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VARIATIONAL APPROACHES FOR ILLUMINATION LEVELING IN IMAGES

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Abstract

This work explores and implements a method for leveling illumination in various types of images. Issues such as insufficient or uneven lighting can lead to the loss of important details, low contrast or incorrect color reproduction. On the other hand, excessive lighting may cause overexposed areas which leads to image information being lost. This problem is particularly relevant in the processing of medical images, where poor lighting can make them look too dark and hide important details, particularly in the shadowed areas. The practical part of the work presents a mathematical model that describes the transition from the variational (optimization) formulation of the problem to its differential form. To evaluate effectiveness of existing numerical methods from the field of mathematical physics in the context of illumination leveling, a comparative analysis is conducted. Experiments were carried out to determine the computation time and number of iterations required to obtain images with leveled illumination.

Keywords: Poisson's equation, variational problem formulation, computational methods, elliptic second order equation, equalization of illumination

Introduction

The issue of image illumination refers to the influence of lighting on the quality and perception of an image [1]. Insufficient or uneven lighting can lead to loss of detail, reduced contrast, and inaccurate color reproduction. In contrast, excessive illumination may cause overexposed regions in which the details of the image are irretrievably lost. In particular, during the acquisition and analysis of medical images, the challenge of insufficient illumination often arises, resulting in dark images with poorly distinguishable details, especially within shadowed regions. A specific and critical challenge in this context is the problem of non-uniform illumination across the object of study, which significantly hinders accurate visual interpretation. To address such illumination-related issues, mathematical methods of varying complexity are employed for the analysis and correction of unevenly illuminated images. One class of techniques utilizes matrix filtering operations, based on convolution principles well-established in linear algebra [2]. These methods are relatively simple to implement in software and require minimal computational resources. More sophisticated approaches, such as those based on the Laplace and Fourier transforms, offer superior visual results in correcting illumination irregularities but require significantly greater computational time and processing power [3]. The most advanced mathematical framework for addressing uneven illumination problems is based on the variational principle. This approach reformulates the correction task as an optimization problem, specifically the minimization of a quadratic functional, which leads to solving the Poisson equation with boundary conditions tailored to the illumination problem. A wide range of methods have been developed to solve both the principal classes of mathematical physics problems and the associated optimization problems with functional constraints defined by process-specific equations [3], [4]. The primary numerical methods for solving the Poisson equation include the finite difference method and the finite element method [2], [4], [5]. It is worth noting that the finite element method has distinct advantages over the finite difference method, particularly when dealing with image regions characterized by complex geometries. In this paper, we present an image analysis method based on the variational principle in combination with the finite difference method.

1. Problem formulation

1.1. Technical formulation of the problem

The technical objective of this work is to analyze unevenly illuminated images using the gradient field derived from these images to reconstruct a visually enhanced representation. Furthermore, the study aims to examine classical computational methods for solving

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second-order elliptic partial differential equations, which form the mathematical foundation of the variational approach employed in Poisson image processing.

1.2. Mathematical formulation of the problem: transition from the variational formulation of the problem to the differential

To correct uneven illumination in an image, it is essential to preserve the boundary features of all objects present within the scene. Effective object differentiation requires access to the image's gradient information. In the case of a full-color digital image represented in the RGB color space, this information is encoded as two gradient matrices for each color channel [6]. The Poisson-based approach retains this gradient information by formulating a quadratic functional $I_1(u)$, which is minimized to correct the illumination.

$$I_1(u) = \iint_G \left(u'_x - v'_x\right)^2 ds + \iint_G \left(u'_y - v'_y\right)^2 ds \to \min,$$

where u – denotes the target image to be reconstructed. The partial derivatives $u'_x = \frac{\partial u}{\partial x}$, $u'_y = \frac{\partial u}{\partial y}$ represent the gradients of u in the horizontal and vertical directions, respectively. v – denotes the input image with known gradient fields, given in the form of matrices computed as $v'_x = \frac{\partial v}{\partial x}$, $v'_y = \frac{\partial v}{\partial y}$. The domain G defines the computational region corresponding to the image.

To effectively correct uneven illumination in the image, it is necessary to minimize an additional functional, defined as follows:

$$I_2(u) = \iint_G \left(u - \overline{u}\right)^2 ds \to \min.$$

Here, $\overline{u}$ denotes the average illumination value. The functional $I_2(u)$ is formulated to minimize the variance, where a decrease in variance indicates a more uniform distribution of pixel brightness across the image. Thus, the problem of equalizing uneven illumination is expressed through the following variational formulation:

$$\begin{aligned} I(u) &= \alpha_1 I_1(u) + \alpha_2 I_2(u) = \\ &= \alpha_1 \left(\iint_G \left(u'_x - v'_x\right)^2 ds + \iint_G \left(u'_y - v'_y\right)^2 ds \right) \\ &+ \alpha_2 \iint_G \left(u - \overline{u}\right)^2 ds \to \min, \quad \alpha_1 + \alpha_2 = 1. \end{aligned}$$

In functional $I(u)$, the parameters α_1 and α_2 are weighting factors that control the relative influence of functionals $I_1(u)$ and $I_2(u)$, respectively. It is evident that this combined functional $I(u)$ can also be expressed using a single parameter:

$$\begin{aligned} I(u) &= \alpha_1 \left(1 \cdot I_1(u) + \frac{\alpha_2}{\alpha_1} \cdot I_2(u)\right) \sim I_1(u) + \\ &+ \lambda \cdot I_2(u) = \iint_G \left(u'_x - v'_x\right)^2 ds + \\ &+ \iint_G \left(u'_y - v'_y\right)^2 ds + \lambda \iint_G \left(u - \overline{u}\right)^2 ds \to \min, \end{aligned}$$

The final equivalence transformation, denoted by «∼», holds under the condition that the extremals of the original and the resulting functionals coincide. It should also be noted that $\lambda = \alpha_2/\alpha_1$, where $\alpha_1 \neq 0$. $\alpha_1 \neq 0$, because if $\alpha_1 = 0$, all information about the original image (the gradient field) would be lost, making the problem meaningless.

To complete the mathematical formulation of the variational problem, it is necessary to specify the boundary conditions. Considering that object boundaries are not accurately captured at the edges of the entire image and that variance minimization for I_2 is better achieved with free (unfixed) boundaries, the following approach is adopted:

$$\left.\frac{\partial u}{\partial x}\right|_{\text{Right}} = \left.\frac{\partial u}{\partial x}\right|_{\text{Left}} = \left.\frac{\partial u}{\partial y}\right|_{\text{Up}} = \left.\frac{\partial u}{\partial y}\right|_{\text{Down}} = \left.\frac{\partial u}{\partial \overline{n}}\right|_{\Gamma} = 0,$$

where Γ – border of the investigated image.

This variational problem can be easily converted into its differential form by applying the Euler–Lagrange equation, which is given by

$$\begin{aligned} &\frac{\partial S\left(u, u'_x, u'_y\right)}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial S\left(u, u'_x, u'_y\right)}{\partial u'_x}\right) - \\ &\qquad - \frac{\partial}{\partial y}\left(\frac{\partial S\left(u, u'_x, u'_y\right)}{\partial u'_y}\right) = 0, \\ &S\left(u, u'_x, u'_y\right) = \left(u'_x - v'_x\right)^2 + \left(u'_y - v'_y\right)^2 + \lambda\left(u - \overline{u}\right)^2. \end{aligned}$$

After simplification, we get the equation:

$$u''_{xx} + u''_{yy} - \lambda u = v''_{xx} + v''_{yy} - \lambda \overline{u}.$$

The resulting equation is the Poisson equation. Taking into account the boundary conditions (where the derivatives along the normal direction are zero), it is evident that if $u^*(x, y)$ is a solution, then the function $u^*(x, y) + C$, where C is an arbitrary constant, is also a solution. We can prove this by verifying:

$$(u^* + C)''_{xx} + (u^* + C)''_{yy} - \lambda(u^* + C) = v''_{xx} + v''_{yy} - \lambda\overline{(u^* + C)}$$

based on the original equation:

$$(u^*)''_{xx} + (u^*)''_{yy} - \lambda u^* = v''_{xx} + v''_{yy} - \lambda \overline{u^*}.$$

In other words, to solve the problem, we can set $\overline{u} = 0$ in order to determine the unique solution. This solution can then be easily constrained within the allowable pixel intensity values (for real numbers, between 0 and 1, or for integers, between 0 and 255), for example, using the classic formula:

$$u = \frac{u - u_{\min}}{u_{\max} - u_{\min}} \ \textit{or}\ u = \left[255 \cdot \frac{u - u_{\min}}{u_{\max} - u_{\min}}\right].$$

2. Computational experiments

2.1. A numerical method for solving the problem

The mathematical model under consideration involves a well-known second-order linear equation from the field of mathematical physics.

$$\lambda u - \Delta u = -\Delta f, \quad \lambda > 0 \tag{1}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

In equation (1), we assume that the function $f(x, y)$ is known. This implies that Δf can be easily computed using the classical and well-known difference schemes. The boundary conditions for equation (1) are given by the vanishing of the derivatives.

We now proceed to derive the difference equation for equation (1). To achieve this we apply central difference schemes at each grid point (x_i, y_i), where $i = \overline{1, N_x + 2}$, $j = \overline{1, N_y + 2}$. We assume uniform step sizes in both the $x-$ and $y-$coordinates, denoted as h_x and h_y respectively. The resulting equation is as follows:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}\Big|_{(x_i,y_j)} &\approx \frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{(h_x)^2}, \\ \frac{\partial^2 u}{\partial y^2}\Big|_{(x_i,y_j)} &\approx \frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{(h_y)^2}. \end{aligned} \tag{2}$$

In a similar manner, we express the second derivatives of the known function $f(x, y)$ on the right-hand side of equation (1):

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}\Big|_{(x_i,y_j)} &\approx \frac{f(x_{i-1}, y_j) - 2f(x_i, y_j) + f(x_{i+1}, y_j)}{(h_x)^2}, \\ \frac{\partial^2 f}{\partial y^2}\Big|_{(x_i,y_j)} &\approx \frac{f(x_i, y_{j-1}) - 2f(x_i, y_j) + f(x_i, y_{j+1})}{(h_y)^2}. \end{aligned} \tag{3}$$

After performing the necessary mathematical transformations and grouping the coefficients corresponding to the points (x_i, y_j), $(x_{i-1}, y_j), (x_{i+1}, y_j), (x_i, y_{j-1})$ and (x_i, y_{j+1}), we obtain the five-point finite difference scheme for equation (1) in the updated notation.

$$\begin{aligned} &\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} - \lambda u_{i,j} \\ &= \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h_x^2} + \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{h_y^2}, \\ &i = 2, N_x + 1, j = 2, N_y + 1. \end{aligned} \tag{4}$$

We introduce new notation into the finite difference equation (4) to simplify its representation.

- coefficient associated with $u_{i-1,j}$: $AX_{i,j} = 1/h_x^2$;
- coefficient associated with $u_{i+1,j}$: $CX_{i,j} = 1/h_x^2$;
- coefficient associated with $u_{i,j-1}$: $AY_{i,j} = 1/h_y^2$;
- coefficient associated with $u_{i,j+1}$: $CY_{i,j} = 1/h_y^2$;
- coefficient associated with $u_{i,j}$: $B_{i,j} = 2/h_x^2 + 2/h_y^2 + \lambda$;
- coefficient of the right side of the equation $D_{i,j}$:

$$D_{i,j} = \frac{f_{i-1,j} - 2f_{i,j} + f_{i+1,j}}{h_x^2} + \frac{f_{i,j-1} - 2f_{i,j} + f_{i,j+1}}{h_y^2}.$$

Thus, equation 4 is transformed into the following finite difference equation:

$$\begin{aligned} &AX_{i,j}u_{i-1,j} + CX_{i,j}u_{i+1,j} + AY_{i,j}u_{i,j-1} + \\ &\quad + CY_{i,j}u_{i,j+1} - B_{i,j}u_{i,j} = D_{i,j}, \\ &\quad i = 2, N_x + 1, j = 2, N_y + 1. \end{aligned}$$

2.2. Testing the numerical method on various images

The developed algorithm was tested on 30 different images. The timing results for the first six images are presented in Tables 1 and 2. The algorithm based on the SciPy library of the Python programming language demonstrated good performance.It should also be noted that computational methods of linear algebra solve systems of linear algebraic equations more efficiently when the system matrices exhibit diagonal dominance. The greater the difference between the absolute value of the diagonal element and the sum of the absolute values of the other elements in the same row (for each row), the faster the convergence of the iterative method or algorithm. The calculation error for the iterative methods, whose results are reported in the tables, was set to $\epsilon = 10^{-7}$.

Table 1. Time characteristics of finding a solution to the problem using the Seidel method for six different images

Im	Height	Width	Total Pixels	Time for λ=0.0001	Time for λ=0.0005
1	216	197	42552	0,21	0,09
2	450	297	133650	0,39	0,14
3	216	218	47088	0,23	0,12
4	249	186	46314	0,08	0,04
5	192	256	49152	0,26	0,19
6	341	404	137764	0,59	0,30
Im	**Height**	**Width**	**Total Pixels**	**Time for** λ=0.001	**Time for** λ=0.01
1	216	197	42552	0,06	0,01
2	450	297	133650	0,08	0,02
3	216	218	47088	0,07	0,01
4	249	186	46314	0,03	0,01
5	192	256	49152	0,10	0,01
6	341	404	137764	0,20	0,03

The visual results obtained from the implementation of the program are shown below in Figures 1–4.

Conclusion

The practical outcome of this work is a comparative analysis of unevenly illuminated images based on the field of given image gradients, aimed at achieving visually improved results. Mathematical formulations were derived to construct an approximate solution to the problem of equalizing the uneven distribution of pixel intensities. The study also includes an analysis of classical computational methods for solving second-order elliptic equations of mathematical physics, which

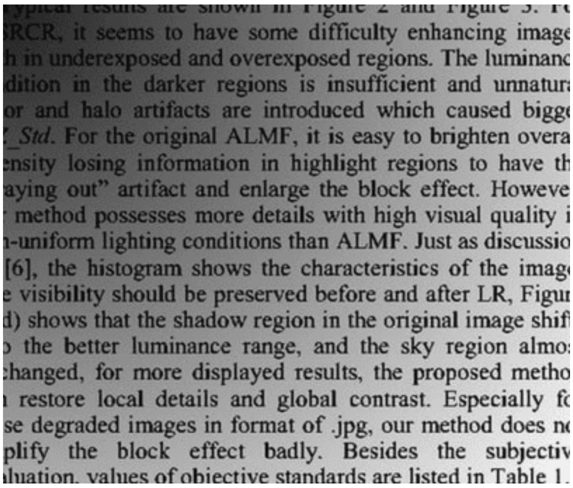
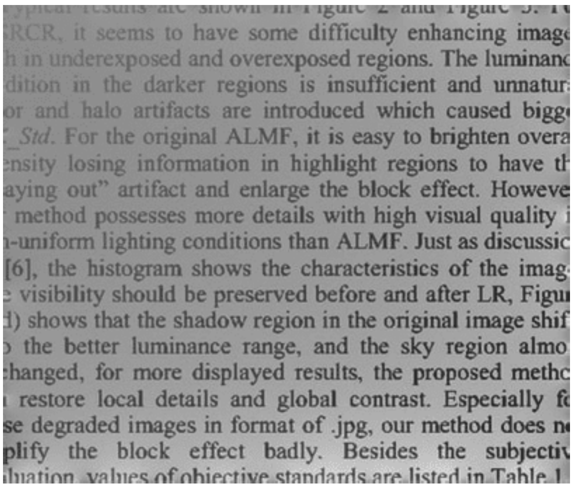

ypical results are shown in Figure 2 and Figure 3. Fo
RCR, it seems to have some difficulty enhancing image
h in underexposed and overexposed regions. The luminanc
dition in the darker regions is insufficient and unnatur
or and halo artifacts are introduced which caused bigge
_Std. For the original ALMF, it is easy to brighten overa
ensity losing information in highlight regions to have th
aying out" artifact and enlarge the block effect. Howeve
method possesses more details with high visual quality i
n-uniform lighting conditions than ALMF. Just as discussio
[6], the histogram shows the characteristics of the imag
e visibility should be preserved before and after LR, Figur
d) shows that the shadow region in the original image shif
the better luminance range, and the sky region almo
changed, for more displayed results, the proposed metho
restore local details and global contrast. Especially fo
se degraded images in format of .jpg, our method does n
plify the block effect badly. Besides the subjectiv
luation, values of objective standards are listed in Table 1.

Figure 1. Original picture «im3.png»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se degraded images in format of .jpg, our method does n
plify the block effect badly. Besides the subjectiv
luation, values of objective standards are listed in Table 1.

Figure 2. Processed picture «im3.png», $\lambda = 0.0005$

Figure 3. Original picture «im5.png»

Table 2. Time characteristics of finding a solution to a problem (for six different images) using built-in Python tools, namely the Scipy library for solving a system of linear equations

Im	**Height**	**Width**	**Total Pixels**	**Time for** λ=0.05	**Time for** λ=0.1
1	216	197	42552	0,003	0,002
2	450	297	133650	0,007	0,005
3	216	218	47088	0,003	0,002
4	249	186	46314	0,003	0,002
5	192	256	49152	0,004	0,002
6	341	404	137764	0,008	0,005
Im	**Height**	**Width**	**Total Pixels**	**Time for** λ=0.15	**Time for** λ=0.2
1	216	197	42552	0,002	0,001
2	450	297	133650	0,003	0,003
3	216	218	47088	0,001	0,001
4	249	186	46314	0,002	0,001
5	192	256	49152	0,002	0,001
6	341	404	137764	0,003	0,003

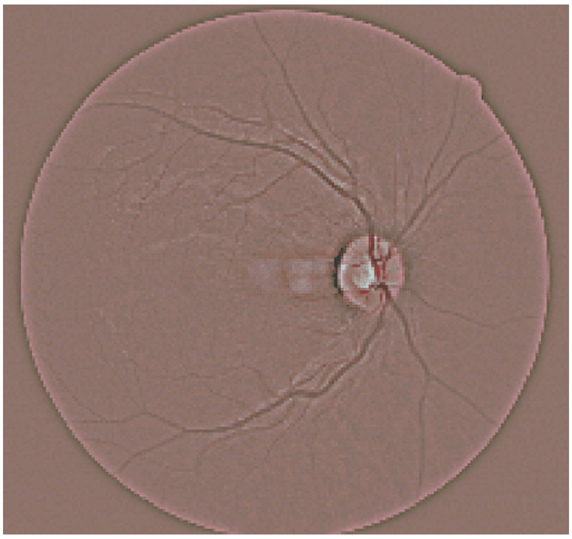

Figure 4. Processed picture «im5.png», $\lambda = 0.01$

form the basis of the variational Poisson image processing approach. Among the evaluated methods, Seidel's method demonstrated the poorest performance in terms of computation time, whereas the block relaxation method implemented through the built-in SciPy library in Python showed the best time efficiency. To achieve faster convergence for a given accuracy, it is recommended to employ multigrid approximation methods, which significantly reduce the number of iterations and thus decrease overall execution time. It is also worth noting that the speed of iterative methods and algorithms can be further improved if calculations are performed in the normalized system (values from 0 to 1) rather than in the traditional scale (0 to 255).

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