BEST WEIGHTED APPROXIMATION OF SOME KERNELS ON THE REAL AXIS

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We calculate the exact value and find the polynomial of the best weighted polynomial approximation of kernels of the form

$$\mathcal{K}_{\lambda,s}(t,A,B) := \frac{A+Bt}{(t^2+\lambda^2)^{s+1}}, \tag{1}$$

where A and B are fixed complex numbers, $\lambda > 0$, $s \in \mathbb{N}$, in the mean square metric.

Let us note, for $A = s = 0$ and $B = \frac{1}{\pi}$, the kernel (1) coincides with the Poisson kernel $P_t(\lambda) = \frac{1}{\pi}\frac{t}{\lambda^2+t^2}$ as function of the variable λ. A special case of kernels (1) are integral kernels defined in the upper half-plane by the well-known biharmonic Poisson integrals

$$\mathcal{B}(f;t;\lambda) := \frac{2\lambda^3}{\pi}\int_{-\infty}^{\infty} \frac{f(x+t)}{(x^2+\lambda^2)^2}\,\mathrm{d}x.$$

which give the solution of the biharmonic equation

$$\nabla^2(\nabla^2 U) = 0, \ \nabla := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \lambda^2},$$

in the upper half-plane of the complex plane ($\lambda > 0$) under the boundary conditions

$$\lim_{\lambda\to 0+} U(x,\lambda) = f(x), \ \lim_{\lambda\to 0+} \frac{\partial}{\partial\lambda} U(x,\lambda) = 0.$$

Denote by P_{n-1} the set of all algebraic polynomials with complex coefficients of degree at most $n-1$ and consider the quantity

$$\mathcal{E}_n(\mathcal{K}_{\lambda,s})_{2,\rho_n} := \inf_{p\in P_{n-1}} \Big(\int_{-\infty}^{\infty} \left| \frac{A+Bt}{(t^2+\lambda^2)^{s+1}} - p(t) \right|^2 \frac{\mathrm{d}t}{|\rho_n(t)|^2} \Big)^{1/2}$$

of best weighted approximation of the kernel $\mathcal{K}_{\lambda,s}(t,A,B)$ by all possible polynomials from the set P_{n-1} in the mean square metric with the weight $\frac{1}{|\rho_n(t)|^2}$, $\rho_n(t) = \rho_0 \prod_{k=1}^{n}(t-a_k)$, where $a_k = \alpha_k + \mathrm{i}\beta_k$, $\beta_k > 0$, $k = 1,2,\ldots,n$, and $\rho_0 \neq 0$ is a constant.

Theorem 1. *Let A, B be any fixed real numbers, $\lambda > 0$ and $s \in \mathbb{N}$. Then for any $n \in \mathbb{N}$*

$$\mathcal{E}_n^2(\mathcal{K}_{\lambda,s})_{2,\rho_n} = \frac{4\pi}{(2\lambda)^{2s+3}\mu_n(\lambda,\mathbf{a})|\rho_0|^2}\sum_{k=0}^{s}\sum_{l=0}^{s}\binom{l+k}{l}G_k\overline{G}_l,$$

where $\mu_n(\lambda,\mathbf{a}) = \prod_{k=1}^{n}\left[\alpha_k^2+(\beta_k+\lambda)^2\right]$, $a_j = \alpha_j + \mathrm{i}\beta_j$, $\beta_j > 0$, $j = 1, 2, \ldots, n$,

$$G_k = G_k(\lambda,\mathbf{a}) = \sum_{j=0}^{s-k}\frac{\binom{s+j}{s}\mathrm{i}^{s-k-j}\nu_{s-k-j}(\mathrm{i}\lambda,\bar{\mathbf{a}})}{(2\lambda)^{k+j}}\left(\frac{B\lambda(s(s+j)-2j)}{s(s+j)} - \mathrm{i}A\right).$$

Similar extremal problems for the kernels (1) was solved: in the case of $s = 0$ in [1] (see, also [2]), in the case of $s = 1$ in [3].

References

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[2] Ahiezer N.I. *Lectures on the theory of approximation.* — New York, 1956. — 326 p.
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