5. **6.19.** Q_1 = = 200 μC, $Q_2 = Q_3$ = 120 μC, Q_4 = 100 μC, 110 V, 60 V, 40 V; Q_{tot} = 220 μC, $\Delta\varphi_1$ = 210 V. **6.20.** $C_4 = \dfrac{C_2C_3}{C_1} = 9\,\text{pF}$. **6.21.** W = 30 μJ. **6.22.** Q = 0.209 J. **6.23.** w = 2.5 J/m^3. **6.24.** A = 50 μJ. **6.25.** 1) U' = 1500 V, 2) A = 0.2 mJ. **6.26.** W = 23 mJ. **6.28.** A = 80 μJ. **6.29.** A = 63.5 nJ. **6.30.** 1) σ' = 5.9 nC/m^2, 2) W = 88.5 pJ. **6.31.** W = 118 pJ. **6.32.** $A = \dfrac{q^2\Delta x}{2\varepsilon_r\varepsilon_0 S} = 11.3\,\mu\text{J}$. **6.33.** a) w increases ε_r times, b) ω decreases ε_r times. **6.34.** ΔP = 7 kPa. **6.35.** $h = \dfrac{\sigma^2}{2\varepsilon_r\varepsilon_0\rho g}(\varepsilon_r - 1)$. **6.36.** W = 30 μJ. **6.37.** W = 225 μJ. **6.38.** R_2 = = 12 cm. **6.39.** W_1 = 7.88 nJ, W_2 = 78.8 nJ. **6.40.** W_1/W_2 = 15. **6.41.** a) $W = \dfrac{1}{2}\left(\dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{r}\right) = 4.5\,\text{nJ}$, b) η = 0.99, c) R = 2cm. **6.42.** A = 0.9 nJ. **6.43.** W = = 27 mJ. **6.44.** $W = \dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_1^2}{2R_1} + \dfrac{q_2^2}{2R_2} + \dfrac{q_1q_2}{R_2}\right)$. **6.45.** $A = \dfrac{q^2(R_2 - R_1)}{8\pi\varepsilon_0 R_1R_2}$.
6.46. $F_x = \dfrac{U^2R\pi\varepsilon_0(\varepsilon_r - 1)}{d}$.

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APPENDICES

Appendix A

Tables of homework variants*

Table A.1

COULOMB'S LAW. INTERACTION OF CHARGED BODIES

Variant	Problem number			
0	1.1	1.18	1.22	1.29
1	1.2	1.17	1.20	1.25
2	1.3	1.16	1.19	1.26
3	1.4	1.15	1.23	1.21
4	1.5	1.14	1.24	1.28
5	1.6	1.13	1.19	1.27
6	1.7	1.12	1.20	1.26
7	1.8	1.11	1.22	1.25
8	1.9	1.18	1.23	1.29
9	1.10	1.17	1.20	1.26

Table A.2

ELECTRIC FIELD.
ELECTRIC DISPLACEMENT FIELD

Variant	Problem number				
0	2.1	2.18	2.24	2.33	2.49
1	2.2	2.17	2.25	2.34	2.48
2	2.3	2.16	2.26	2.35	2.50
3	2.4	2.15	2.27	2.36	2.51
4	2.5	2.14	2.28	2.37	2.52
5	2.6	2.13	2.29	2.38	2.53
6	2.7	2.12	2.31	2.39	2.54
7	2.8	2.11	2.22	2.40	2.55
8	2.9	2.10	2.23	2.35	2.56
9	2.1	2.17	2.32	2.34	2.57

*Variant number is the number of the last digit of the student's card

Table A.3

ELECTRIC POTENTIAL. ENERGY OF THE SYSTEM OF ELECTRIC CHARGES. POTENTIAL GRADIENT. WORK IN MOVING AN ELECTRIC CHARGE. MOTION OF CHARGED PARTICLES IN ELECTRIC FIELD

Variant	Problem number				
0	3.1	3.15	3.23	4.12	4.20
1	3.2	3.14	3.24	4.13	4.21
2	3.3	3.13	3.25	4.14	4.22
3	3.4	3.12	3.26	4.15	4.23
4	3.5	3.21	4.10	4.16	4.24
5	3.6	3.20	4.9	4.17	4.25
6	3.7	3.19	4.8	4.18	4.26
7	3.8	3.18	4.7	4.19	4.27
8	3.9	3.17	4.6	4.17	4.28
9	3.10	3.16	4.5	4.18	4.29

Table A.4

ELECTRIC DIPOLE

Variant	Problem number			
0	5.1	5.9	5.10	5.18
1	5.2	5.8	5.11	5.17
2	5.3	5.7	5.12	5.16
3	5.4	5.9	5.13	5.18
4	5.5	5.8	5.14	5.17
5	5.6	5.9	5.15	5.16
6	5.1	5.8	5.10	5.18
7	5.2	5.7	5.11	5.17
8	5.3	5.9	5.12	5.16
9	5.4	5.8	5.13	5.18

Table A.5

CAPACITANCE, CAPACITORS. ENERGY STORED IN A CHARGED CONDUCTOR. ENERGY STORED IN ELECTRIC FIELD

Variant	Problem number			
0	6.1	6.17	6.23	6.33
1	6.2	6.16	6.24	6.44
2	6.3	6.15	6.25	6.34
3	6.4	6.14	6.26	6.43
4	6.5	6.13	6.18	6.35
5	6.6	6.12	6.19	6.42
6	6.7	6.11	6.20	6.36
7	6.8	6.10	6.21	6.41
8	6.1	6.9	6.22	6.37
9	6.2	6.17	6.27	6.40

Appendix B

Tables of physical quantities

Table B.1

Prefixes and multipliers for the formation of multiple and fractional units

Name	Symbol	Multiplier	Name	Symbol	Multiplier
Exa	E	10^{18}	deci	d	10^{-1}
Peta	P	10^{15}	centi	c	10^{-2}
Tera	T	10^{12}	milli	m	10^{-3}
Giga	G	10^{9}	micro	μ	10^{-6}
Mega	M	10^{6}	nano	n	10^{-9}
Kilo	k	10^{3}	pico	p	10^{-12}
Hecto	h	10^{2}	femto	f	10^{-15}
Deca	da	10^{1}	atto	a	10^{-18}

Table B.2

Fundamental physical constants

Name	Symbol	Numerical value
Electric constant	ε_0	$8.854 \cdot 10^{-12}$ F·m^{-1}
Gravitational constant	G	$6.672 \cdot 10^{-11}$ m^3·kg^{-1}·s^{-2}
Elementary charge	e	$1.602 \cdot 10^{-19}$ C
Electron rest mass	m_e	$9.109 \cdot 10^{-31}$ kg
Proton rest mass	m_p	$1.673 \cdot 10^{-27}$ kg
Neutron rest mass	m_n	$1.675 \cdot 10^{-27}$ kg
Gravitational acceleration	g	9.807 m/c^2
Specific charge of an electron	e/m_e	$1.76 \cdot 10^{11}$ C/kg

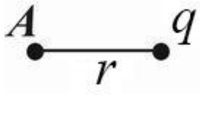
Table B.3

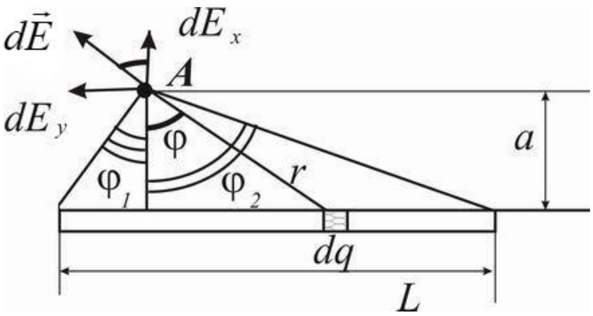
Relative permittivity ε_r

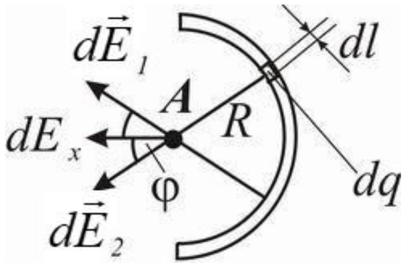
Material	ε_r	Material	ε_r
Water	81	Transformer oil	2.2
Wax paper	3.7	Paraffin	2
Kerosene	2	Glass	5.5...10
Ebonite	2.6	Mica	6
Quartz	2.7	Ceramics	6

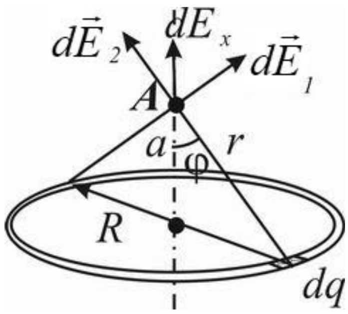
Appendix C

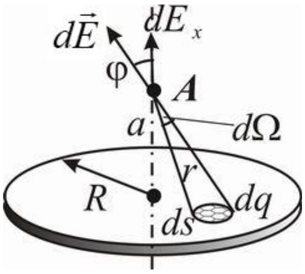
Scheme of gradual increase in complexity in the formation of the concept of electric field

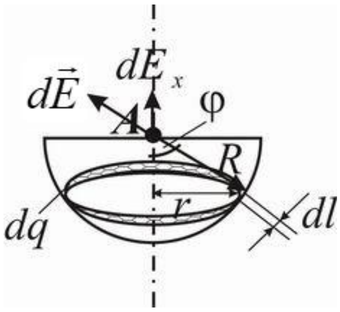
	Example 1. Material point
Example	Material **point** of charge q at a distance r from the point A
Figure	
Magnitude of electric field	$E = k \cdot \frac{q}{r^2}$

	Example 2. Thin thread
Example	A thin **thread** of length L carrying charge Q uniformly distributed along the length. Point A is located at a distance a from the thread
Figure	
Element	A **point** of infinitesimal length dl and charge dq
Elementary magnitude of electric field in projections onto the coordinate axes	$dE_x = k\dfrac{dq}{r^2}\cos\varphi;\quad dE_y = k\dfrac{dq}{r^2}\sin\varphi$
Elementary charge	$dq = \tau dl = \dfrac{Q}{L}dl$
Components of electric field	$dE_x = k\dfrac{Q}{L}\cdot\dfrac{dl}{r^2}\cos\varphi = \left\vert\begin{matrix} l = a\cdot tg\varphi \\ r = a/\cos\varphi \end{matrix}\right\vert = k\dfrac{Q}{L}\cdot\dfrac{d\varphi}{a}\cos\varphi$ $dE_y = k\dfrac{Q}{L}\cdot\dfrac{dl}{r^2}\sin\varphi = k\dfrac{Q}{L}\cdot\dfrac{d\varphi}{a}\sin\varphi$
Calculation formula	$E_x = \int_{\varphi_1}^{\varphi_2} k\dfrac{Q}{L}\cdot\dfrac{\cos\varphi}{a}d\varphi;\quad E_y = \int_{\varphi_1}^{\varphi_2} k\dfrac{Q}{L}\cdot\dfrac{\sin\varphi}{a}d\varphi;$ $E = \sqrt{E_x^2 + E_y^2}$
Result (Particular cases)	1) $\begin{matrix}\varphi_1 = -\pi/2 \\ \varphi_2 = \pi/2\end{matrix}$ $(L \to \infty)$ $\quad E = 2k\dfrac{\tau}{a}$ 2) $\begin{matrix}\varphi_1 = 0 \\ \varphi_2 = \pi/2\end{matrix}$ $(L \to \infty)$ $\quad E = \sqrt{2}k\dfrac{\tau}{a}$ 3) $\begin{matrix}\varphi_1 = -\varphi_0 \\ \varphi_2 = \varphi_0\end{matrix}$ $tg\varphi_0 = \dfrac{L}{2a}$ $\quad E = 2k\dfrac{Q}{La}\sin\varphi_0$ 4) $\begin{matrix}\varphi_1 = 0 \\ \varphi_2 = \varphi_0\end{matrix}$ $tg\varphi_0 = \dfrac{L}{a}$ $\quad E = 2k\dfrac{Q}{La}\sin\dfrac{\varphi_0}{2}$

Example 3. Thin semicircle	
Example	A thin **semicircle** of radius R carrying charge Q uniformly distributed along the length. Point A is located in the geometric center of the semicircle
Figure	
Element	A **point** of infinitesimal length dl and charge dq
Elementary magnitude of electric field in projections onto the coordinate axes	$dE_x = k\dfrac{dq}{R^2}\cos\varphi$; $dE_y = k\dfrac{dq}{R^2}\sin\varphi$
Elementary charge	$dq = \tau dl = \dfrac{Q}{\pi R}dl = \dfrac{Q}{\pi R}\cdot Rd\varphi$
Components of electric field	$dE_x = k\dfrac{Q}{\pi R}\cdot\dfrac{Rd\varphi}{R^2}\cos\varphi$; $dE_y = k\dfrac{Q}{\pi R}\cdot\dfrac{Rd\varphi}{R^2}\sin\varphi$
Calculation formula	$E_x = \int\limits_{-\pi/2}^{\pi/2} k\dfrac{Q}{\pi}\cdot\dfrac{\cos\varphi}{R^2}d\varphi$; $E_y = \int\limits_{-\pi/2}^{\pi/2} k\dfrac{Q}{\pi}\cdot\dfrac{\sin\varphi}{R^2}d\varphi$; $E = \sqrt{E_x^2 + E_y^2}$
Result	$E = 2k\dfrac{Q}{\pi R^2}$

	Example 4. Thin ring
Example	A thin **ring** of radius *R* carrying charge *Q* uniformly distributed along the length. Point *A* is located on the axis of the ring at a distance *a* from its plane
Figure	
Element	A **point** of infinitesimal length *dl* and charge *dq*
Elementary magnitude of electric field in projections onto the coordinate axes	$dE_x = k\dfrac{dq}{r^2}\cos\varphi;\;\; dE_y = k\dfrac{dq}{r^2}\sin\varphi$
Elementary charge	$dq = \tau dl = \dfrac{Q}{2\pi R}dl$
Components of electric field	$dE_x = k\dfrac{Q}{2\pi R}\cdot\dfrac{dl}{r^2}\cos\varphi;\;\; dE_y = k\dfrac{Q}{2\pi R}\cdot\dfrac{dl}{r^2}\sin\varphi$
Calculation formula	$E_x = \int\limits_{(l)} k\dfrac{Q}{2\pi R}\cdot\dfrac{dl}{r^2}\cos\varphi;\;\; E_y = \int\limits_{(l)} k\dfrac{Q}{2\pi R}\cdot\dfrac{dl}{r^2}\sin\varphi;$ $E = \sqrt{E_x^2 + E_y^2}$
Result	$E = k\dfrac{Q}{r^2} = k\dfrac{Q}{(R^2 + a^2)}$

	Example 5. Thin disk
Example	A thin **disk** of radius *R* carrying charge *Q* uniformly distributed over the surface. Point *A* is located on the axis of the disk at a distance *a* from its plane, and the disk is visible from that point at a solid angle Ω
Figure	
Element	An infinitely thin **ring** of radius *r* and charge *dq*
Elementary magnitude of electric field in projections onto the coordinate axes	$dE_x = k\frac{dq}{r^2}\cos\varphi$
Elementary charge	$dq = \sigma ds = \frac{Q}{\pi R^2}ds$
Components of electric field	$dE_x = k\frac{Q}{\pi R^2}\cdot\frac{ds}{r^2}\cos\varphi = \left\lvert\frac{ds}{r^2}\cos\varphi = d\Omega\right\rvert = k\frac{Q}{\pi R^2}\cdot d\Omega$
Calculation formula	$E_x = \int_0^{\Omega} k\frac{Q}{\pi R^2}\cdot d\Omega$
Result (Particular cases)	$E = k\frac{Q}{\pi R^2}\cdot\Omega$ 1) $a\to\infty \Rightarrow \Omega = \frac{S}{a^2} = \frac{\pi R^2}{a^2}$ $\quad E = k\frac{Q}{a^2}$ 2) $a\to 0 \Rightarrow \Omega = 2\pi$ $\quad E = k\frac{2Q}{R^2}$

	Example 6. Hemisphere
Example	A **hemisphere** of radius R carrying charge Q uniformly distributed over the surface. Point A is located in the center of the hemisphere
Figure	
Element	An infinitely thin **ring** of radius r and charge dq
Elementary magnitude of electric field in projections onto the coordinate axes	$$dE_x = k\frac{dq}{R^2}\cos\varphi;\;\; dE_y = k\frac{dq}{R^2}\sin\varphi$$
Elementary charge	$$dq = \sigma ds = \frac{Q}{2\pi R^2}ds = \frac{Q}{2\pi R^2}\cdot 2\pi r dl = \lvert r = R\sin\varphi\rvert = = \frac{Q}{2\pi R^2}\cdot 2\pi R\sin\varphi\cdot R d\varphi$$
Components of electric field	$$dE_x = k\frac{Q}{2\pi R^2}\cdot\frac{2\pi R^2\sin\varphi\cos\varphi d\varphi}{R^2}$$
Calculation formula	$$E_x = \int_0^{\pi/2} k\frac{Q}{R^2}\cdot\sin\varphi\cos\varphi d\varphi$$
Result	$$E = k\frac{Q}{2R^2}$$

Appendix D
Comparison of the DI-method based calculation of the moment of inertia and the electric field

Table D.1

A point body

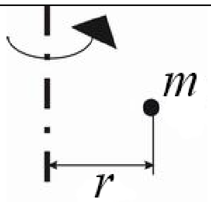
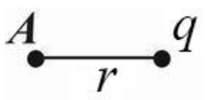
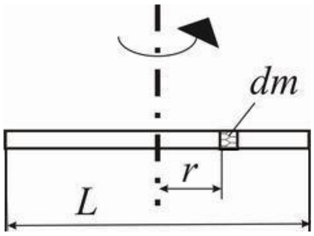
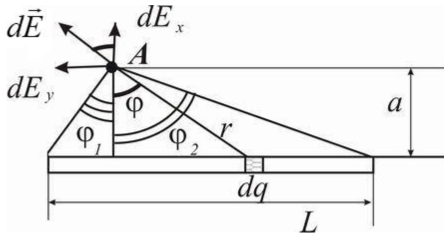
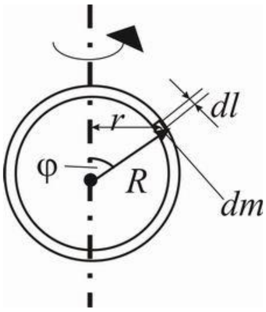
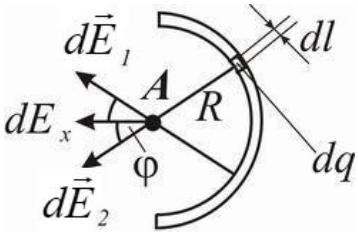
	Moment of inertia $I = \int\limits_{(m)} r^2 dm$	**Magnitude of electric field** $E_n = k \int\limits_{(q)} \frac{dq}{r^2} \cdot e_{\vec{n}}$
Example	A material **point** of mass m located at a distance r from the axis of rotation	A material **point** of charge q located at a distance r from the point A
Figure		
Compared quantities	Mass m	Charge q
Final formula	$I = mr^2$	$E = k \cdot \frac{q}{r^2}$

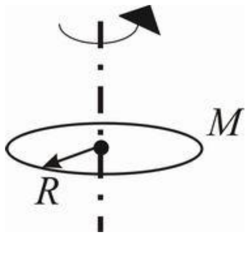
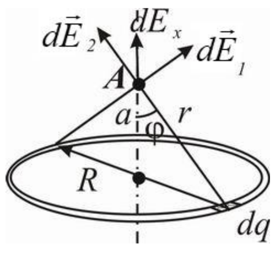
Table D.2

A linear body

	Moment of inertia $I = \int\limits_{(m)} r^2 dm$	**Magnitude of electric field** $E_n = k \int\limits_{(q)} \frac{dq}{r^2} \cdot e_{\vec{n}}$
		Example 1. A thin rod
Example	A thin **rod** of length L and mass M. The axis of rotation is perpendicular to the rod and passes through its middle	A thin **thread** of length L carrying charge Q uniformly distributed along the length. Point A is located at a distance a from the thread

Figure		
Element	An infinitesimal **point** of mass *dm* located at a distance *r* from the axis of rotation	A **point** of infinitesimal length *dl* and charge *dq*
Compared elementary quantities	$dm = \tau dl = \frac{M}{L} \cdot dr$	$dq = \tau dl = \frac{Q}{L} dl$
General formula	$dI = r^2 dm = r^2 \cdot \frac{M}{L} \cdot dr$	$dE_n = k \cdot \frac{dq}{r^2} \cdot e_{\vec{n}} = k \frac{Q}{L} \cdot \frac{dl}{r^2} \cdot e_{\vec{n}} =$ $= k \frac{Q}{L} \cdot \frac{d\varphi}{a} \cdot e_{\vec{n}};$ $e_x = \cos\varphi; \quad e_y = \sin\varphi$
Calculation formula	$I = \int_{-L/2}^{L/2} \frac{M}{L} \cdot r^2 dr$	$E_x = \int_{\varphi_1}^{\varphi_2} k \frac{Q}{L} \cdot \frac{\cos\varphi}{a} d\varphi;$ $E_y = \int_{\varphi_1}^{\varphi_2} k \frac{Q}{L} \cdot \frac{\sin\varphi}{a} d\varphi;$ $E = \sqrt{E_x^2 + E_y^2}$
Result (Particular cases)	$I = \frac{1}{12} MR^2$	1) $\varphi_1 = -\pi/2$, $\varphi_2 = \pi/2$ $(L \to \infty)$ $E = 2k\frac{\tau}{a}$ 2) $\varphi_1 = 0$, $\varphi_2 = \pi/2$ $(L \to \infty)$ $E = \sqrt{2}k\frac{\tau}{a}$

		3) $\begin{matrix}\varphi_1 = -\varphi_0 \\ \varphi_2 = \varphi_0\end{matrix}$ $tg\varphi_0 = \dfrac{L}{2a}$ $E = 2k\dfrac{Q}{La}\sin\varphi_0$ 4) $\begin{matrix}\varphi_1 = 0 \\ \varphi_2 = \varphi_0\end{matrix}$ $tg\varphi_0 = \dfrac{L}{a}$ $E = 2k\dfrac{Q}{La}\sin\dfrac{\varphi_0}{2}$
Example 2. A thin ring, the axis coincides with one of the diameters		
Example	A thin **ring** of radius R and mass M. The axis of rotation coincides with one of the diameters	A thin **semicircle** of radius R carrying charge Q uniformly distributed along the length. Point A is located in the geometric center of the semicircle
Figure		
Element	An infinitesimal **point** of mass dm located at a distance r from the axis of rotation	A **point** of infinitesimal length dl and charge dq
Compared elemen-tary quantities	$dm = \tau dl = \dfrac{M}{2\pi R}\cdot dl =$ $= \dfrac{M}{2\pi R}\cdot Rd\varphi$	$dq = \tau dl = \dfrac{Q}{\pi R}dl =$ $= \dfrac{Q}{\pi R}\cdot Rd\varphi$

General formula	$dI = r^2 dm = r^2 \cdot \dfrac{M}{2\pi} \cdot d\varphi$	$dE_n = k \cdot \dfrac{dq}{r^2} \cdot e_{\vec{n}} =$ $= k \dfrac{Q}{\pi R} \cdot \dfrac{Rd\varphi}{R^2} \cdot e_{\vec{n}};$ $e_x = \cos\varphi;\quad e_y = \sin\varphi$
Calcula-tion formula	$I = \int\limits_0^{2\pi} \dfrac{M}{2\pi} \cdot r^2 d\varphi = \lvert r = R\sin\varphi \rvert =$ $= \dfrac{M}{2\pi} \int\limits_0^{2\pi} R^2 \sin^2 \varphi d\varphi$	$E_x = \int\limits_{-\pi/2}^{\pi/2} k \dfrac{Q}{\pi} \cdot \dfrac{\cos\varphi}{R^2} d\varphi;$ $E_y = \int\limits_{-\pi/2}^{\pi/2} k \dfrac{Q}{\pi} \cdot \dfrac{\sin\varphi}{R^2} d\varphi;$ $E = \sqrt{E_x^2 + E_y^2}$
Result	$I = \dfrac{1}{2} MR^2$	$E = 2k \dfrac{Q}{\pi R^2}$
Example 3. A thin ring, the axis is perpendicular to the plane of the ring		
Example	An infinitely thin **ring** of radius *R* and mass *M*. The axis of rotation passes through the center of the ring perpendicular to its plane	A thin **ring** of radius *R* carrying charge *Q* uniformly distributed along the length. Point *A* is located on the axis of the ring at a distance *a* from its plane
Figure		
Element	An infinitesimal **point** of mass *dm* located at a distance *r* from the axis of rotation	A **point** of infinitesimal length *dl* and charge *dq*
Compared elemen-tary quantities	$dm = \tau dl = \dfrac{M}{2\pi R} \cdot dl$	$dq = \tau dl = \dfrac{Q}{2\pi R} dl$

General formula	$dI = r^2 dm = r^2 \dfrac{M}{2\pi R} dl$	$dE_n = k \cdot \dfrac{dq}{r^2} \cdot e_{\vec{n}} = k \dfrac{Q}{2\pi R} \cdot \dfrac{dl}{r^2} \cdot e_{\vec{n}}$; $e_x = \cos\varphi$; $e_y = \sin\varphi$
Calculation formula	$I = \int\limits_{(l)} r^2 \dfrac{M}{2\pi R} dl$	$E_x = \int\limits_{(l)} k \dfrac{Q}{2\pi R} \cdot \dfrac{dl}{r^2} \cos\varphi$; $E_y = \int\limits_{(l)} k \dfrac{Q}{2\pi R} \cdot \dfrac{dl}{r^2} \sin\varphi$; $E = \sqrt{E_x^2 + E_y^2}$
Result	$I = MR^2$	$E = k \dfrac{Q}{r^2} = k \dfrac{Q}{(R^2 + a^2)}$

Table D.3

A planar body

	Moment of inertia	Magnitude of electric field
	$I = \int\limits_{(m)} r^2 dm$	$E_n = k \int\limits_{(q)} \dfrac{dq}{r^2} \cdot e_{\vec{n}}$
		Example 1. A thin disk
Example	A thin **disk** of radius *R* and mass *M*. The axis of rotation is perpendicular to the plane of the disk and passes through its center	A thin **disk** of radius *R* carrying charge *Q* uniformly distributed over the surface. Point *A* is located on the axis of the disk at a distance *a* from its plane, and the disk is visible from that point at a solid angle Ω

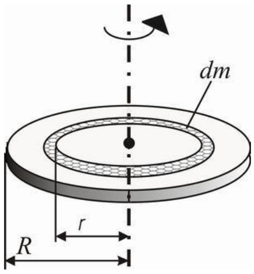
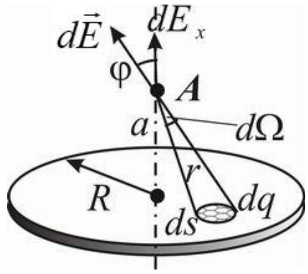
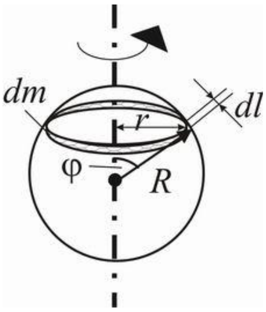
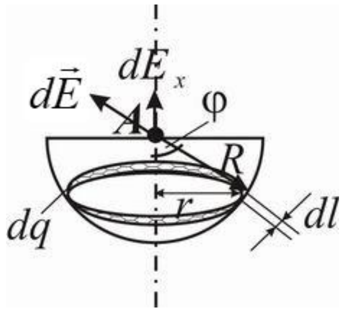
Figure		
Element	An infinitely thin **ring** of mass *dm* and radius *r*	An infinitely thin **ring** of charge *dq* and radius *r*
Compared elemen-tary quantities	$dm = \sigma ds = \dfrac{M}{\pi R^2} \cdot 2\pi r dr$	$dq = \sigma ds = \dfrac{Q}{\pi R^2} ds$
General formula	$dI = r^2 dm = r^2 \cdot \dfrac{M}{\pi R^2} \cdot 2\pi r dr$	$dE_x = k \cdot \dfrac{dq}{r^2} \cdot \cos\varphi =$ $= k \dfrac{Q}{\pi R^2} \cdot \dfrac{ds}{r^2} \cos\varphi =$ $= k \dfrac{Q}{\pi R^2} \cdot d\Omega$
Calcula-tion formula	$I = \int_0^R \dfrac{2M}{R^2} \cdot r^3 dr$	$E_x = \int_0^\Omega k \dfrac{Q}{\pi R^2} \cdot d\Omega$
Result (Particular cases)	$I = \dfrac{1}{2} MR^2$	$E = k \dfrac{Q}{\pi R^2} \cdot \Omega$ 1) $a \to \infty \Rightarrow \Omega = \dfrac{S}{a^2} = \dfrac{\pi R^2}{a^2}$ 2) $a \to 0 \Rightarrow \Omega = 2\pi$
Example 2. Spherical surface		
Example	A **spherical surface** of radius *R* and mass *M*. The axis of rotation coincides with one of the diameters	A **hemisphere** of radius *R* carrying charge *Q* uniformly distributed over the surface. Point *A* is located in the center of the hemisphere

Figure		
Element	An infinitely thin **ring** of mass *dm* and radius *r*	An infinitely thin **ring** of charge *dq* and radius *r*
Compared elemen-tary quantities	$dm = \sigma ds = \frac{M}{4\pi R^2} \cdot 2\pi r dl =$ $= \frac{M}{4\pi R^2} \cdot 2\pi R \sin\varphi \cdot R d\varphi$	$dq = \sigma ds = \frac{Q}{2\pi R^2} ds =$ $= \frac{Q}{2\pi R^2} \cdot 2\pi r dl =$ $= \frac{Q}{2\pi R^2} \cdot 2\pi R \sin\varphi \cdot R d\varphi$
General formula	$dI = r^2 \cdot \frac{M}{4\pi R^2} \cdot 2\pi R^2 \sin\varphi d\varphi$	$dE_x = k \cdot \frac{dq}{r^2} \cdot \cos\varphi =$ $= k \frac{Q}{2\pi R^2} \cdot \frac{2\pi R^2 \sin\varphi \cos\varphi d\varphi}{R^2}$
Calcula-tion formula	$I = \int_0^{\pi} \frac{M}{2} \cdot r^2 \sin\varphi d\varphi =$ $= \frac{M}{2} \int_0^{\pi} R^2 \sin^3\varphi d\varphi$	$E_x = \int_0^{\pi/2} k \frac{Q}{R^2} \cdot \sin\varphi \cos\varphi d\varphi$
Result	$I = \frac{2}{3} MR^2$	$E = k \frac{Q}{2R^2}$