

ON DISCRETIZATION OF SOME EXTREMAL PROBLEMS

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We solve two continuous extremal problems on the classes of monotone functions: in the first problem we find extremal values for a line integral of a coordinate-wise monotone function of two variables from a rearrangement-invariant class of functions; in the second one we find extremal values for the expectation of a random process with monotone trajectories at a random time. In both cases we reduce the continuous problems to their discrete counterparts. The obtained discrete problems are on the one hand interesting on their own, and on the other hand give a natural explanation of the structure of the extremal functions for the continuous problems.

Recall that for a measurable function $f: E \rightarrow \mathbb{R}$ defined on a measurable space (E, μ) with a finite measure μ , the function

$$m_f: [0, \infty) \rightarrow [0, \mu E], \quad m_f(t) = \mu\{x \in E: |f(x)| > t\}$$

is called the distribution function of f . The function

$$r_f: [0, \mu E] \rightarrow [0, \infty), \quad r_f(t) = \inf\{s: m_f(s) \leq t\}$$

is called the non-increasing rearrangement of f . A function $f: [0, 1]^2 \rightarrow \mathbb{R}$ is called coordinate-wise non-decreasing, if it is non-decreasing with respect to the partial order $(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$. Let $m: [0, 1] \rightarrow [0, 1]$ be an increasing bijection. Denote by $\mathcal{M}_m$ the class of all measurable (with respect to the two-dimensional Lebesgue measure μ) coordinate-wise non-decreasing functions $f: [0, 1]^2 \rightarrow [0, 1]$ such that $m_f(t) \geq 1 - m(t)$, $t \in [0, 1]$.

Theorem 1 ([1]). *Let $m: [0, 1] \rightarrow [0, 1]$ be an increasing bijection. For each continuous non-decreasing function $t: [0, 1] \rightarrow [0, 1]$ one has*

$$\inf_{f \in \mathcal{M}_m} \int_0^1 f(t(s), s) ds = \int_0^1 m^{-1}(t(s) \cdot s) ds.$$

A natural analogue of the class $\mathcal{M}_m$ for functions of two variables that are monotone with respect to the partial order

$$(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 = y_2,$$

can be defined as follows. Assume an atomless probability space $\{\Omega, \mathcal{F}, P\}$ is given. For a function m as above, let $\mathfrak{M}_m$ be the space of all measurable

random processes $\{\xi_t, t \in [0, 1]\}$ defined on $\{\Omega, \mathcal{F}, P\}$ such that

$$s < t \implies 0 \leq \xi_s(\omega) \leq \xi_t(\omega) \leq 1 \text{ for almost all } \omega \in \Omega,$$

and

$$\mathbf{E}\mu\{t \in [0, 1]: \xi_t > s\} \geq 1 - m(s) \text{ for all } s \in [0, 1],$$

where $\mathbf{E}$ denotes the expectation of a random variable, and μ is the one-dimensional Lebesgue measure.

Theorem 2 ([1]). *Let $m: [0, 1] \rightarrow [0, 1]$ be an increasing bijection, and τ be a random variable on $\{\Omega, \mathcal{F}, P\}$ that takes values in $[0, 1]$. Then*

$$\inf_{\xi_t \in \mathfrak{M}_m} \mathbf{E}\xi_\tau = \int_0^1 m^{-1} \left(\int_{1-y}^1 r_\tau(s) ds \right) dy.$$

If in addition $m(t) = t$ for all $t \in [0, 1]$, then one has

$$\inf_{\xi_t \in \mathfrak{M}_m} \mathbf{E}\xi_\tau = \int_0^1 r_\tau(s) s ds.$$

Classes $\mathcal{M}_m$ and $\mathfrak{M}_m$ are examples of rearrangement-invariant classes, which appear in many extremal problems of approximation theory. Note that in the situation, when τ is not constant, the extremal random process for Theorem 2 that we construct in the proof can not be represented as a product $\xi_t(\omega) = x(t) \cdot \eta(\omega)$ of a function $x: [0, 1] \rightarrow \mathbb{R}$ and a random variable η .

We illustrate the discretization approach for Theorem 1 in the partial case when $m(s) = s$, $s \in [0, 1]$ and $t(s) = \alpha$, $s \in [0, 1]$ for some fixed number $\alpha \in [0, 1]$. For large $n \in \mathbb{N}$ divide the unit square $[0, 1]^2$ into n^2 equal squares, and approximate a function $f \in \mathcal{M}_m$ by a coordinate-wise non-decreasing piecewise-constant function $f_n: [0, 1]^2 \rightarrow \Xi_n := \left\{ \frac{1}{n^2}, \frac{2}{n^2}, \dots, \frac{n^2}{n^2} \right\}$. The function f_n can be identified with a bijection from $\{(i, j): 1 \leq i, j \leq n\}$ to Ξ_n . If $k = k(n)$ is the number of the column that contains the set $\{\alpha\} \times [0, 1]$, then $\frac{1}{n} \sum_{\nu=1}^n f_n(k, \nu)$ is an approximation of the integral $\int_0^1 f(\alpha, s) ds$, and we arrive at an extremal problem to find

$$\min \sum_{\nu=1}^n f_n(k, \nu)$$

over all non-decreasing bijections f_n .

REFERENCES

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