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LIE SYMMETRIES OF THE FOKKER-PLANCK EQUATION WITH POWER DIFFUSIVITY

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Abstract

We study Lie symmetries of the class of Fokker-Planck equations with power diffusivity, parametrized by real parameter, which describe the growth of the cell population. The kernel invariance algebra and a complete list of its extensions are obtained. Non-trivial equivalence transformation is constructed. Structural properties of the maximal invariance algebras are discussed.

Keywords: Lie symmetries, Fokker-Planck equation.

Introduction

The problem of cell population evolution plays a significant role in biology, medicine, especially in the application of treating tumor diseases. A fundamental understanding of the process of tumor evolution could be derived with help of mathematical and physical methods. The main experimental ideas of tumor researches were embodied in so-called two–compartment models.

Extensive studying of two–compartment models [1] has led to a deep understanding of the process of a proliferation cell population. The main idea of this model is that cells could exist in one of two states: proliferative state and quiescent state. In the proliferative state, cell is undergoing active reproduction. In general theory [2], the cells could be transferred to the quiescent state after finishing the proliferative state. In the [2], Rottenberg considered that quiescent state is distributed throughout the whole proliferative state (thus, the cell could be dropped out of the proliferative state in any moment with probability $\propto dy$, where y is a degree of cell maturity). This model is particularly relevant to solid tumors, where there is reasonable evidence that cells drop out of cycle throughout the cycle.

A tremendously difficult sequence of the reasoning and the generalization of experimental data, which were made in the [2, 3], has led to derivation of the main equation for cell number density, which we describe below.

Let y be the independent variable that describes the degree of maturity. The maturation rate $x = \mathrm{d}y/\mathrm{d}t$ is also considered to be independent non-negative variable. Since a large number of individuals is being considered, the dependent variable is contemplated to be cell number density $v(t, x, y)$. Here and further we use the notation for dependent and independent variables, which are common in the theory of partial differential equations and do not relate to the physical interpretation of that variables.

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The number density $v(t, x, y)$ is the number of cells between y and $y + dy$ that is maturing at velocity x at the time t. It is governed by the equation [3]

$$\frac{\partial v}{\partial t} + x\frac{\partial v}{\partial y} = -\lambda v + \sum_k r_i(y, x_j, x_k)v(y, x_k, t) - \\ - v\sum_k r_0(y, x_k, x_j), \qquad (1)$$

where λ is the death-rate velocity, r_i the rate that cells change their velocities from any x_k to specific x_j under consideration, that is, into the cohort characterized by x_j; r_0 is the rate at which cells drop out of the cohort with x_j to join the cohort x_k.

In case of continuous x (which we are precisely interested in) the equation (1) could be rewritten in the following form:

$$\frac{\partial v}{\partial t} + x\frac{\partial v}{\partial y} = r\int_0^\infty r_i(y; x, x')v(y, x', t)\mathrm{d}x' - \\ - v\int_0^\infty r_0(y; x', x)\mathrm{d}x', \qquad (2)$$

where the death set to be zero. If $r(y; x, x') = r(x-x')$ is sharply peaked about $x-x'$, the classical Fokker-Planck equation for the Green's function of the operator (2) can be obtained. If the transition rate is sharply peaked it means, that velocity changes slowly, making possible a Taylor expansion in the velocity variable the Green function $P_w(t, x, y)$ of the integrodifferential operator [3]

$$P_w(t, x', y) \simeq P_w(t, x, y) - (x - x')\frac{\partial P_w(t, x, y)}{\partial x} + \\ + \frac{1}{2}(x - x')^2\frac{\partial^2 P_w(t, x, y)}{\partial x^2}. \qquad (3)$$

After putting (3) to the (2) we can obtain the equation for the Green function and generalize it to the case of symmetric $r(x)$ [3]

$$\frac{\partial P_w}{\partial t} + x\frac{\partial P_w}{\partial y} = D(x)\frac{\partial^2 P_w}{\partial x^2}, \qquad (4)$$

where D depends on the x polynomially. The coefficient D plays the role of diffusion coefficient, and further we refer it as to diffusivity (or diffusion coefficient).

Studying symmetry properties of partial differential equations could lead us to numerous fundamental effects, which take place in the considered system. Symmetry is described in terms of Lie algebras and Lie groups and provides widely applicable techniques to study partial differential equations, for example constructing exact solutions and finding conservation laws.

1. Fokker-Planck equation with power diffusivity

In the [4], the symmetry properties of the equation with the diffusion coefficient being proportional to $(x-\alpha)^\beta$ were described, albeit not without mistakes. As well, this partial case could be reduced to $D\sim x^\beta$ by the shifts $\tilde x = x-\alpha$, $\tilde y = y$ and $\tilde t = t + y/\alpha$.

Partial cases of the considered equation were described also in the [5], and the exact solutions for the boundary value problems were obtained.

In the present paper, we refine and purify the results from the [4], and construct non-trivial equivalence transformation. We also describe structural properties of the maximal invariance algebra, admitted by the considered equation.

In this paper, we consider a class of Fokker-Planck equations with power diffusion coefficients, depending on x,

$$\frac{\partial v}{\partial t} + x\frac{\partial v}{\partial y} - x^\beta\frac{\partial^2 v}{\partial x^2} = 0. \tag{5}$$

Here v is the cell number density, t is time, y the degree of maturity, x the velocity of cell growth. The constant power β is the arbitrary element of the class (5).

2. Equivalence transformation

The class of Fokker-Planck equations (5) admits nontrivial discrete equivalence transformation

$$\begin{aligned}&\tilde t = -y,\ \tilde y = -t,\ \tilde x = -\frac{1}{x},\\ &\tilde v = \frac{v}{x},\ \tilde\beta = -\beta + 5.\end{aligned} \tag{6}$$

Thus, the equivalence transformation (6) generates a discrete point symmetry transformation for equation (5) with $\beta = 5/2$.

3. Lie symmetries classification

According to the classical Lie algorithm of computing point symmetries [6], we consider a Lie symmetry vector field of the equations (5), which is of the form:

$$Y = \tau\partial_t + \xi^x\partial_x + \xi^y\partial_y + \eta\partial_v,$$

where τ, ξ^x, ξ^y and η are smooth functions of (t,x,y,v). Hereafter we denote $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $\partial_v = \partial/\partial v$. Then we take the second prolongation $\mathrm{pr}_2\,Y$ of Y and apply it to the equation to obtain the invariance condition

$$\mathrm{pr}_2\,Y\Big(v_t + xv_y - x^\beta v_{xx}\Big)\Big|_{v_{xx}=x^{-\beta}(v_t+xv_y)} = 0.$$

Note that

$$\begin{aligned}\mathrm{pr}_2\,Y = Y &+ \zeta^t\partial_{v_t} + \zeta^x\partial_{v_x} + \zeta^y\partial_{v_y}\\ &+ \zeta^{tt}\partial_{v_{tt}} + \zeta^{tx}\partial_{v_{tx}} + \zeta^{ty}\partial_{v_{ty}}\\ &+ \zeta^{xx}\partial_{v_{xx}} + \zeta^{xy}\partial_{v_{xy}} + \zeta^{yy}\partial_{v_{yy}},\end{aligned}$$

where the components of prolongation are computed according to the general prolongation formula,

$$\begin{aligned}\zeta^t &= D_t\eta - v_tD_t\tau - v_xD_t\xi^x - v_yD_t\xi^y,\\ \zeta^x &= D_x\eta - v_tD_x\tau - v_xD_x\xi^x - v_yD_x\xi^y,\\ \zeta^y &= D_y\eta - v_tD_y\tau - v_xD_y\xi^x - v_yD_y\xi^y,\\ \zeta^{tt} &= D_t\zeta^t - v_{tt}D_t\tau - v_{tx}D_t\xi^x - v_{ty}D_t\xi^y,\\ \zeta^{tx} &= D_t\zeta^x - v_{tt}D_t\tau - v_{tx}D_t\xi^x - v_{ty}D_t\xi^y,\\ \zeta^{xx} &= D_x\zeta^x - v_{tx}D_x\tau - v_{xx}D_x\xi^x - v_{xy}D_x\xi^y,\end{aligned}$$

etc., D_t, D_x and D_y denote the total derivative operators with respect to t, x and y, respectively,

$$\begin{aligned}D_t &= \partial_t + v_t\partial_v + v_{tt}\partial_{v_t} + v_{tx}\partial_{v_x} + v_{ty}\partial_{v_y} + \cdots,\\ D_x &= \partial_x + v_x\partial_v + v_{tx}\partial_{v_t} + v_{xx}\partial_{v_x} + v_{xy}\partial_{v_y} + \cdots,\\ D_y &= \partial_y + v_y\partial_v + v_{ty}\partial_{v_t} + v_{xy}\partial_{v_x} + v_{yy}\partial_{v_y} + \cdots.\end{aligned}$$

Evaluating the action of $\mathrm{pr}_2\,Y$ on the equation from the class (5), we obtain

$$\zeta^t + \xi^x v_y + x\zeta^y - \beta x^{\beta-1}\xi^x v_{xx} - x^\beta\zeta^{xx} = 0,$$

where v_{xx} should be substituted by $x^{-\beta}(v_t + xv_y)$. We split the last equation with respect to v, v_t and v_x, which leads to a system of so-called determining equations for the components of Y,

$$\begin{aligned}&\tau_v = \tau_x = 0,\quad \xi^x_v = 0,\quad \xi^y_v = \xi^y_x = 0,\\ &\eta_{vv} = 0,\quad \eta_t + x\eta_y - x^\beta\eta_{xx} = 0,\\ &-2x^\beta\eta_{vx} = \xi^x_t + x\xi^x_y - x^\beta\xi^x_{xx},\\ &\xi^y_t + x\xi^y_y = 2x\xi^x_x + (1-\beta)\xi^x,\\ &\xi^y_t + x\xi^y_y = \tau_tx + \tau_yx^2 + \xi^x.\end{aligned} \tag{7}$$

Directly from the exact form of the system of determining equations (7) we can find the kernel Lie invariance algebra of the class (5), which is $\mathfrak{g}^\cap = \langle\mathcal{P}^t, \mathcal{P}^y, \mathcal{I}\rangle$, where

$$\mathcal{P}^t = \partial_t,\quad \mathcal{P}^y = \partial_y,\quad \mathcal{I} = v\partial_v.$$

Unlike the group classification in the [4], we carry out the group classification of the class (5) up to the equivalence generated by the action of the group constiting of the transformations (6) and the identity transformation. In other words, without loss of generality we can impose the gauge $\beta < 5/2$, which in its turn reduces the number of cases we need to consider.

Case 0. For regular values of β, the general solution of the determining equations is

$$\tau = C_2t + C_3,\quad \xi^x = -\frac{C_2x}{\beta-2},\quad \xi^y = C_2\frac{\beta-3}{\beta-2}y + C_4,$$
$$\eta = C_1v + f^\beta(t,x,y),$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and f^β is an arbitrary solution of the equation of the form (5) for the fixed β. The correspondent maximal Lie invariance algebra $\mathfrak{g}^{\rm gen}_\beta$ is spanned by $\mathfrak{g}^\cap$ and the vector fields

$$\mathcal{D}_\beta = (\beta - 2)t\partial_t - x\partial_x + (\beta - 3)y\partial_y,$$
$$\mathcal{Z}(f^\beta) = f^\beta(t, x, y)\partial_v.$$

$\mathcal{Z}(f^\beta)$ forms the abelian ideal of the algebra $\mathfrak{g}_\beta^{\rm gen}$. In other words, we can decompose the maximal Lie invariance algebra $\mathfrak{g}_\beta^{\rm gen}$ into its essential part and infinitydimensional ideal $\mathfrak{g}_\beta^{\rm lin} = \langle \mathcal{Z}(f^\beta)\rangle$: $\mathfrak{g}_\beta^{\rm gen} = \mathfrak{g}_\beta^{\rm lin} \Subset \mathfrak{g}_\beta^{\rm ess}$. Essential part of the considered algebra is spanned by the vector fields $\{\mathcal{P}^t,\ \mathcal{P}^y,\ \mathcal{I},\ \mathcal{D}_\beta\}$.

For some specific values of β, the maximal Lie invariance algebra $\mathfrak{g}_\beta$ is wider than $\mathfrak{g}_\beta^{\rm gen}$. Below we describe these cases of Lie symmetry extension.

Case 1: $\beta = 2$. The algebra $\mathfrak{g}_2$ is spanned by $\mathfrak{g}_2^{\rm gen}$ and the vector field

$$\mathcal{Y} = 2yx\partial_x + y^2\partial_y - vx\partial_v.$$

Case 2: $\beta = 0$. A complement subspace to $\mathfrak{g}_0^{\rm gen}$ in $\mathfrak{g}_0$ is spanned by the vector fields

$$\mathcal{P}^x = \partial_x + t\partial_y,$$
$$\mathcal{G} = 2t\partial_x + t^2\partial_y - vx\partial_v,$$
$$\mathcal{F} = t^2\partial_x + \tfrac{1}{3}t^3\partial_y - (tx - y)v\partial_v,$$
$$\mathcal{A} = t^2\partial_t + (tx + 3y)\partial_x + 3yt\partial_y - (vx^2 + 2vt)\partial_v.$$

4. Structural properties of the maximal Lie invariance algebras

We describe the structural properties of the maximal invariance algebras $\mathfrak{g}_\beta^{\rm gen}$, $\mathfrak{g}_2$ and $\mathfrak{g}_0$.

Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations between the vector fields spanning $\mathfrak{g}_\beta^{\rm gen}$ are exhausted by the following:

$$[\mathcal{P}^t, \mathcal{D}_\beta] = (\beta - 2)\mathcal{P}^t, \quad [\mathcal{P}^y, \mathcal{D}_\beta] = (\beta - 3)\mathcal{P}^t,$$
$$[\mathcal{P}^t, \mathcal{Z}(f^\beta)] = f_t^\beta\partial_v, \quad [\mathcal{P}^y, \mathcal{Z}(f^\beta)] = f_y^\beta\partial_v,$$
$$[\mathcal{I}, \mathcal{Z}(f^\beta)] = -\mathcal{Z}(f^\beta),$$
$$[\mathcal{D}_\beta, \mathcal{Z}(f^\beta)] = ((\beta - 2)tf_t^\beta - xf_x^\beta + (\beta - 3)yf_y^\beta)\partial_v.$$

Therefore, the algebra $\mathfrak{g}_\beta^{\rm gen}$ is solvable, and its nilradical is $\mathfrak{n}_\beta^{\rm gen} = \{\mathcal{P}^t, \mathcal{P}^y, \mathcal{I}, \mathcal{Z}(f^\beta)\}$.

In the algebra $\mathfrak{g}_2$ the nonvanishing commutation relations between vector fields spanning $\mathfrak{g}_2^{\rm gen}$ and vector field $\mathcal{Y}$ are following:

$$[\mathcal{P}^y, \mathcal{Y}] = -2D_2, \quad [\mathcal{D}_2, \mathcal{Y}] = -\mathcal{Y},$$
$$[\mathcal{Z}(f^2), \mathcal{Y}] = -\mathcal{Y}[f^2]\partial_v.$$

Hereafter as $\mathcal{Q}[g]$ we denote the characterisitic of the differential operator, which is corresponded to the vector field $\mathcal{Q}$. Note that $v\partial_v$ plays the role of the unitary operator.

The analogous commutation relations for the algebra $\mathfrak{g}_0$ are

$$[\mathcal{P}^t, \mathcal{P}^x] = \mathcal{P}^y, \quad [\mathcal{P}^t, \mathcal{G}] = 2\mathcal{P}^x, \quad [\mathcal{P}^t, \mathcal{F}] = \mathcal{G},$$
$$[\mathcal{P}^t, \mathcal{A}] = -\mathcal{D}_0 - 2\mathcal{I},$$
$$[\mathcal{P}^y, \mathcal{D}_0] = -3\mathcal{P}^y, \quad [\mathcal{P}^y, \mathcal{F}] = \mathcal{I}, \quad [\mathcal{P}^y, \mathcal{A}] = 3\mathcal{P}^x,$$
$$[\mathcal{D}_0, \mathcal{G}] = -\mathcal{G}, \quad [\mathcal{D}_0, \mathcal{F}] = -3\mathcal{F}, \quad [\mathcal{D}_0, \mathcal{A}] = -2\mathcal{A},$$
$$[\mathcal{P}^x, \mathcal{G}] = -\mathcal{I}, \quad [\mathcal{F}, \mathcal{A}] = 2\mathcal{G}, \quad [\mathcal{G}, \mathcal{A}] = 3\mathcal{F},$$
$$[\mathcal{Z}(f^0), \mathcal{P}^x] = -\mathcal{P}^x[f^0]\partial_v, \quad [\mathcal{Z}(f^0), \mathcal{G}] = -\mathcal{G}[f^0]\partial_v,$$
$$[\mathcal{Z}(f^0), \mathcal{F}] = -\mathcal{F}[f^0]\partial_v,$$
$$[\mathcal{Z}(f^0), \mathcal{A}] = -\mathcal{A}[f^0]\partial_v.$$

As it was mentioned in the previous section, the maximal Lie invariance algebra for the arbitrary parameter β can be decomposed into its essential part and the abelian ideal, namely $\mathfrak{g}_0 = \mathfrak{g}_0^{\rm lin} \Subset \mathfrak{g}_0^{\rm ess}$. The Levi–Maltsev decomposition for the algebra $\mathfrak{g}_0^{\rm ess}$ takes the following form: $\mathfrak{g}_0^{\rm ess} = \mathfrak{r}_0^{\rm ess} \Subset \mathfrak{l}_0^{\rm ess}$, where $\mathfrak{r}_0^{\rm ess}$ is the radical of the essential subalgebra, spanned by the vector fields $\{\mathcal{P}^y, \mathcal{I}, \mathcal{P}^x, \mathcal{G}, \mathcal{F}\}$ and $\mathfrak{l}_0^{\rm ess}$ is the Levi factor: $\mathfrak{l}_0^{\rm ess} = \langle \mathcal{P}^t, \mathcal{D}_0 + 2\mathcal{I}, \mathcal{A}\rangle$. The subalgebra $\mathfrak{l}_0^{\rm ess}$ is isomorphic to the algebra $\mathfrak{sl}(2, \mathbb{R})$.

5. Conclusion

In the present paper, we have carried out the group classification of the class of Fokker-Planck equations (5) with power diffusion coefficients. Unlike the classification in [4], we have found the nontrivial discrete equivalence transformation (6) of the class (5) and classified extensions of the kernel Lie symmetry algebra $\mathfrak{g}^\cap$ up to this equivalence transformation. As well, we have described structural properties of the algebras $\mathfrak{g}_\beta^{\rm gen}$, $\mathfrak{g}_2$ and $\mathfrak{g}_0$.

With the knowledge of the group classification of the class (5), exact solutions of the equations from this class can be found by using the Lie reduction method, the method of noncomutating integration and by considering higher symmetries.

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