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THEORETICAL FUNDAMENTALS of ELECTRICAL ENGINEERING

SINGLE PHASE AC CIRCUITS

*Recommended by the Methodological Board
of Igor Sikorsky Kyiv Polytechnic Institute
as a tutorial for students doing Bachelor's degree programmes in
speciality 141 “Electric Power Engineering, Electrical Engineering and
Electromechanics”*

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THEORETICAL FUNDAMENTALS of ELECTRICAL ENGINEERING SINGLE PHASE AC CIRCUITS

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The tutorial “Theoretical fundamentals of electrical engineering: Single phase AC circuits” is intended for students doing Bachelor's degree in specialty 141 “Electric Power Engineering, Electrical Engineering and Electromechanics”.

This educational publication deals with one section of theoretical fundamentals of electrical engineering – single phase AC circuits. It includes a brief theoretical overview and provides some examples to make the explanation simple and clear.

Being written in English, the tutorial can be useful for both international and Ukrainian students.

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CONTENTS

INTRODUCTION.....	4
1. SINUSOIDAL FUNCTIONS: TERMINOLOGY	5
2. EFFECTIVE VALUES OF HARMONIC QUANTITY	8
3. REPRESENTATION FORMS OF SINUSOIDAL QUANTITIES	10
4. MATHEMATICAL OPERATIONS WITH THE COMPLEX VALUES.	15
5. THE ELEMENTS OF A SINUSOIDAL CURRENT CIRCUIT	18
5.1. Sinusoidal Current Through a Resistor	18
5.2. Sinusoidal Current Through an Induction Coil	21
5.3. Sinusoidal Current Through a Capacitor	26
5.4. Resume.....	32
6. Ohm's and Kirchhoff's Laws in Complex Notation.....	34
7. CONNECTIONS OF R , L AND C ELEMENTS.....	37
7.1. Series Connection of R , L and C	37
7.2. Parallel Connection of R , L and C	41
8. INSTANTANEOUS AND AVERAGE POWER. POWER FACTOR.....	
9. POWER IN COMPLEX FORM.....	44
10. RESONANCE IN ELECTRICAL CIRCUITS UNDER HARMONIC OSCILLATIONS ACTION.....	53
10.1. Series oscillatory circuit. Voltage resonance.....	53
10.2. Parallel oscillatory circuit. Current resonance.....	59
REFERENCES.....	65

INTRODUCTION

The theory of the sinusoidal steady-state response of circuits occupies a position of pre-eminence in electric-circuit theory. The analysis of many circuits and devices throughout all branches of electrical engineering is accomplished by the techniques embodied in the sinusoidal theory. Particularly impressive in this regard is the fact that the sinusoidal circuit theory is applicable not only in situations involving sinusoidal forcing functions, but equally in those situations where the forcing functions have a non-sinusoidal character.

It is not by chance that the bulk of the electric power generated in power plants throughout the world and distributed to the consumer appears in the form of sinusoidal variations of voltage and current. There are many technical and economical advantages of using sinusoidal voltages and currents. A significant appreciation of this statement will be gained upon the completion of the study of this book. For example, it will be learned that the use of sinusoidal voltages applied to appropriately designed coils results in a revolving magnetic field which has the capacity to do work. As a matter of fact, it is this principle that underlies the operation of almost all electric motors found in home appliances and about 90% of all electric motors found in commercial and industrial applications. Although other waveforms can be used in such devices, none leads to an operation which is as efficient and economical as that achieved through the use of sinusoidal functions.

In addition to these practical aspects, however, the sinusoidal function offers some very important and significant advantages from a mathematical perspective.

1. SINUSOIDAL FUNCTIONS: TERMINOLOGY

In dealing with sinusoidal functions we must become familiar with the nomenclature before proceeding with the sinusoidal steady-state analysis of circuits. This makes it easier to describe and to interpret the results.

Harmonic oscillations in electrical circuits are the changes of currents, voltages and electromotive forces (EMFs) over the time.

Instantaneous values of current $i(t)$, voltage $u(t)$ and EMF $e(t)$ at the time t are changed under harmonic law.

Sinusoidal alternating current, voltage and EMF are the types of harmonic oscillations, which are a sine function. Electrical circuits where alternating currents (AC) flow are termed *AC circuits*.

The mathematical equations of variable single-phase harmonic oscillations of

- current is $i(t) = I_m \sin\left(\frac{2\pi}{T}t + \psi_i\right) = I_m \sin(\omega t + \psi_i);$ (1.1)

- voltage is $u(t) = U_m \sin\left(\frac{2\pi}{T}t + \psi_u\right) = U_m \sin(\omega t + \psi_u);$ (1.2)

- EMF is $e(t) = E_m \sin\left(\frac{2\pi}{T}t + \psi_e\right) = E_m \sin(\omega t + \psi_e).$ (1.3)

For example, the graph in Fig. 1.1 shows the *instantaneous current $i(t)$* at the time t .

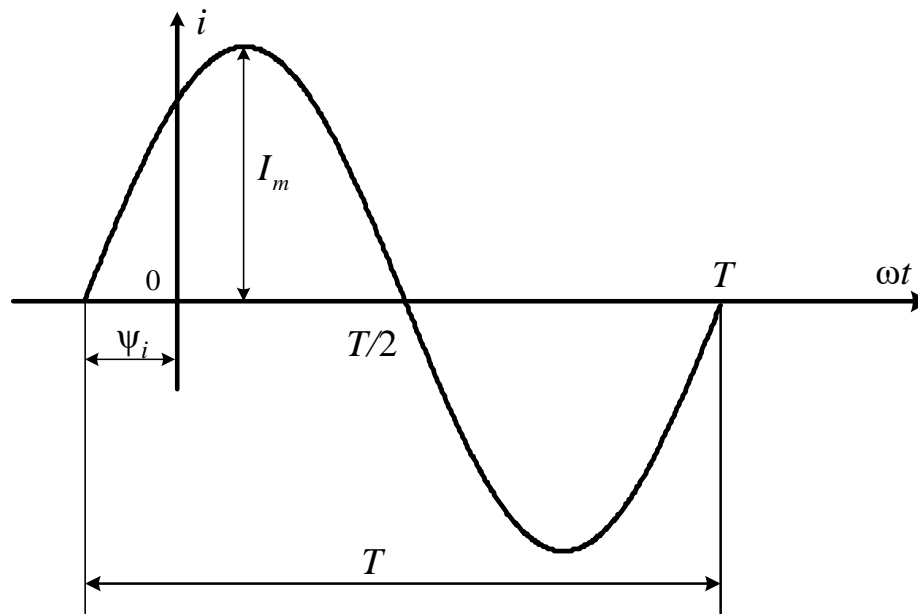


Fig. 1.1

Currents, voltages and EMFs in AC circuits cannot be completely characterized by algebraic values, like it was in DC circuits.

According to equations (1.1)-(1.3), the harmonic variable values are characterized by the following parameters:

- I_m, U_m, E_m are *maximum (amplitude) values*;
- $T = \frac{1}{f}$ is the time occupied by one complete cycle of change, or the *period*; it is measured in seconds (s);
- f is *cycle frequency* which is inversely proportional to period; it is measured in hertz (Hz);
- $\omega = 2\pi f$ is *radian (or angular) frequency*; the unit of measurement is radian per second;
- $\omega t + \psi$ is *argument (or oscillations phase)* which determines the value of sinusoidal function at the specified moment of time; oscillations phase is measured in radians (rad) or degree ($^{\circ}$);
- $\psi = (\omega t + \psi)|_{t=0}$ is *initial phase angle*. It is specified from the origin of coordinates ($t = 0$) to the beginning of a sinusoidal function. If the function of

harmonic oscillation is shifted to the left, then angle $\psi > 0$. If it is shifted to the right, then $\psi < 0$.

The sinusoidal functions of current $i(\omega t)$ and voltage $u(\omega t)$ which are changing with equal frequency are given in Fig. 1.2. They are shifted relative to each other's phase. The *phase shift* is measured by *phase difference* φ , which is equal to the difference of initial phases $\varphi = \psi_u - \psi_i$. The numerical value of the phase shift angle is selected in the range $-\pi \leq \varphi \leq \pi$.

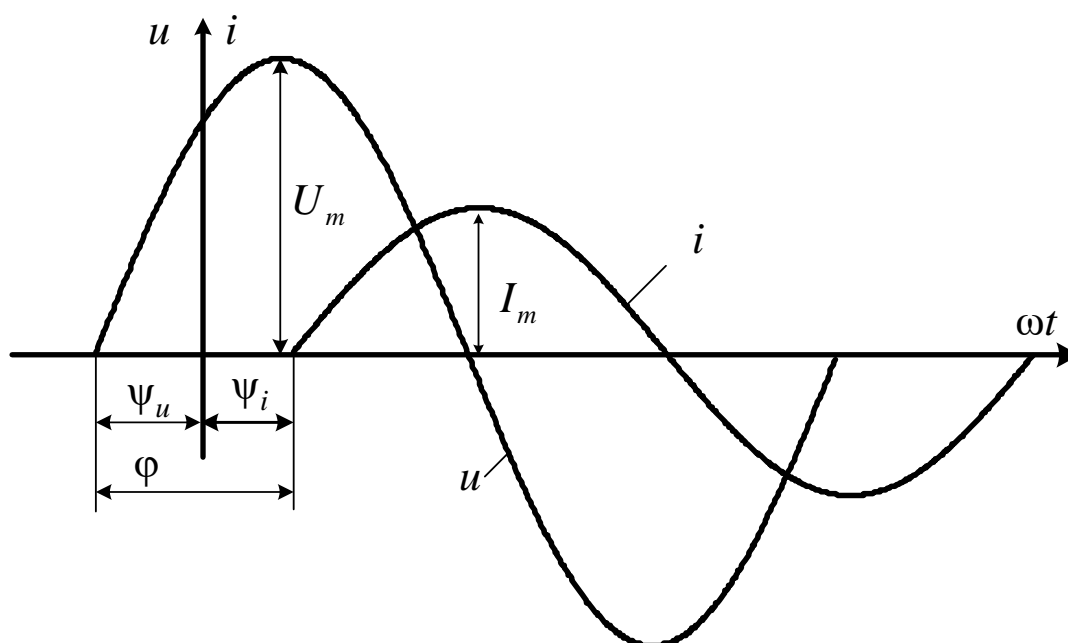


Fig. 1.2

If

$\varphi > 0$, the voltage *leads in phase* with the current;

$\varphi = 0$, the voltage and the current are *in phase*;

$\varphi < 0$, the voltage *lags in phase* with the current.

2. EFFECTIVE VALUES OF HARMONIC QUANTITY

In linear DC electric circuit theory all values are constants, i.e. they are not dependent on time. Any constant character of the response makes it a simple matter to identify the number of amperes flowing in the circuit and thereby to describe the energy-transferring capability of the circuit. Moreover, the computation of the power absorbed by each circuit element is accomplished in a direct manner through the use of the equation $P=I \times U$ by inserting the constant values of voltage and current.

The sinusoidal current is an *alternating* current, i.e. one which has positive and negative values. In this case, the manner of describing the energy-transferring capability of the current is not at all obvious, as it is when direct sources are used; in the latter case the average current flow is identical to the direct (or constant) value.

In view of the fact that the average current serves as a useful criterion in determining the energy transfer in circuits involving direct sources, it is important to explore its usefulness in situations involving sinusoidal (periodic) driving functions.

The *average* (or *mean*) value of sinusoidal quantity over one cycle, or an integral number of cycles, is obviously zero. A significant mean will therefore be the average of the values prevailing during one positive (or negative) half-cycle. Thus, the average current during one half-cycle is given by

$$I_{av} = \frac{1}{T/2} \int_0^{T/2} i dt = \frac{1}{T/2} \int_0^{T/2} I_m \sin(\omega t + \psi_i) dt = \frac{2}{\pi} I_m.$$

Similarly, for voltage $U_{av} = \frac{2}{\pi} U_m$ and EMF $E_{av} = \frac{2}{\pi} E_m$.

Thus, the average value of either the positive or negative half of a sine function can be found simply by multiplying the amplitude of the wave by 0.636. When taken over a full cycle, the equal and opposite average values cancel out.

The *effective* value of current is the magnitude of the direct current which has the same heating effect in a given resistive circuit as the alternating current in question.

For the practical purposes, a convenient mean value is the *effective* value of current. It is the *square root of the mean of the currents squared* (which are all positive).

Hence the root mean square (r.m.s. or effective) value of current is given by

$$I = \sqrt{\frac{1}{T} \int_0^T i^2 dt} = 0,707 I_m = \sqrt{2} I_m .$$

Similarly,

$$E = \sqrt{\frac{1}{T} \int_0^T e^2 dt} = 0,707 E_m = \sqrt{2} E_m, \quad U = \sqrt{\frac{1}{T} \int_0^T u^2 dt} = 0,707 U_m = \sqrt{2} U_m .$$

A direct voltage source applied to the resistor R causes an average power dissipation of $I^2 R$. Here I denotes the direct or average current. The flow of a periodic function of current through the same resistor yields an average power dissipation of $I_{ef}^2 R$. Comparing this result with that of the direct-current case yields another interpretation of effective current: it is the current which produces the same heating effect as the direct current.

Most electrical measuring instruments are constructed to indicate the effective values of currents and voltages being measured.

3. REPRESENTATION FORMS OF SINUSOIDAL QUANTITIES

Analyzing the sinusoidal steady-state response of circuits, it is necessary to perform algebraic operations, such as addition, subtraction, multiplication and division, on two or more sinusoidal quantities of the same frequency. The sinusoids usually differ in amplitude and phase.

There are three representation forms of sinusoidal quantities:

- *analytical* (trigonometric form), for example $i = I_m \sin(\omega t + \psi_i)$,
 $u = U_m \sin(\omega t + \psi_u)$, $e = E_m \sin(\omega t + \psi_e)$;
- *graphical* (Fig. 1.1);
- *vectorial* (Fig. 3.1), which presents generating the sine function from the vertical component of the rotating line.

Let us consider the addition of two sinusoidal currents whose equations are as follows:

$$i_1 = I_{m1} \sin(\omega t), \quad i_2 = I_{m2} \sin(\omega t + \psi_2).$$

The current i_2 leads i_1 by the angle ψ_2 of relative phase. The resultant current i_3 can obviously be written as

$$i_3 = i_1 + i_2 = I_{m1} \sin(\omega t) + I_{m2} \sin(\omega t + \psi_2).$$

The addition of two sinusoids of the same frequency always results in another sinusoid. It is necessary to understand how a resultant appropriate amplitude and phase of the current i_3 are expressed. Note that any sinusoid at a given frequency is exactly specified once its amplitude and phase are known. Using graphical forms of sine functions, we can plot each sinusoid and then make a point-by-point summation of the two sine waves. The amplitude and phase of the resultant sinusoid can be measured, thus allowing i_3 to be written in the more useful form:

$$i_3 = I_{m3} \sin(\omega t + \psi_3),$$

where ψ_3 is the phase angle measured with respect to the same reference point used for ψ_2 . Needless to say, such a procedure is laborious and time-consuming.

An alternative to the graphical solution is an analytical one which simplifies the procedure through the use of trigonometric identities.

Although this analytical procedure requires less effort than the graphical one, the method is too cumbersome, in particular in situations where more than two sinusoidal quantities have to be summed.

Furthermore, multiplication and division present additional complications. Clearly, we need a simpler and more direct method of treating sinusoidal quantities. In 1893, such a method was introduced when Charles P. Steinmetz advanced the idea of using a constant amplitude line rotating at a frequency ω to represent a sinusoid. Let us consider how this idea is effective in simplifying the algebraic operations involving sinusoidal quantities.

This method is based on the replacement of trigonometric function with rotation vectors.

The main idea of this method is that any sine function, for example $i_1 = I_{m1} \sin(\omega t)$ can be rewritten as $\text{Im} \left[I_{m1} e^{j\omega t} \right]$.

It is important to keep in mind that the exponential function $e^{j\omega t}$ may be treated as a rotational operator. Its amplitude is always unity, but the cosine and sine components vary when time progresses. This is illustrated in Fig. 3.1, *a*. When ωt moves through one full period of 2π radians (i.e. one complete cycle), the line OA makes one complete traversal of the circle in a counter clockwise direction. Line OA is fixed in value of the sine function amplitude. Note that the vertical projection of line OA is the sine function, and the horizontal one is the cosine function.

As a matter of fact, this is the meaning of the notation $\text{Im}[\dots]$ – it refers to the values generated by taking the projections of a rotating line on a pre-established reference line (the vertical in this case).

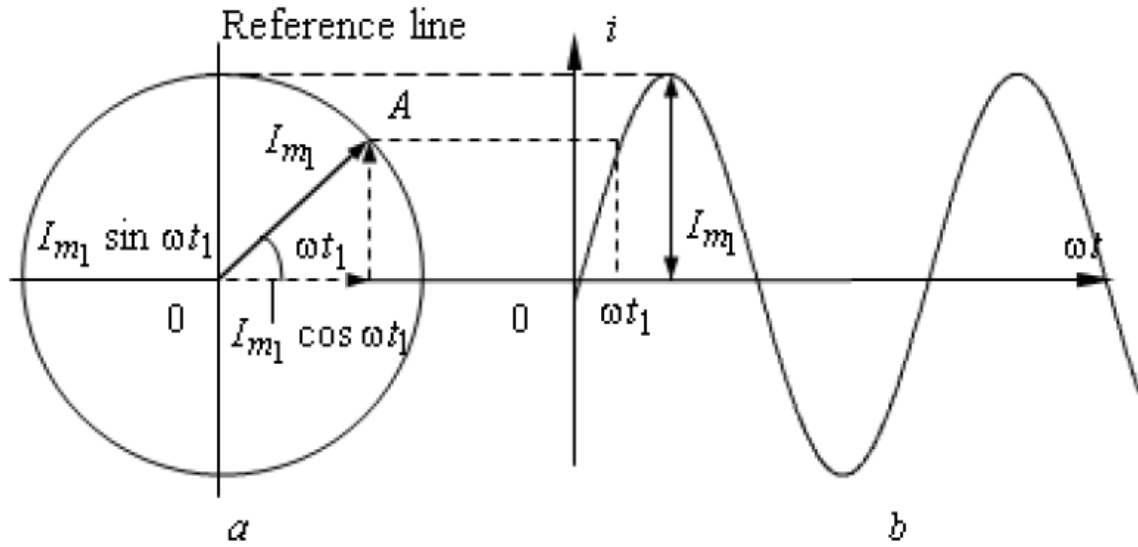


Fig. 3.1

Accordingly, if we plot the vertical components of OA as it makes one complete revolution, the sine function shown in Fig. 3.1, b is generated. When $\omega t = 0$, the position of OA is on the horizontal axis directed towards the right. Its vertical component at this instant is zero, as it should be for the sine function.

The current i_2 can be represented in a similar fashion. Thus, in terms of the exponential notation, we have

$$i_2 = I_{m2} \sin(\omega t + \psi_2) = \text{Im} \left[I_{m2} e^{j\omega t} e^{j\psi_2} \right].$$

To simplify this notation, it is convenient to use the *phasor method*. The *phasor* of the sinusoidal function of i_2 is defined as

$$\dot{I}_{m2} = I_{m2} e^{j\psi_2} = I_{m2} \cos(\psi_2) + j I_{m2} \sin(\psi_2).$$

In general, the phasor can be represented by a *complex number* which is a result of locating a line in a plane, which means that each point in the complex plane is determined by the radius vector of this point, i.e. by vector whose beginning coincides with the coordinate origin and its ending is at a point corresponding to a given complex number. The complex plane with arbitrary vector is shown in Fig. 3.2.

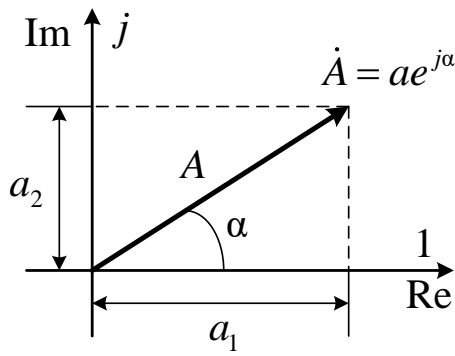


Fig.3.2

$a_1 = \text{Re}[\dot{A}] = \text{Re}[ae^{j\alpha}]$ is the projection of a vector on the horizontal axis called an axis of real numbers;

$a_2 = \text{Im}[\dot{A}] = \text{Im}[ae^{j\alpha}]$ is the projection of a vector on vertical axis called an axis of imaginary numbers.

Where $j = \sqrt{-1}$ is the imaginary unit;

Im is the axis of imaginary numbers;

Re is the axis of real numbers.

There are three forms of a complex number.

Using *exponential* or *polar form* of a complex number, we have

$$\dot{A} = Ae^{j\alpha} = A\angle\alpha,$$

where A is the modulus of a complex number; α is the argument or phase.

Applying *Euler's formula*, we obtain a *trigonometric form* of a complex number:

$$\dot{A} = A\cos\alpha + jA\sin\alpha$$

or the respective *algebraic form*:

$$\dot{A} = a_1 + ja_2,$$

$$\text{where } a_1 = \text{Re}[\dot{A}] = A\cos\alpha \text{ and } a_2 = \text{Im}[\dot{A}] = A\sin\alpha. \quad (3.1)$$

Hypotenuse and both cathetuses of a right triangle shown in Fig. 3.2 are related by Pythagorean Theorem:

$$A = \sqrt{a_1^2 + a_2^2}; \quad \alpha = \arctg \frac{a_2}{a_1}. \quad (3.2)$$

The complex value of instantaneous values of the current is

$$I_m e^{j(\omega t + \psi)} = I_m e^{j\psi} e^{j\omega t} = \dot{I}_m e^{j\omega t}.$$

The complex of current amplitude $\dot{I}_m = I_m e^{j\psi}$ presents the current i in the complex plane for the instant $\omega t = 0$.

The complex values of the effective values of both current and voltage are as follows:

$$\dot{U} = \frac{\dot{U}_m}{\sqrt{2}}; \dot{I} = \frac{\dot{I}_m}{\sqrt{2}} .$$

Vector diagrams are used to analyse the circuits. Vector diagram is a set of vectors representing sinusoidal magnitudes of equal frequency.

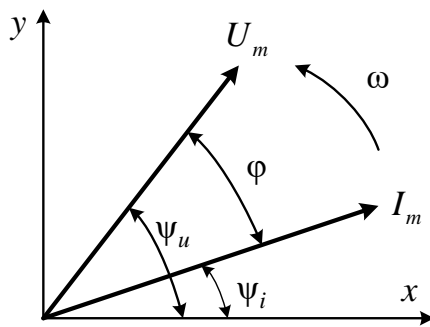


Fig.3.3

The length of the vector is equal to the amplitude or effective value, the angle with the axis OX is the initial phase angle ψ_u or ψ_i . The angle between the vectors (phase shift angle φ) remains unchanged.

4. MATHEMATICAL OPERATIONS WITH THE COMPLEX VALUES

1. Finding the sum and difference by using the algebraic form:

$$\dot{A}_1 = A_1 e^{j\alpha_1} = a_1 + jb_1, \dot{A}_2 = A_2 e^{j\alpha_2} = a_2 + jb_2;$$

$$\dot{A}_1 \pm \dot{A}_2 = (a_1 \pm a_2) + j(b_1 \pm b_2) = \dot{A} = a + jb.$$

For example, if $\dot{A}_1 = A_1 e^{j\alpha_1} = 12e^{j25^\circ}$, $\dot{A}_2 = A_2 e^{j\alpha_2} = 17e^{-j30^\circ}$; then using (3.1), we obtain:

$$\begin{aligned} \dot{A}_1 + \dot{A}_2 &= 12e^{j25^\circ} + 17e^{-j30^\circ} = 10,8757 + j5,0714 + 14,7224 - j8,5 = \\ &= 25,5981 - j3,4286; \end{aligned}$$

$$\begin{aligned} \dot{A}_1 - \dot{A}_2 &= 12e^{j25^\circ} - 17e^{-j30^\circ} = 10,8757 + j5,0714 - 14,7224 + j8,5 = \\ &= -3,8467 + j13,5714. \end{aligned}$$

2. Multiplying and dividing using exponential or polar form:

$$\dot{A}_1 \cdot \dot{A}_2 = A_1 e^{j\alpha_1} \cdot A_2 e^{j\alpha_2} = A_1 \cdot A_2 e^{j(\alpha_1 + \alpha_2)};$$

$$\frac{\dot{A}_1}{\dot{A}_2} = \frac{A_1 e^{j\alpha_1}}{A_2 e^{j\alpha_2}} = \frac{A_1}{A_2} e^{j(\alpha_1 - \alpha_2)}.$$

For example, if $\dot{A}_1 = a_1 + jb_1 = 11 + j19$, $\dot{A}_2 = a_2 + jb_2 = 25 - j17$; then using (3.2), we obtain:

$$\begin{aligned} \dot{A}_1 \cdot \dot{A}_2 &= (11 + j19) \cdot (25 - j17) = (21,9545e^{j59,93}) \cdot (30,2324e^{-j34,21}) = \\ &= 633,7379e^{j25,71} = 598 + j288; \end{aligned}$$

$$\frac{\dot{A}_1}{\dot{A}_2} = \frac{11 + j19}{25 - j17} = \frac{21,9545e^{j59,93}}{30,2324e^{-j34,21}} = 0,7262e^{j94,14} = -0,0524 + j0,7243.$$

3. Raising to a power by using the exponential form:

$$(\dot{A})^n = A^n e^{j(n\alpha)}; \quad \sqrt[n]{\dot{A}} = \sqrt[n]{A} \cdot e^{j\frac{\alpha + 2k\pi}{n}}, \text{ where } k = 0, 1, \dots, (n-1).$$

$$\text{For example, } (22e^{j60})^2 = 484e^{j120}; \quad \sqrt[2]{22e^{j60}} = 4,69 \cdot e^{j30}.$$

4. Conjugate complex values for $\dot{A} = A e^{j\alpha} = a + jb$ are the complex value $\dot{A}^* = A e^{-j\alpha} = a - jb$ which is a different sign of the argument α .

$$\dot{A} \cdot \dot{A}^* = a^2 + b^2; \quad \dot{A} + \dot{A}^* = 2a; \quad \dot{A} - \dot{A}^* = 2jb; \quad \dot{A} / \dot{A}^* = e^{j2\alpha}.$$

5. Multiplication of the complex value on $\pm j$ corresponds to its turn by the angle $\pm 90^\circ$:

$$\pm j = e^{\pm j90^\circ}; \quad \dot{A} \cdot (\pm j) = Ae^{j\alpha} \cdot 1e^{\pm j90^\circ} = Ae^{j(\alpha \pm 90^\circ)}.$$

$$j \cdot j = j^2 = e^{j180^\circ} = -1; \quad \frac{1}{j} = -j.$$

6. Representation of the integral in a complex form.

To integrate a sine wave function, you need to split its complex image into $j\omega$:

$$\int idt \rightarrow \frac{\dot{I}_m}{j\omega}.$$

For example, the complex voltage on the capacitor is

$$u_C = \frac{1}{C} \int idt \rightarrow \dot{U}_{Cm} = \frac{\dot{I}_m}{j\omega C} = -j \frac{\dot{I}_m}{\omega C} = -j \frac{1}{\omega C} \dot{I}_m.$$

7. Representation of the derivative in a complex form.

To represent a derivative of a sinusoidal function, its complex image must be multiplied by $j\omega$:

$$\frac{di}{dt} \rightarrow j\omega \dot{I}_m.$$

For example, the complex voltage on inductance is

$$u_L = L \frac{di}{dt} \rightarrow \dot{U}_{Lm} = j\omega L \dot{I}_m.$$

Example 4.1. Two sinusoidal currents are described as follows:

$$i_1 = 10\sqrt{2} \sin(\omega t) \text{ A and } i_2 = 20\sqrt{2} \sin(\omega t + 60^\circ) \text{ A}.$$

Find the expression for the sum of these currents.

Solution: The solution is found by applying the phasor method. First, we rewrite equations i_1 and i_2 as complex values:

$$i_1 = 10\sqrt{2} \sin(\omega t) \text{ A} \leftrightarrow \dot{I}_{m1} = 10\sqrt{2};$$

$$\begin{aligned} i_2 &= 20\sqrt{2} \sin(\omega t + 60^\circ) \text{ A} \leftrightarrow \dot{I}_{m2} = 20\sqrt{2} e^{j60^\circ} = \\ &= 20\sqrt{2} \left(\cos(60^\circ) + j \sin(60^\circ) \right) = 20\sqrt{2} \left(\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = 10\sqrt{2} + j10\sqrt{6}. \end{aligned}$$

Hence

$$i_3 = i_1 + i_2 \leftrightarrow \dot{I}_{m3} = \dot{I}_{m1} + \dot{I}_{m2} = 10\sqrt{2} + 10\sqrt{2} + j10\sqrt{6} = 20\sqrt{2} + j10\sqrt{6}.$$

So,

$$I_{m3} = \sqrt{(20\sqrt{2})^2 + (10\sqrt{6})^2} = 37,4, \quad \psi = \arctg \frac{10\sqrt{6}}{20\sqrt{2}} = 41^\circ.$$

$$\text{Thus, } \dot{I}_{m3} = 37,4 e^{j41^\circ} \leftrightarrow i_3 = 37,4 \sin(\omega t + 41^\circ) \text{ A}.$$

5. THE ELEMENTS OF A SINUSOIDAL CURRENT CIRCUIT

The response of each element of the circuit to a sustained sinusoidal source function is found individually for two reasons. First, it provides an opportunity to illustrate the manner in which the response can be found easily and directly by using the phasor representation of sinusoids. Second, it allows establishing the phase-angle relationships existing between the current and voltage for each element of the circuit. These relationships are fixed and must always be satisfied irrespective of whether a given circuit element is in a series or parallel arrangement with other circuit elements.

5.1. A Sinusoidal Current Through a Resistor

The resistor and the resistance parameter

A resistor is a passive two-terminal electrical component that implements electrical resistance as a circuit element. The resistor helps to control/ reduce the flow of current in a circuit. Inside a resistor, electrons collide with ions, slowing the flow of electricity and lowering the current while producing heat.

Resistance is measured in ohms, symbolized by the Greek letter omega (Ω). Ohms are named after Georg Simon Ohm (1784-1854), the German physicist who studied the relationship between voltage, current and resistance.

Because resistors convert electrical energy into heat, they are good heating elements for toasters, heaters, electric stoves and similar devices. Traditional light bulbs work because the very high temperature due to their resistance makes a metal filament white-hot, producing light. The formula $P = I^2 \cdot R$, where P is the heating power in watts, I is the current in amps, and R is the resistance in ohms, determines the amount of heat given off by a resistor.

Many components, such as heating elements and resistors, have a fixed-resistance value. These values are often printed on the nameplates of components or in the manuals for reference.

When a sinusoidal current $i_R = I_m \sin(\omega t + \psi_i)$ flows through a resistor R (Fig. 5.1, a), a voltage drop occurs in it:

$$u_R = Ri_R = RI_m \sin(\omega t + \psi_i) = U_m \sin(\omega t + \psi_u).$$

Consequently, the voltage u_R across the resistance R and the current i_R passing through this resistance are in phase. They reach their maxima U_m and I_m and pass through zero together (Fig. 5.1, b); the phase angle between the current i_R and the voltage u_R equals zero $\varphi = 0$.

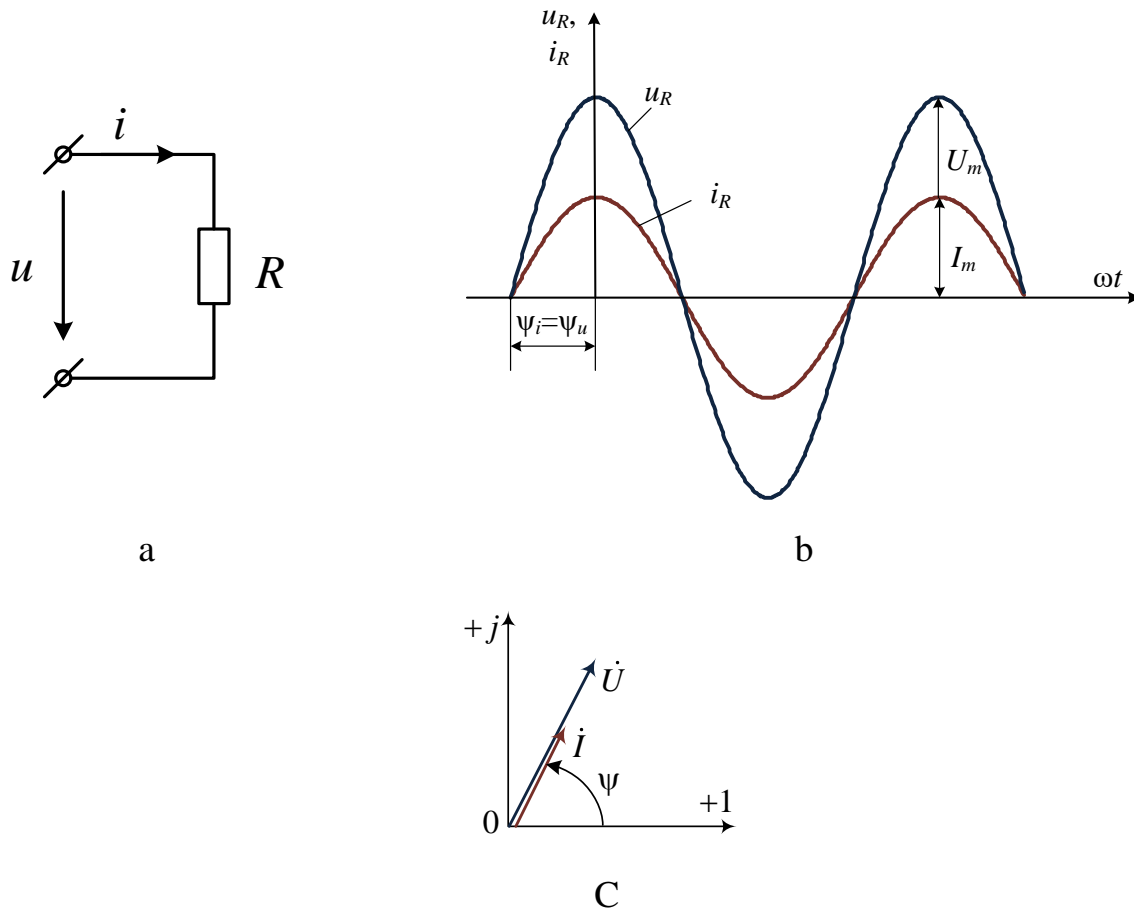


Fig. 5.1

Instantaneous amplitude and effective values of voltage and current are related by Ohm's law:

$$u_R = i_R R, \quad U_m = I_m R \quad \text{and} \quad U = IR.$$

The vector diagram for voltage and current resistor is shown in Fig. 5.1, c.

By using the conductance $G = 1/R$, we obtain

$$i_R = u_R G, \quad I_m = U_m G \quad \text{and} \quad I = UG.$$

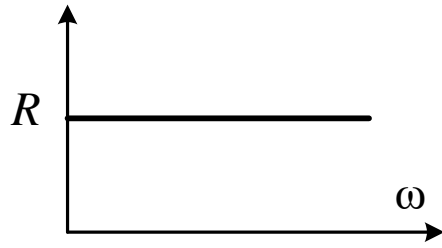


Fig. 5.2

If we neglect the skin effect, it is supposed that parameter R is not the function of frequency (Fig. 5.2).

Let us rewrite the equation for instantaneous values of voltage and current using the phasor method:

$$i \stackrel{\cdot}{=} \dot{I}_m(t) = I_m e^{j\psi_i} e^{j\omega t} = \dot{I}_m e^{j\omega t},$$

$$i \stackrel{\cdot}{=} \dot{I}_m = I_m e^{j\psi_i}, \quad u_R \stackrel{\cdot}{=} \dot{U}_{Rm} = R \cdot \dot{I}_m = R \cdot I_m e^{j\psi_i};$$

$$\dot{U}_{Rm} = R \cdot \dot{I}_m, \quad \dot{U}_R = R \cdot \dot{I}.$$

We obtain Ohm's law written in terms of phasor quantities.

The instantaneous power is given by

$$\begin{aligned} p_R = u_R \cdot i &= R \cdot i^2 = R \cdot I_m^2 \sin^2(\omega t + \psi_i) = 2R \cdot I^2 \sin^2(\omega t + \psi_i) = \\ &= R \cdot I^2 (1 - \cos(2\omega t + \psi_i)). \end{aligned}$$

Thus, the instantaneous power has an unvarying component $R \cdot I^2 = UI = \frac{1}{2} U_m I_m$ and a varying component $\frac{1}{2} U_m I_m \cos(2\omega t)$, varying at frequency 2ω . The energy input in course of the time dt is $p dt$.

The *average* value of sinusoidal power over one cycle p_R is given by

$$P_R = \frac{1}{T} \int_0^T p_R dt = \frac{1}{T} \int_0^T R I^2 dt - \frac{1}{T} \int_0^T R I^2 \cos(2\omega t + \psi_i) dt = R I^2.$$

$P_R = R I^2$ is the active (average) power.

The diagrams of the instantaneous current and power are shown in Fig. 5.3.

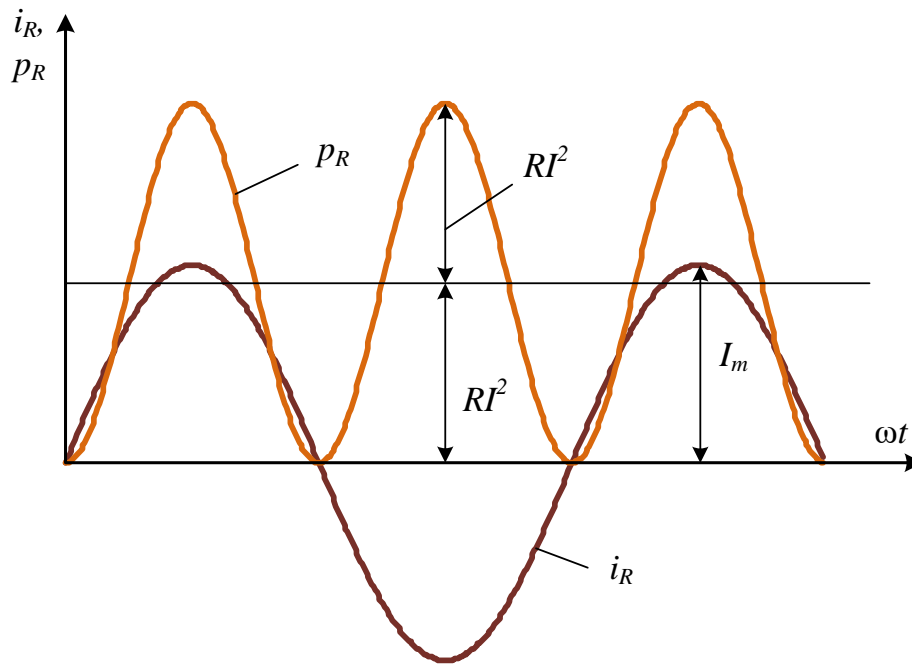


Fig. 5.3

5.2. A Sinusoidal Current Through an Induction Coil

The inductor and the inductance parameter

An inductor, also called a coil, is a passive two-terminal electrical component that stores energy in a magnetic field when electric current flows through it. An inductor typically consists of an insulated wire wound into a coil around a core.

Any change in the current flowing through an inductor creates a changing flux, inducing an electromotive force (EMF) (voltage) across the inductor.

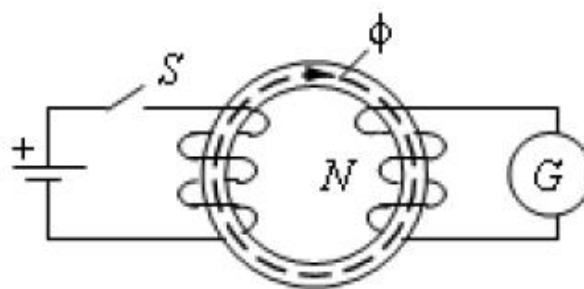


Fig. 5.4

According to Faraday's law of induction, the voltage induced by any change in the magnetic flux through the circuit is given by

$$e = -\frac{d\Psi}{dt} = -L \frac{di_L}{dt}.$$

The quantity e denotes the instantaneous value of electromotive force induced in a closed circuit having a flux linkage Ψ . In those instances where the magnetic flux Φ penetrates all the turns of the coil, as shown in Fig. 5.4, Faraday's law can be rewritten as:

$$e = -w \frac{d\Phi}{dt}.$$

According to Lenz's law, the induced voltage has a polarity (direction) which opposes the change in the current (i_L) creating this voltage. The negative sign in the equation indicates that the induced voltage is in a direction which opposes the change in the current that created it. If the current is increasing, the voltage is positive at the end of the conductor through which the current enters and negative at the end through which it leaves, tending to reduce the current. If the current is decreasing, the voltage is positive at the end through which the current leaves the conductor, tending to maintain the current.

The American inventor Joseph Henry also independently discovered the current of self-induction, but not before Faraday. Both experimenters were able to demonstrate that a changing current produced an EMF of self-induction in a coil of wire which varied directly with the time rate of change of current. The mathematical expression is given by

$$u_L = -e_L = L \frac{di}{dt},$$

where L is the factor of proportionality called the coefficient of self-inductance which is dependent upon the medium and some physical dimensions. This result is equivalent to Faraday's law of induction as expressed in the equation above.

Self-inductance, usually just called inductance L , is the ratio between the induced voltage and the rate of change of the current. Any alteration to a circuit which increases the flux (total magnetic field) through the circuit produced by a given

current increases the inductance, because inductance is also equal to the ratio of magnetic flux to current:

$$L = \frac{\Psi}{i_L}.$$

In the International System of Units (SI), the unit of inductance is henry (H) named after Joseph Henry, the American scientist of 19th century.

The same method of analysis is used to find the response of a purely inductive circuit to a sinusoidal source function in the steady state.

When a sinusoidal current $i_L = I_m \sin(\omega t + \psi_i)$ flows through the inductance L (Fig. 5.5, *a*), on the basis of expression $u_L = -e_L = L \frac{di}{dt}$ we find the voltage drop:

$$\begin{aligned} u_L &= L \frac{di_L}{dt} = L \frac{d}{dt} (I_m \sin(\omega t + \psi_i)) = \omega L I_m \sin\left(\omega t + \psi_i + \frac{\pi}{2}\right) = \\ &= U_m \sin(\omega t + \psi_u). \end{aligned}$$

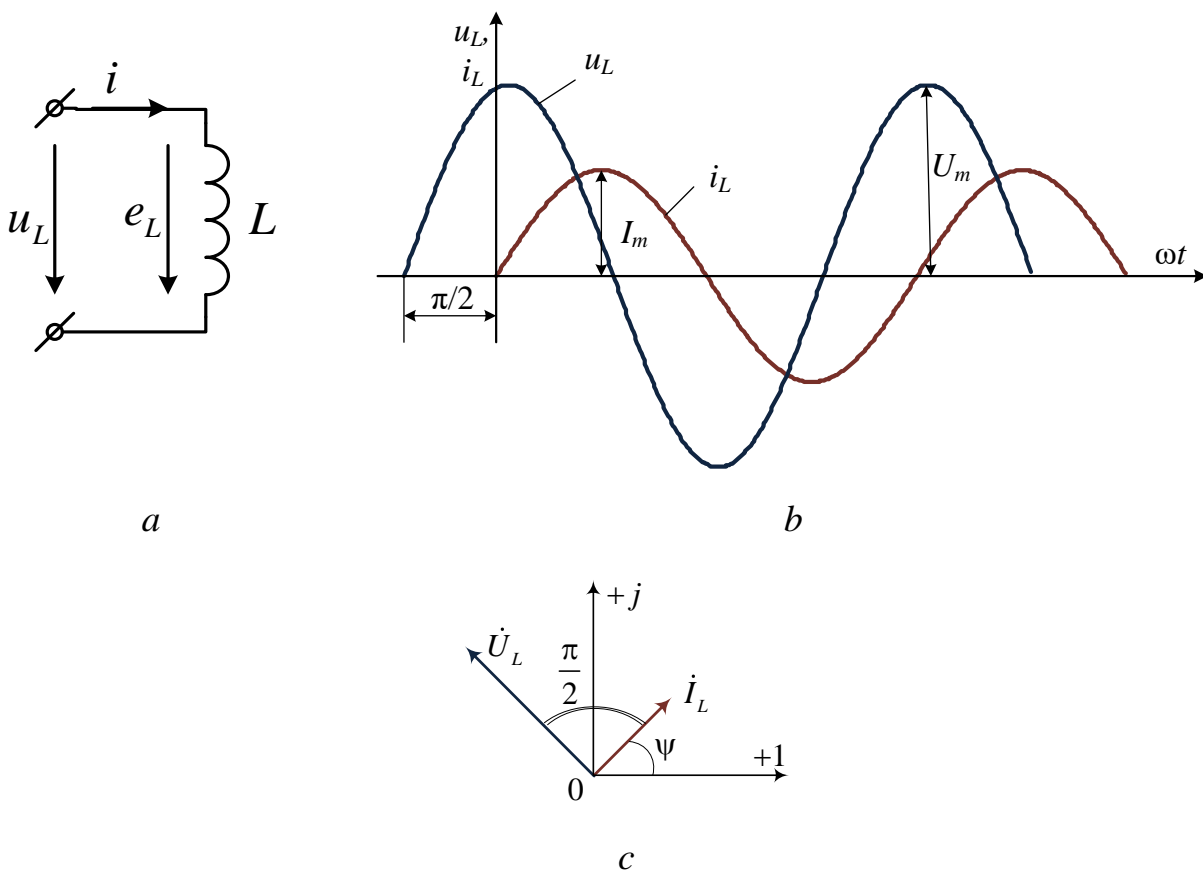


Fig. 5.5

The obtained expression shows that voltage u_L leads current i_L by angle $\pi/2$: the maximum voltage u_L is displaced to the left with respect to the maximum current i_L by $\pi/2$ (Fig. 5.5). When the current passes through zero, voltage u_L reaches a positive or a negative maximum, it is proportional to the rate of change of current di/dt which is maximum at the instant the current passes through zero (the sine curve of the current has maximum steepness at this instant); when the current reaches maximum, the voltage u_L become zero. The vector diagram given in Fig. 8b corresponds to this phenomenon.

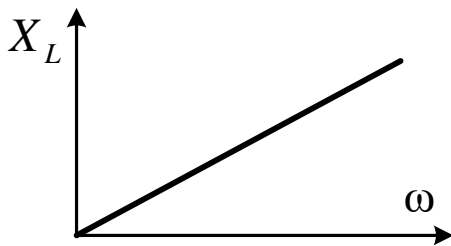
By *phase-angle difference* φ between voltage and current we understand, by convention, the *difference between the initial phases* of voltage and current.

Consequently, in this case $\varphi = \psi_u - \psi_i = \frac{\pi}{2}$.

The relation of the amplitudes or the effective values of voltage and current is similar to Ohm's law:

$$U_m = \omega L I_m \text{ and } U = \omega L I.$$

The quantity $x_L = \omega L$, measured like a resistance, in ohms, is called *inductive reactance*, and the reciprocal $b_L = 1/\omega L$ is called *inductive susceptance*.



The quantity $x_L = \omega L$ is directly proportional to the source radian frequency ω (Fig. 5.6).

Fig. 5.6

Let us replace the trigonometric functions of the inductive element with the phasor values:

$$i = I_m \sin(\omega t + \psi_i) \stackrel{\cdot}{=} \dot{I}_m = I_m e^{j\psi_i},$$

$$u_L = X_L I_m \sin(\omega t + \psi_i + \frac{\pi}{2}) \stackrel{\cdot}{=} \dot{U}_{mL} = X_L \dot{I}_m e^{j\psi_i} e^{j\frac{\pi}{2}};$$

Since $e^{j\frac{\pi}{2}} = \cos \frac{\pi}{2} + j \sin \frac{\pi}{2} = j$,

then

$$\dot{U}_{m_L} = jX_L \dot{I}_m, \quad \dot{U}_L = jX_L \dot{I}$$

$$\frac{\dot{U}_{m_L}}{\dot{I}_{m_L}} = \frac{\dot{U}_L}{\dot{I}_L} = \underline{Z}_L,$$

where $\underline{Z}_L = jX_L = X_L e^{j\frac{\pi}{2}}$ is the complex impedance of the inductance.

The instantaneous values of current, voltage and power are given by the graphs in Fig. 5.7.

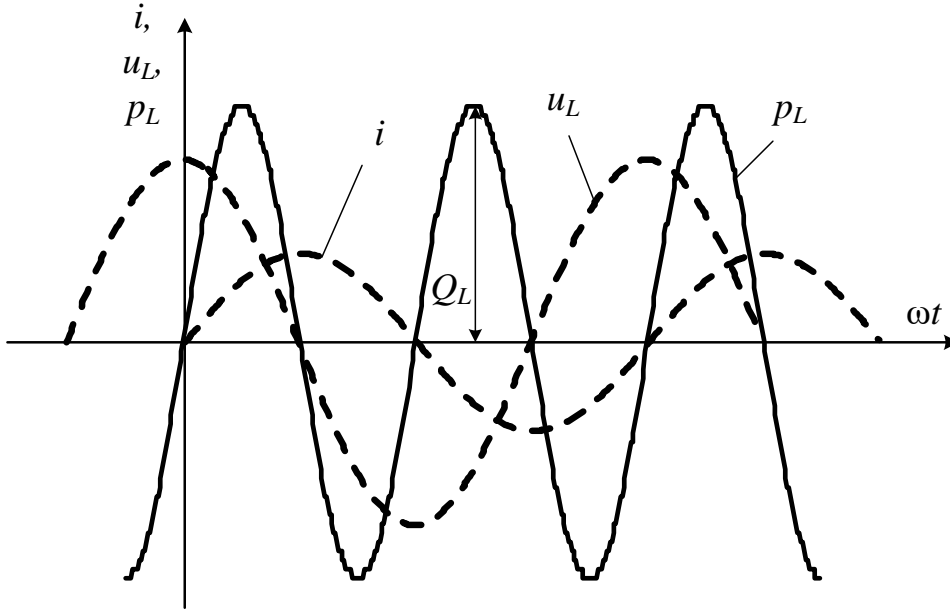


Fig. 5.7

The instantaneous power is calculated as

$$\begin{aligned} p_L &= u_L \cdot i = X_L I_m \cos(\omega t + \psi_i) \cdot I_m \sin(\omega t + \psi_i) = \\ &= 2X_L I^2 \cos(\omega t + \psi_i) \sin(\omega t + \psi_i) = X_L I^2 \sin(2\omega t + \psi_i). \end{aligned}$$

$$p_L = X_L I^2 \sin(2\omega t + \psi_i).$$

It is zero when either voltage or current is zero. During the first quarter-cycle, when the voltage and the current are positive, the power is also positive. The area that is enclosed by the curve p and the axis of abscissa during this period of time represents the energy taken from the source and spent to establish a magnetic field in the inductance.

During the second quarter-cycle, when the current in the circuit decreases from a maximum to zero, the energy of the magnetic field is returned to the supply, and the instantaneous power is negative. During the third quarter-cycle, the energy is again taken from the supply and is returned to it during the fourth quarter-cycle, and so on.

Thus, an inductance alternately draws energy from the supply and gives it back.

5.3. A Sinusoidal Current Through a Capacitor

The Capacitor and the Capacitance Parameter

The capacitor is a passive two-terminal electronic component which has the ability or “capacity” to store energy in the form of an electric charge producing a potential difference (*Static Voltage*) across its plates.

In its basic form, a capacitor consists of two or more parallel conductive (metal) plates not connected or touching each other but electrically separated by an insulating layer called *Dielectric*. Due to this insulating layer, DC current cannot flow through the capacitor as the insulating layer blocks DC current allowing a voltage drop across the plates to be present.

The conductive metal plates of a capacitor can be either square or circular, or rectangular, or they can be a cylindrical or spherical shape with the general shape, size and construction of a parallel plate capacitor depending on its application and voltage rating.

The parallel plate capacitor is the simplest form of a capacitor. It can be constructed using two metal plates at a distance from each other, with its capacitance value in Farads, being dependent on the surface area (A) of the conductive plates and the distance (d) of separation between them. Altering any two of these values alters the value of its capacitance.

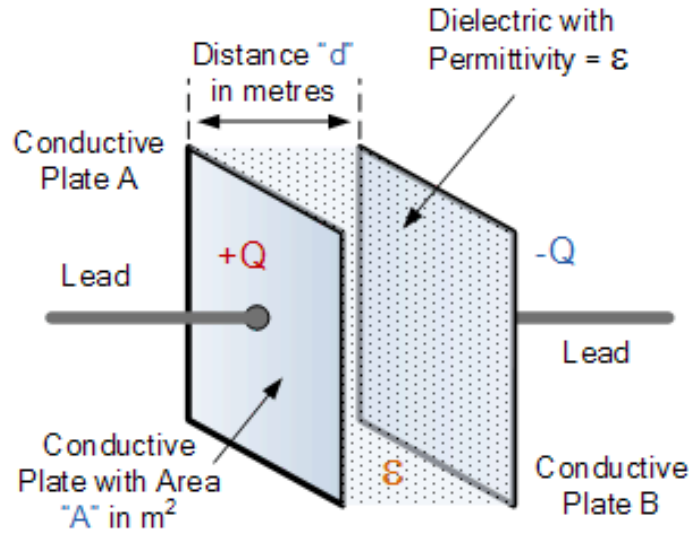


Fig. 5.8

The generalized equation for the capacitance of a parallel plate capacitor is given as:

$$C = \epsilon \frac{A}{d},$$

where

C is the capacitance of the capacitor, in farad (F);

ϵ is the permittivity of the dielectric material used in the capacitor, in farad per meter (F/m);

A is the area of the capacitor plate in square meters (m²);

d is the distance between the capacitor plates, in meters (m).

The common property of a capacitor to store charge on its plates in the form of an electrostatic field is called the **Capacitance** of the capacitor. In other words, capacitance is the electrical property of a capacitor and is the measure of the capacitor ability to store an electric charge onto its two plates with the unit of capacitance being the Farad (abbreviated to F).

Mathematically, the capacitance (C) of the capacitor is equal to the electric charge (Q) divided by the voltage (u_C):

$$C = \frac{Q}{u_C},$$

where C is the capacitance in farad (F);

Q is the electric charge in coulombs (C) stored on the capacitor;

u_C is the voltage between the capacitor plates in volts (V).

The capacitance of a parallel plate capacitor is proportional to the smallest area A in square metres of the two plates and inversely proportional to the distance d (i.e. the dielectric thickness) given in metres between these two conductive plates.

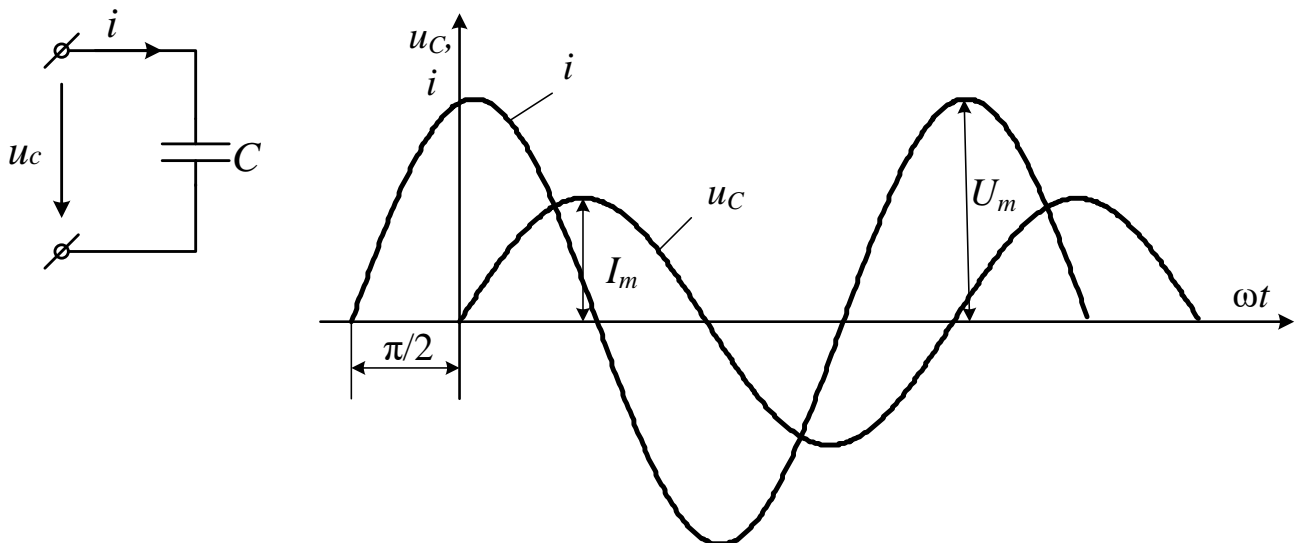
The instantaneous current $i_C(t)$ flowing through the capacitor is equal to the capacitance of the capacitor multiplied by the derivative of the instantaneous capacitor voltage $u_C(t)$:

$$i_C(t) = \frac{dQ}{dt} = \frac{d(Cu_C)}{dt} = C \frac{du_C}{dt}.$$

This expression shows the manner in which the current flowing through a capacitance parameter is related to the potential difference across it.

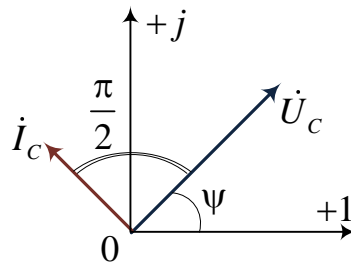
If the voltage across a capacitance is sinusoidal, i.e. $u_C = U_m \sin(\omega t + \psi_u)$, then the expression for the current given above is as follows:

$$\begin{aligned} i_C &= C \frac{d}{dt} (U_m \sin(\omega t + \psi_u)) = \omega U_m \sin\left(\omega t + \psi_u - \frac{\pi}{2}\right) = \\ &= I_m \sin\left(\omega t + \psi_u - \frac{\pi}{2}\right). \end{aligned}$$



a

b



c

Fig. 5.9

The electric charge changes in accordance with a sinusoidal law, corresponding to the applied voltage u_C . The alternate storage of positive and negative charges on the plates of the capacitive element allows a sinusoidal current i_C to flow into the circuit in which the capacitance is connected. Its value is determined by the time rate of change of the charge of the capacitance dq/dt .

The expression for i_C shows that the current leads the applied voltage u_C by angle $\pi/2$ (Fig. 5.9, *b*). The maximum (positive and negative) values of voltage u_C correspond to the zero values of current. This is explained by the fact that when the current passes through zero, the electric charge q of the capacitance reaches a maximum value (positive or negative) while voltage u_C is directly proportional to the charge.

As in the previous case, by the angular phase difference between the voltage and the current we understand the difference between the initial phases of voltage and current, i.e. $\varphi = \psi_u - \psi_i = -\frac{\pi}{2}$. The vector diagram of voltage and current of the capacitance is shown in Fig. 5.9.

Thus, angle φ represents an algebraic quantity: it is positive in the case of inductance and negative in the case of capacitance.

The relation of the amplitudes, correspondingly, the effective values of voltage and current is similar to Ohm's law

$$U_{mC} = x_C I_m \text{ and } U_C = x_C I.$$

The quantity $x_C = 1/\omega C$, measured like a resistance, in ohms, is called *capacitive reactance*, and its reciprocal value $b_C = \omega C$ is called *capacitive susceptance*. Therefore, for the current, the expressions given above are as follows:

$$I_m = b_C U_{mC} \text{ and } I = b_C U_C.$$

Thus, there are various parameters in sinusoidal current circuits, depending on the types of elements: resistance R , inductive reactance $x_L = \omega L$ and capacitive reactance $x_C = 1/\omega C$.

The current flowing through the inductance lags behind the voltage, and the current flowing through the capacitance leads the applied voltage by angle $\pi/2$. Consequently, *the current flowing through a capacitor in the sinusoidal steady state always leads the potential difference across the capacitor by 90° .*

It should be noted that *Ohm's law is applicable to the instantaneous values of voltage and current only in the case of resistance R .* In the remaining cases, the ratio of the instantaneous values of u and i do not represent reactance, but reflect a certain time function of no practical interest and no use.

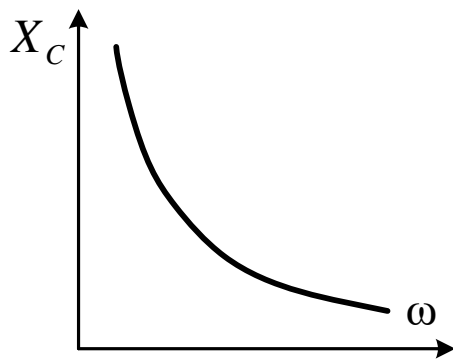


Fig. 10

The quantity $x_C = \frac{1}{\omega C}$ is inversely proportional to the source radian frequency ω (Fig. 5.10).

Let us use the phasor method:

$$i = I_m \sin(\omega t + \psi_i) \stackrel{\cdot}{=} \dot{I}_m = I_m e^{j\psi_i};$$

$$u_C = U_{cm} \sin(\omega t + \psi_i - \frac{\pi}{2}) \stackrel{\cdot}{=} \dot{U}_{cm} = X_C I_m e^{j\psi_i} e^{-j\frac{\pi}{2}} = \dot{I}_m X_C e^{-j\frac{\pi}{2}} = -jX_C \dot{I}_m,$$

$$\dot{U}_{cm} = -jX_C \dot{I}_m, \quad \dot{U}_C = -jX_C \dot{I}.$$

$$\frac{\dot{U}_{Cm}}{\dot{I}_m} = \frac{\dot{U}_C}{\dot{I}} = \underline{Z}_C.$$

where $\underline{Z}_C = j - X_C = X_C e^{-j\frac{\pi}{2}}$ is the complex impedance of the capacitance.

The instantaneous values of current, voltage and power are given by the graphs in Fig. 5.11.

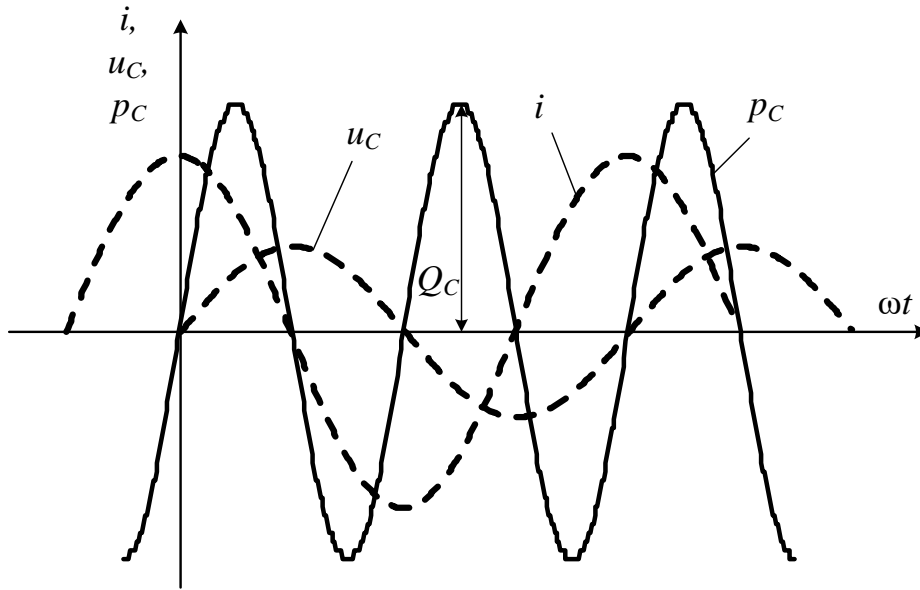


Fig. 5.11

The math equation for the instantaneous power is written as:

$$\begin{aligned} p_C &= u_C \cdot i = -X_C I_m \cos(\omega t + \psi_i) \cdot I_m \sin(\omega t + \psi_i) = \\ &= -2X_C I^2 \cos(\omega t + \psi_i) \sin(\omega t + \psi_i) = -X_C I^2 \sin(2\omega t + \psi_i). \end{aligned}$$

It is zero when either voltage or current is zero. During the first quarter-cycle, when the voltage and the current are positive, the power is also positive. The area enclosed by the curve p and the axis of abscissa during this period of time represents the energy taken from the source and spent to establish the electric field in the capacitance.

During the second quarter-cycle, when the current in the circuit decreases from a maximum to zero, the energy of the electric field is returned to the supply, and the instantaneous power is negative. During the third quarter-cycle, the energy is again taken from the supply and is returned to it during the fourth quarter-cycle, and so on.

Thus, the capacitance alternately draws energy from the supply and gives it back like inductance.

5.4. Resume

Electric circuit devices capable to reserve the energy of electric or magnetic field are called reactive.

The symbol of a resistor is given in Fig.5.12, *a*, the symbol of an inductor is in Fig. 5.12, *b*, and the symbol of a capacitor is in Fig. 5.12, *c*.

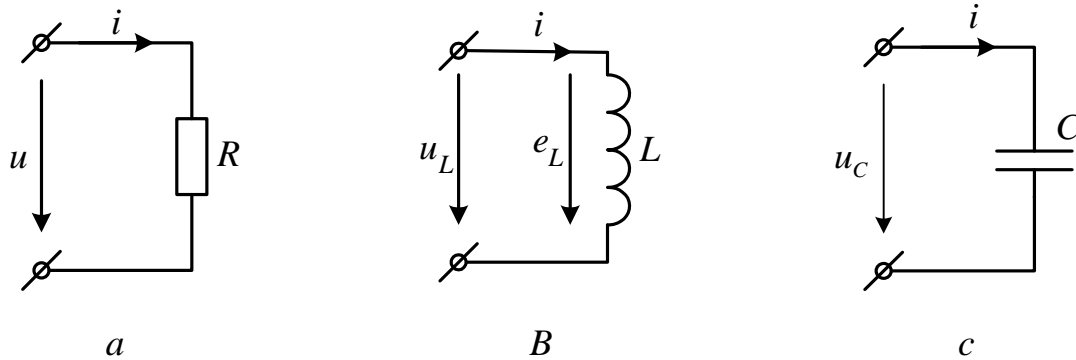


Fig.5.12.

All basic formulas which describe the properties of reactive elements of an electrical circuit are represented in Table 5.1.

Table 5.1.

	Current	Voltage	Power	Energy
R	$i = \frac{u_R}{R}$	$u_R = Ri$	$p_R = i_R^2 \cdot R$	$W_R = \int_0^t i_R^2 \cdot R dt$
L	$i_L(t) = \frac{1}{L} \int_{-\infty}^t u_L dt$	$u_L = L \frac{di_L}{dt}$	$p_L = i_L \cdot L \frac{di_L}{dt}$	$W_L = \frac{Li^2}{2}$
C	$i_C(t) = \frac{dQ}{dt} = C \frac{du_C}{dt}$	$u_C(t) = u_C(0) + \frac{1}{C} \int_{-\infty}^t i_C dt$	$p_C = u_C \cdot C \frac{du_C}{dt}$	$W_C = \frac{Cu_C^2}{2}$

As is seen from this table, only in a resistive element the current and voltage are directly proportional to each other and algebraically related. But there are integral-differential parities between current and voltage in inductive and capacitive elements.

6. KIRCHHOFF'S LAWS IN COMPLEX NOTATION

By *Kirchhoff's first law*, the algebraic sum of the instantaneous currents at any node of network is zero:

$$\sum_{k=1}^n \pm i_k = 0.$$

When replacing phasor $\dot{I}_{m_k} = I_{m_k} e^{j\psi_k} e^{j\omega t} = \dot{I}_{m_k} e^{j\omega t}$ with i_k in the previous equation and putting $e^{j\omega t}$ before the brackets, we have

$$e^{j\omega t} \sum_{k=1}^n \pm \dot{I}_{m_k} = 0.$$

Since $e^{j\omega t} \neq 0$ for any t , it follows that

$$\sum_{k=1}^n \pm \dot{I}_{m_k} = 0. \quad (6.1)$$

Equation (6.1) is Kirchhoff's current law (KCL) in a complex form.

Suppose the instantaneous currents (fig. 6.1) are described by equations

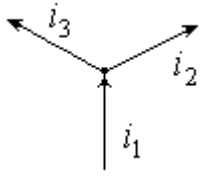


Fig. 6.1

$$\begin{aligned} i_1 &= I_{m1} \sin(\omega t + \psi_1); \\ i_2 &= I_{m2} \sin(\omega t + \psi_2); \\ i_3 &= I_{m3} \sin(\omega t + \psi_3), \end{aligned}$$

then

$$\begin{aligned} i_1 &= i_2 + i_3 = \\ &= I_{m2} \sin(\omega t + \psi_2) + I_{m3} \sin(\omega t + \psi_3) = I_{m1} \sin(\omega t + \psi_1). \end{aligned}$$

Let the given sinusoidal currents (i_1, i_2, i_3) be symbolized by the complex functions

$\dot{I}_{m1} = I_{m1} e^{j\psi_1} e^{j\omega t}$, $\dot{I}_{m2} = I_{m2} e^{j\psi_2} e^{j\omega t}$, $\dot{I}_{m3} = I_{m3} e^{j\psi_3} e^{j\omega t}$. The complex currents amplitudes, correspondingly, equal $\dot{I}_{m1} = I_{m1} e^{j\psi_1}$, $\dot{I}_{m2} = I_{m2} e^{j\psi_2}$, $\dot{I}_{m3} = I_{m3} e^{j\psi_3}$. Using

KCL, we can write $\dot{I}_{m1} = \dot{I}_{m2} + \dot{I}_{m3}$.

It is worth considering **Kirchhoff's voltage law**. The Kirchhoff's second law, also known as the mesh or voltage law (its acronym being KVL from Kirchhoff's voltage law), states that the algebraic sum of the instantaneous voltages across individual electrical components is equal to the algebraic sum of the AC source voltages in a mesh.

Suppose a mesh has n branches and each (k -th) branch in the general case contains an electromotive force (e_k), a resistance (R_k), an inductance (L_k), and a capacitance (C_k), with a current i_k flowing through all of them. Then by Kirchhoff's voltage law

$$\sum_{k=1}^n \left(i_k R_k + L_k \frac{di_k}{dt} + \frac{1}{C_k} \int i_k dt \right) = \sum_{k=1}^n e_k.$$

In accordance with substituting complex values for their instantaneous ones described above, each term on the left-hand side can be replaced by $\dot{I}_k \underline{Z}_k$, and each term on the right-hand side can be replaced by \dot{E}_k . Then Kirchhoff's second law for the network with n branches may be written in the complex form:

$$\sum_{k=1}^n \pm \dot{I}_k \underline{Z}_k = \sum_{k=1}^m \pm \dot{E}_k \quad \text{or} \quad \sum_{k=1}^n \pm \dot{U}_k = 0.$$

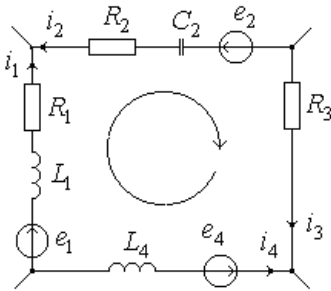


Fig. 6.2

Let us consider the part of network in Fig. 6.2 and write Kirchhoff's voltage law for it:

$$\begin{aligned} i_1 R_1 + L_1 \frac{di_1}{dt} - i_2 R_2 - \frac{1}{C_2} \int i_2 dt + i_3 R_3 - L_4 \frac{di_4}{dt} &= \\ &= e_1 - e_2 - e_4. \end{aligned}$$

Using Kirchhoff's second law in the complex form, we obtain:

$$\begin{aligned} \dot{E}_1 - \dot{E}_2 - \dot{E}_4 &= \dot{I}_1 R_1 + j\omega L_1 \dot{I}_1 - \dot{I}_2 R_2 - \frac{1}{j\omega C_2} \dot{I}_2 + \dot{I}_3 R_3 - j\omega L_4 \dot{I}_4 = \\ &= \dot{I}_1 \underbrace{(R_1 + j\omega L_1)}_{\underline{Z}_1} - \dot{I}_2 \underbrace{\left(R_2 + \frac{1}{j\omega C_2}\right)}_{\underline{Z}_2} + \dot{I}_3 \underbrace{R_3}_{\underline{Z}_3} - \dot{I}_4 \underbrace{j\omega L_4}_{\underline{Z}_4}. \end{aligned}$$

$$\text{or } \dot{I}_1 \underline{Z}_1 - \dot{I}_2 \underline{Z}_2 + \dot{I}_3 \underline{Z}_3 - \dot{I}_4 \underline{Z}_4 = \dot{E}_1 - \dot{E}_2 - \dot{E}_4.$$

Since Kirchhoff's current and voltage laws hold for a sinusoidal current as well, one might write down the equations for the values in a complex form, solve them by Kirchhoff's laws and check the result of calculation with the help of a vector diagram.

7. CONNECTIONS OF R , L AND C ELEMENTS

Let us examine the application of the complex method for a case of R , L and C elements connected in series and in parallel.

7.1. Series Connection of R , L and C

When a current i flows through a circuit consisting of passive R , L and C elements connected in series (Fig. 7.1), a voltage drop is produced on the terminals of this circuit equal, in accordance with Kirchhoff's voltage law, to the algebraic sum of the voltage drops on the individual elements

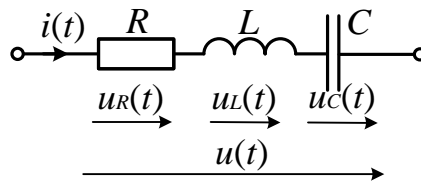


Fig. 7.1

$$u = u_R + u_L + u_C$$

or in the integral-differential form:

$$u = iR + L \frac{di}{dt} + \frac{1}{C} \int i dt .$$

In view of the fact that for the moment we consider steady-state conditions for an electrical circuit with a sinusoidal current, let us take $i = I_m \sin(\omega t + \psi_i)$. Voltage u_R is in phase with current i , voltage u_L leads, and voltage u_C lags behind i by $\pi/2$.

Consequently, the voltage across the terminals of the circuit is calculated as:

$$u = RI_m \sin(\omega t + \psi_i) + \omega LI_m \sin\left(\omega t + \psi_i + \frac{\pi}{2}\right) + \frac{1}{\omega C} I_m \sin\left(\omega t + \psi_i - \frac{\pi}{2}\right).$$

This equation is a trigonometrical form of Kirchhoff's voltage law for a circuit with R , L and C connected in series:

$$U_{mR} = RI_m; U_{mL} = \omega LI_m; U_{mC} = \frac{1}{\omega C} I_m .$$

The complex-number method is also called *symbolic method* for the reason that the currents and voltages are represented by their complex transforms or symbols. The use of complex notation gives

$$\dot{U} = \dot{U}_R + \dot{U}_L + \dot{U}_C = R\dot{I} + jX_L\dot{I} - jX_C\dot{I} = (R + j(X_L - X_C))\dot{I} = \underline{Z}\dot{I}, \quad (7.1)$$

where $\dot{I} = \frac{\dot{U}}{R + j(X_L - X_C)} = \frac{\dot{U}}{\underline{Z}}$ is the complex value of effective current;

$\underline{Z} = R + j(X_L - X_C) = Ze^{j\varphi}$ is the *complex impedance* of the circuit;

$Z = \sqrt{R^2 + (X_L - X_C)^2}$ is the *impedance* of the electrical circuit;

$X = X_L - X_C$ is the reactive resistance;

$\varphi = \arctg \frac{X_L - X_C}{R}$ is the phase difference between the initial phases of the voltage and the current.

The equation $\dot{U} = \underline{Z}\dot{I}$ is *Ohm's law* for alternating current circuits in the complex form.

Equation (7.1) reveals that the complex current may either lead or lag the complex voltage, depending on the relative values of the inductive and capacitive reactance.

Whenever the inductive reactance is more than the capacitive one, i.e. $\omega L > 1/\omega C$, the *RLC* circuit essentially behaves as an inductive circuit insofar as the current is concerned. Interestingly, this condition can be satisfied either by having a large inductance or by operating at a high frequency.

On the other hand, whenever $\omega L < 1/\omega C$, the current leads the voltage, thereby indicating that the *RLC* circuit behaves as a capacitive circuit as far as the current is concerned. However, there are some differences that must be discussed below.

Fig. 7.2 presents the vector diagrams for three cases of the *RLC* circuit shown in Fig. 7.1.

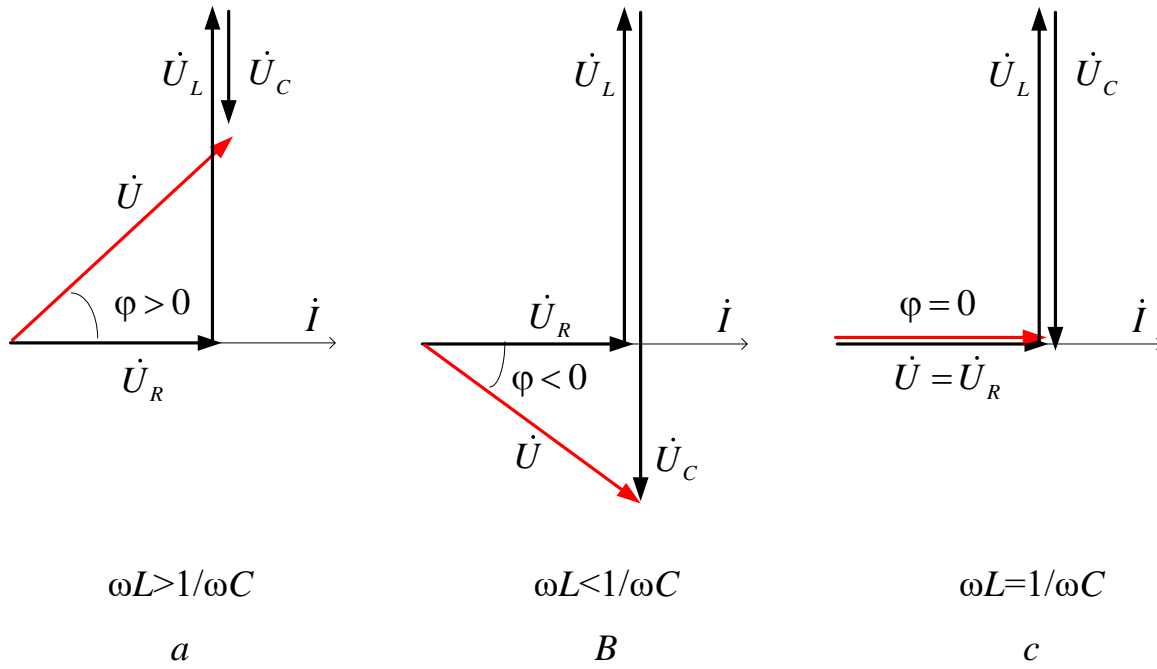


Fig. 7.2

Fig. 7.2, *a* shows the case where $\omega L > 1/\omega C$. Hence the vector of the current must lag the vector of voltage. The component values of the effective potential difference across each circuit element are also depicted. Note that the voltage drop across the resistor terminals \dot{U}_R must be in phase with the current \dot{I} . It is represented by the line, which is parallel to the vector of current. In vector diagrams, any line drawn parallel to another line means that the quantities represented by the two lines are in phase. The current \dot{I} leads the voltage across the capacitor terminals \dot{U}_C by 90° , or the capacitive voltage lags behind the current by 90° . Finally, the effective potential difference across the inductor terminals \dot{U}_L leads the current \dot{I} by 90° .

Although the vector diagram depicted in Fig. 7.2, *a* is for an *RLC* circuit in which the inductive reactance predominates, note that the circuit behaves differently from the straight *RL* circuit. Firstly, the potential difference across the inductor contains a component which is equal and opposite to the total voltage across the capacitor terminals. This leaves a net reactive voltage, as ‘seen’ by the source. Secondly, the voltage across the inductor can be several times greater in magnitude than the source voltage U . This cannot occur in the simple *RL* circuit. However, the large value of U_L should not be disturbing because Kirchhoff's voltage law continues

to be satisfied. Keep in mind that with AC circuits it is the vector sum which is important. An algebraic sum is meaningless except in those instances where only elements of the same type appear in the circuit. The angle ϕ is positive if the circuit has inductive character. Thus, for the first vector diagram (Fig. 7.2, *a*) the current lags the input voltage in phase, and the angle ϕ is read off on the abscissa axis to the left from the current to the voltage.

Fig. 7.2, *b* is drawn for the case where $\omega L < 1/\omega C$. The angle ϕ is negative at the capacity character of the circuit. Thus, voltage lags behind the current in phase, and the angle ϕ is read off on a real axis to the right from the current to the voltage.

Now we will pass from vector diagrams (Fig. 7.2, *a*, *b*) to impedance triangles. One can see the case of active-inductive resistance of an electrical circuit in Fig. 7.3, *a* and the case of active-capacitive resistance of an electrical circuit in Fig. 7.3, *b*.

As it follows from the formula for impedance, the resistance and the reactance may be depicted by a rectangular triangle, similar to the triangle of the vectors of voltages in Fig. (7.2, *a*, *b*). The impedance triangle graphically shows the relationship between the impedance magnitude Z and the associated resistance and reactance of the circuit.

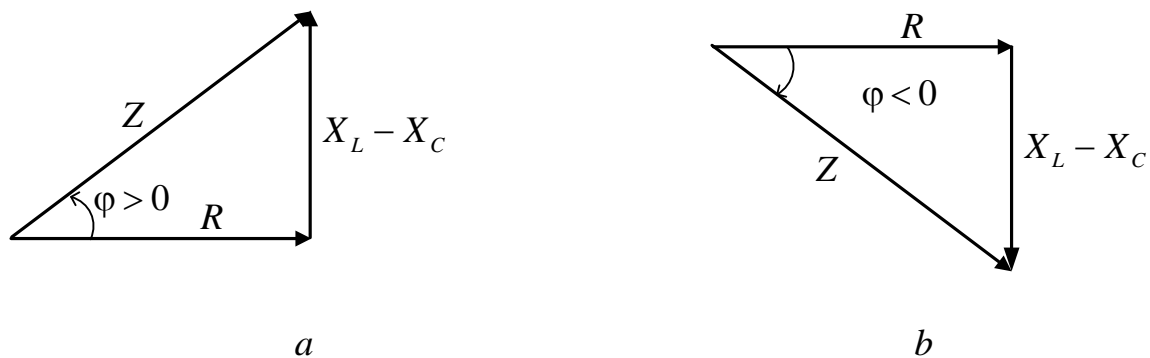


Fig. 7.3

From impedance triangles, it follows that the resistance and reactance are connected with the impedance:

$$R = Z \cos \phi \text{ and } X = Z \sin \phi.$$

7.2. Parallel Connection of R , L and C

The circuit configuration of parallel RLC connection is shown in Fig. 7.4.

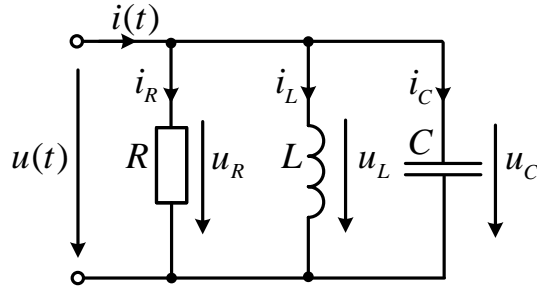


Fig. 7.4

If we apply sinusoidal voltage $u(t) = U_m \sin(\omega t + \psi_u)$ to input terminals, in a non-ramified part there appear a current whose instantaneous value, according to the first Kirchhoff's law, is equal to an algebraic sum of instantaneous values of the branch currents:

$$i = i_R + i_L + i_C.$$

If we rewrite the equation given above in a complex notation, we will get the equation that represents a trigonometrical notation of Kirchhoff's first law for the values of the currents:

$$\dot{I} = \dot{I}_R + \dot{I}_L + \dot{I}_C = G\dot{U} + j(B_C\dot{U} - B_L\dot{U}) = (G + j(B_C - B_L))\dot{U} = \underline{Y}\dot{U} = Ie^{-j\varphi}. \quad (7.1)$$

Here

$\dot{I}_R = G\dot{U}$ is the current through resistance R (in phase with voltage U);

$\dot{I}_L = jB_L\dot{U}$ is the current through the inductance (lags behind the voltage by $\pi/2$);

$\dot{I}_C = -jB_C\dot{U}$ is the current through the capacitance (leads the voltage by $\pi/2$),

where G is called the *conductance*,

B_L is called the *inductive susceptance*,

B_C is called the *capacitive susceptance*.

The value $B = B_C - B_L$ is called the *susceptance*.

The *complex admittance* of the circuit is given by:

$$\underline{Y} = G + j(B_C - B_L).$$

Vector diagram for parallel RLC connection in Fig. 7.5, a is a geometrical interpretation of equation (7.1). Fig. 7.5, a is drawn for the case where $B_C > B_L$. Hence the current phasor must lead the voltage phasor.

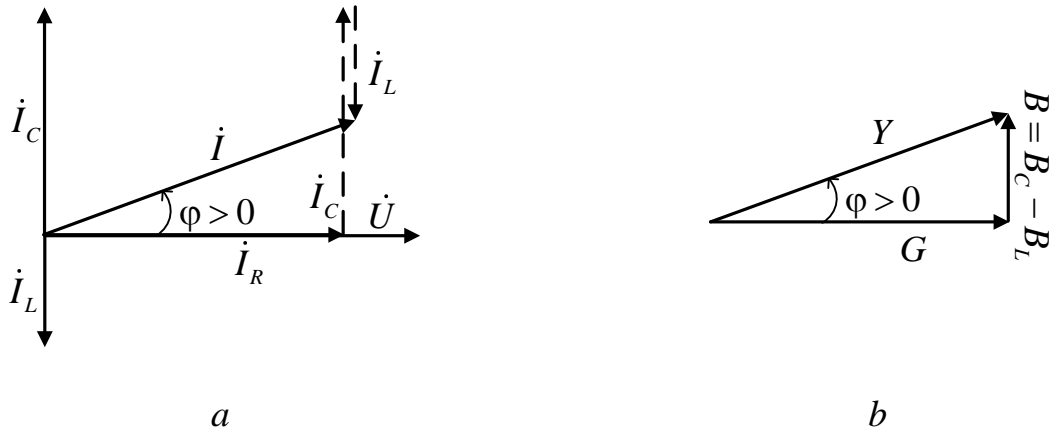


Fig. 7.5

Note that the voltage drop across the resistor terminals \dot{U}_R must be in phase with the current \dot{i}_R . It is represented by the line which is parallel to phasor \dot{i}_R . In vector diagrams, any line drawn parallel to another line means that the quantities represented by the two lines are in phase.

The current \dot{i}_C leads the voltage across the capacitor terminals \dot{U}_C by 90° , or the capacitive voltage lags behind the current by 90° .

Finally, the effective potential difference across the inductor terminals \dot{U}_L leads the current \dot{i}_L by 90° as expected.

Now we will pass from the vector diagram (Fig. 7.5, a) to the triangle of conductivities. Figure 7.5, b shows us a right-angled triangle, called admittance triangle, for the case of active-capacity load of an electrical circuit.

The magnitude of the complex admittance is given by

$$Y = \sqrt{G^2 + (B_C - B_L)^2}.$$

The admittance may be interpreted in terms of the role of an operator, too. It is the quantity which (when multiplied by the voltage phasor) yields the current phasor. In general, \underline{Y} is a complex number which often is calculated as:

$$\underline{Y} = \frac{1}{R} + j \left(\omega C - \frac{1}{\omega L} \right) = G + j(B_C - B_L) = G + jB = Ye^{-j\varphi},$$

where G is the real part of the admittance and is called *conductance*, and B is the quadrature component and is called *susceptance*.

Then we can write that for the parallel RLC case

$$G = \frac{1}{R} \text{ and } B = \frac{1}{X},$$

where $B = B_C - B_L$ and $X = X_L - X_C$.

The active and reactive conductivities of a circuit are connected with admittance by the following relationships:

$$G = Y \cos \varphi ;$$

$$B = Y \sin \varphi ;$$

$$\varphi = \operatorname{arctg} \frac{B_C - B_L}{G}.$$

The admittance triangle graphically shows the relationship between the magnitude of the admittance of a circuit, Y , and its inphase and quadrature components.

8. INSTANTANEOUS AND AVERAGE POWER. POWER FACTOR. POWER IN COMPLEX FORM

Our interest in this section is to develop a general expression for the average power associated with a voltage and current in an AC circuit. In this connection, let $u = U_m \sin(\omega t)$ represent the potential difference across the branch terminals of a given circuit, and let $i = U_m \sin(\omega t - \varphi)$ denote the corresponding current flowing through that branch. The relative phase angle is given by: $\varphi = \psi_u - \psi_i$.

The voltage and current sinusoid are shown in Fig.8.1. It follows then that the expression for the instantaneous power is calculated as:

$$\begin{aligned} p &= ui = U_m I_m \sin(\omega t - \varphi) \sin \omega t = \\ &= \frac{U_m I_m}{2} [\cos \varphi - \cos(2\omega t - \varphi)] = UI \cos \varphi - UI \cos(2\omega t - \varphi). \end{aligned} \quad (8.1)$$

The graphs of Eq. (8.1) are shown in Fig. 8.1.

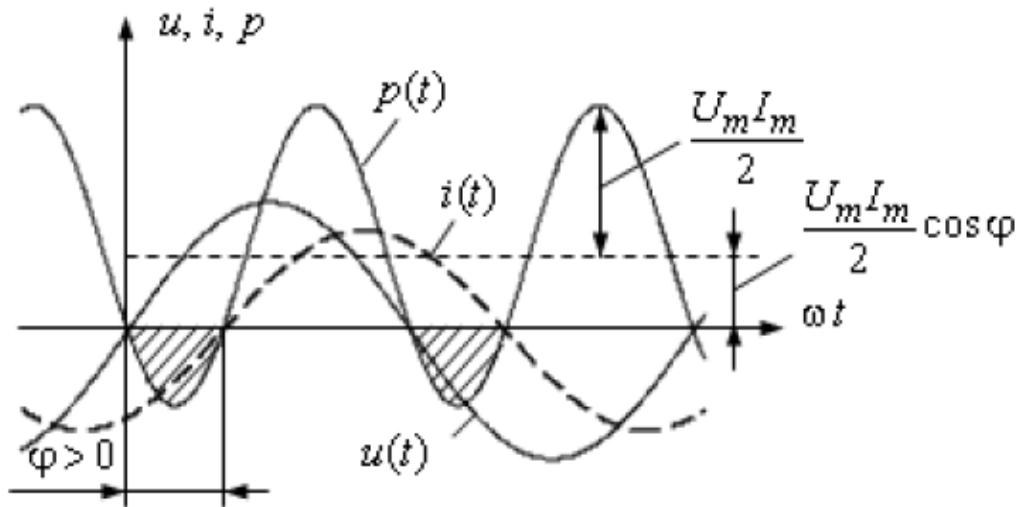


Fig. 8.1

Note that for a fixed φ the instantaneous power consists of two components: a constant part and a time-varying part. The varying part has a frequency that is twice more than ones of the voltage and current sinusoids. The shaded portions of the plot of $p(t)$ refer to those time intervals when the power is negative. In effect, this means that the circuit returns power to the source during these intervals.

Figure 8.1 shows that the instantaneous power is negative whenever the voltage and current are of opposite sign. However, for the case plotted in Fig. 8.1 the positive

area under the $p(t)$ curve exceeds the negative area. Therefore, the average power is positive. The smaller ϕ , when i is brought more closely in phase with u , the smaller negative areas of the $p(t)$ curve in Fig. 8.1, so the average power increases. This is equivalent to raising the $p(t)$ curve higher above the abscissa axis. When $\phi = 0$, the current and voltage are in phase. There are no negative areas associated with the $p(t)$ curve. Hence all the power is consumed between the circuit branch terminals. The circuit may then be called purely resistive. On the other hand, when ϕ is increased, the negative areas become larger, and the lower power is consumed between the terminals, whereas the greater one returns to the source. At the extreme value of ϕ , the $p(t)$ curve is dropped to that position which makes the negative and positive areas equal. In this instance there is no average power consumed between the circuit terminals.

The relative phase angle ϕ is determined by the values of the circuit parameters that appear between the circuit branch terminals across which voltage $u(t)$ exists. Because of the passive nature of these circuit parameters, the value of ϕ is restricted to the range expressed by $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$. The general expression for the instantaneous power in an AC circuit is described by Eq. (8.1). The really useful quantity in terms of the capability of the circuit to do work is the average value of the power over one cycle.

The average power during a period, called *active power*, is equal to the constant item of expression (8.2), since the average value of the sinusoidal item completing two cycles during period T is equal to zero, i.e.:

$$P = \frac{1}{T} \int_0^T p dt. \quad (8.2)$$

Inserting equation (8.1) for $p(t)$ gives

$$P = \frac{1}{T} \left[\int_0^T UI \cos \phi dt - \int_0^T UI \cos(2\omega t - \phi) dt \right].$$

Since the second term on the right side in the equation involves the integration of a simple sine function over a time interval equal to two complete periods of the

double frequency of sine function, the value is always identically equal to zero. This leaves just the first term and, since φ is independent of t , the average power called active is given by

$$P = UI \cos \varphi. \quad (8.3)$$

Active power is measured in watts (W). The factor $\cos \varphi$ is called the *power factor*:

$$\cos \varphi = \frac{P}{UI}.$$

As is seen from (8.3), the active power is equal to the product of the effective values of the voltage and the current with the power factor. The power factor of an electrical energy receiver depends on the impedance angle of the given receiver; the nearer the angle φ to zero, the nearer $\cos \varphi$ to unity and, consequently, the greater is the active power transmitted by the source to the receiver, for the given values of U and I .

Let us convert the expression of active power:

$$P = ZI^2 \cos \varphi = RI^2, \quad P = YU^2 \cos \varphi = GU^2.$$

If the circuit contains reactive (energy-storing) components, energy will circulate to-and-from between circuit and supply.

The expression for the instantaneous inductive power is written as:

$$p = U_L I_L \cos \frac{\pi}{2} - U_L I_L \cos \left(2\omega t - \frac{\pi}{2} \right) = U_L I_L \sin 2\omega t.$$

For the instantaneous capacitive power, it is as follows:

$$p = U_C I_C \cos \frac{\pi}{2} - U_C I_C \cos \left(2\omega t - \frac{\pi}{2} \right) = U_C I_C \sin 2\omega t.$$

The amplitude of the sinusoidal component of instantaneous power is equal to the product of the effective values of voltage and current. This quantity bears the name of apparent power and is measured in voltamperes (VA).

$$S = UI. \quad (8.4)$$

On the basis of (8.3) and (8.4), the power factor equals the ratio of active power to apparent power:

$$\cos \varphi = \frac{P}{S}.$$

In electrical circuit calculations the concept of *reactive power* is also used

$$Q = \sqrt{(UI)^2 - (UI \cos \varphi)^2} = UI \sin \varphi.$$

and measured in reactive volt-amperes (VAR).

Let us convert the expression of reactive power:

$$Q = ZI^2 \sin \varphi = XI^2, \quad Q = YU^2 \sin \varphi = BU^2.$$

Apparent power can be written in a complex form. Let us suppose that a sinusoidal current is flowing through an electrical circuit, the positive directions of the current and the voltage at the circuit terminals being chosen so as to coincide (Fig. 8.2).

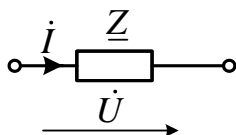


Fig. 8.2

The complex values of the effective current and voltage are, respectively:

$$\dot{U} = Ue^{j\psi_u} \text{ and } \dot{I}$$

The angular phase difference between voltage and current equals the difference between their initial phases $\varphi = \psi_u - \psi_i$.

Let us multiply complex voltage \dot{U} by the complex current \dot{I}^* , conjugate with \dot{I} ,

$$S = \dot{U} \dot{I}^* = Ue^{j\psi_u} \cdot Ie^{-j\psi_i} = UIe^{j\varphi} = UI \cos \varphi + jUI \sin \varphi = P + jQ.$$

Then it is apparent that

$$P = \operatorname{Re}\{S\} = \operatorname{Re}\{\dot{U} \dot{I}^*\}; \quad Q = \operatorname{Im}\{S\} = \operatorname{Im}\{\dot{U} \dot{I}^*\}.$$

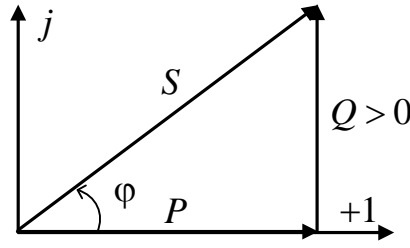


Fig. 8.3

Thus, the complex value S has a real part equaled to the active power and an imaginary part equaled to the reactive power. It is known as complex power and, being a phasor quantity, it can be represented graphically in a complex plane (Fig. 8.3).

Modulus $S=UI$ equals apparent power that is measured in *volt-amperes* .

Power balance

From the law of conservation of energy, it follows that the sum of the instantaneous powers given by all sources of the circuit should be equal to the sum of the instantaneous powers consumed by all receivers of energy:

$$\sum_{k=1}^n p_{ks} = \sum_{k=1}^m p_{kc} .$$

In sinusoidal current circuits, the balance of complex, active and reactive powers is considered. The condition for the balance of complex power is

$$\sum_{k=1}^n \tilde{S}_{ks} = \sum_{k=1}^m \tilde{S}_{kc} .$$

Let us consider the complex power by means of active and reactive power: the active power supplied by all energy sources is equal to the active power of all its consumers (it is completely consumed in the resistive elements of the circuit):

$$\sum_{k=1}^n P_{ks} = \sum_{k=1}^m P_{kc} , \quad \sum_{k=1}^n \operatorname{Re}[\tilde{S}] = \sum_{k=1}^m I_{kc}^2 R_k .$$

the reactive power of all sources is equal to the reactive power of all consumers (it circulates between energy sources and its consumers):

$$\sum_{k=1}^n Q_{ks} = \sum_{k=1}^m Q_{kc} , \quad \sum_{k=1}^n \operatorname{Im}[\tilde{S}] = \sum_{k=1}^m I_{kc}^2 X_k .$$

Example 8.1. Consider sinusoidal alternating current circuit (Fig. 8.4). The instantaneous input voltage is $u = 141 \sin(\omega t)$, where $\omega = 314 \text{ s}^{-1}$. The circuit resistances $R_1 = 10 \Omega$, $R_2 = 40 \Omega$, $R_3 = 20 \Omega$; inductances and capacitances of reactive elements $L_1 = 15,9 \text{ mH}$, $L_2 = 95,5 \text{ mH}$, $C_1 = 212 \mu\text{F}$, $C_3 = 79,6 \mu\text{F}$.

Find the branch currents, the voltages across the sections of the circuit, and write them in instantaneous forms. Determine the powers of the supply and the load. Draw a vector diagram of currents and voltages.

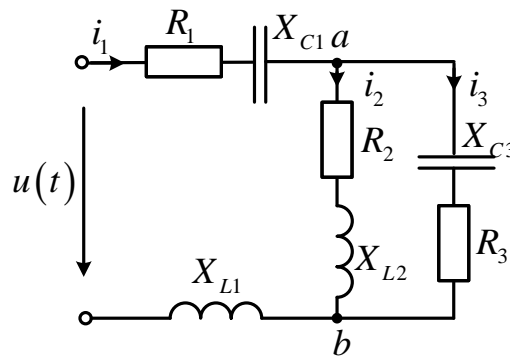


Fig. 8.4

First, determine inductive and capacitive reactances of the branches:

$$X_{L1} = \omega L_1 = 5 \Omega; \quad X_{L2} = \omega L_2 = 30 \Omega;$$

$$X_{C1} = \frac{1}{\omega C_1} = 15 \Omega; \quad X_{C3} = \frac{1}{\omega C_3} = 40 \Omega.$$

Use a symbolic method for calculation.

All complex values must be written both in algebraic and in exponential form.

The complex effective input voltage is given by

$$\dot{U} = \frac{U_m}{\sqrt{2}} e^{j\psi_u} = \frac{141}{\sqrt{2}} e^{j0^\circ} = 100 \text{ V}.$$

The complex impedances of branches are:

$$\underline{Z}_1 = R_1 + j(X_{L1} - X_{C1}) = 10 - j10 = 14.1 e^{-j45^\circ} \Omega;$$

$$\underline{Z}_2 = R_2 + jX_{L2} = 40 + j30 = 50 e^{j37^\circ} \Omega;$$

$$\underline{Z}_3 = R_3 - jX_{C3} = 20 - j40 = 44.7 e^{-j63^\circ} \Omega.$$

Then pass from the initial circuit to the circuit with complex impedances (Fig. 8.5).

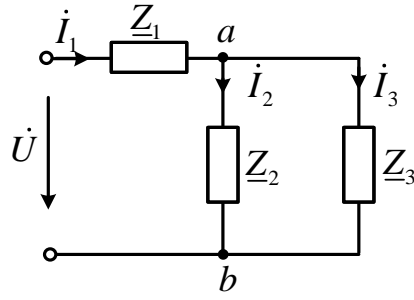


Fig. 8.5

In order to calculate the circuit, we use the method of equivalent transformations. The section ab with a parallel connection of branch impedances \underline{Z}_2 and \underline{Z}_3 is changed by an equivalent complex impedance:

$$\underline{Z}_{23} = \frac{\underline{Z}_3 \cdot \underline{Z}_2}{\underline{Z}_3 + \underline{Z}_2} = \frac{(40 + j30)(20 - j40)}{40 + j30 + 20 - j40} = 35.1 - j10.8 = 36.8e^{-j17^\circ} \Omega.$$

Then the equivalent impedance of the circuit is given by

$$\underline{Z}_e = \underline{Z}_1 + \underline{Z}_{23} = 10 - j10 + 35.1 - j10.8 = 45.1 - j20.8 = 49.7e^{-j25^\circ} \Omega.$$

The current in a non-ramified part of the electrical circuit is found by Ohm's law:

$$\dot{I}_1 = \frac{\dot{U}}{\underline{Z}_e} = \frac{100}{49.7e^{-j25^\circ}} = 2.01e^{j25^\circ} = 1.83 + j0.84 \text{ A}.$$

The voltages across circuit sections are:

$$\dot{U}_1 = \dot{I}_1 \underline{Z}_1 = 2.01e^{j25^\circ} \cdot (10 - j10) = 28.5e^{-j20^\circ} \text{ V};$$

$$\dot{U}_{ab} = \dot{I}_1 \underline{Z}_{23} = 2.01e^{j25^\circ} \cdot 36.8e^{-j17^\circ} = 73.9e^{j8^\circ} \text{ V}.$$

The currents through the parallel branches are calculated by:

$$\dot{I}_2 = \frac{\dot{U}_{ab}}{\underline{Z}_2} = \frac{73.9e^{j8^\circ}}{50e^{j37^\circ}} = 1.48e^{-j29^\circ} \text{ A};$$

$$\dot{I}_3 = \frac{\dot{U}_{ab}}{\underline{Z}_3} = \frac{73.9e^{j8^\circ}}{44.7e^{-j63^\circ}} = 1.65e^{j71^\circ} \text{ A}.$$

The solution can be checked using Kirchhoff's voltage and current-laws:

$$\dot{I}_1 = \dot{I}_2 + \dot{I}_3 = 1.48e^{-j29^\circ} + 1.65e^{j71^\circ} = 2.01e^{j25^\circ} \text{ A};$$

$$\dot{U} = \dot{U}_1 + \dot{U}_{ab} = 28.5e^{-j20^\circ} + 73.9e^{j8^\circ} = 100.2 \text{ V}.$$

The relative error does not exceed one per cent.

Let us pass from complex currents and voltages to their instantaneous values:

$$u_1 = \sqrt{2}U_1 \sin(\omega t + \psi_{u_1}) = \sqrt{2} \cdot 28.5 \sin(314t - 20^\circ) \text{ V};$$

$$u_{ab} = \sqrt{2}U_{ab} \sin(\omega t + \psi_{u_{ab}}) = \sqrt{2} \cdot 73.9 \sin(314t + 8^\circ) \text{ V};$$

$$i_1 = \sqrt{2}I_1 \sin(\omega t + \psi_{i_1}) = \sqrt{2} \cdot 2.01 \sin(314t + 25^\circ) \text{ A};$$

$$i_2 = \sqrt{2}I_2 \sin(\omega t + \psi_{i_2}) = \sqrt{2} \cdot 1.48 \sin(314t - 29^\circ) \text{ A};$$

$$i_3 = \sqrt{2}I_3 \sin(\omega t + \psi_{i_3}) = \sqrt{2} \cdot 1.65 \sin(314t + 71^\circ) \text{ A}.$$

The complexes of apparent powers of the energy source:

$$S = \dot{E} I_1^* = 100 \cdot (1.83 - j0.84) = 183 - j84 \text{ VA},$$

where $P_E = 183 \text{ Wt}$ – the active powers for the supply;

$Q_E = -84 \text{ VAr}$ – the reactive powers for the supply.

The active powers for the branches of the electrical circuit:

$$P_{load} = R_1 I_1^2 + R_2 I_2^2 + R_3 I_3^2 = 10 \cdot 2.01^2 + 40 \cdot 1.48^2 + 20 \cdot 1.65^2 = 182.5 \text{ Wt}.$$

The reactive powers for the branches of the electrical circuit are calculated as:

$$\begin{aligned} Q_{load} &= (X_{L1} - X_{C1}) I_1^2 + X_{L2} I_2^2 - X_{C3} I_3^2 = \\ &= (5 - 15) \cdot 2.01^2 + 30 \cdot 1.48^2 - 40 \cdot 1.65^2 = -83.6 \text{ VAr}. \end{aligned}$$

Then check the balance of powers:

$$P_E \approx P_{load} \Rightarrow 183 = 182.5 \text{ Wt};$$

$$Q_E \approx Q_{load} \Rightarrow -84 = -83.6 \text{ VAr}.$$

Finally, draw a combined vector diagram of the branch currents and voltages (in the corresponding scale):

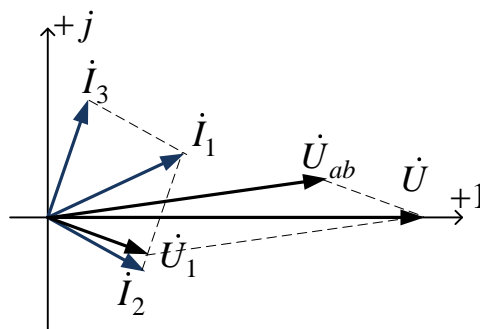


Fig. 8.6

9. RESONANCE IN ELECTRICAL CIRCUITS UNDER HARMONIC OSCILLATIONS ACTION

Resonance is a steady-state operation mode of electrical circuit comprising heterogeneous reactive elements (energy-storing elements) in case when the phase shift between input current and applied voltage of this circuit is equal to zero. Resonance cannot take place when only one type of energy-storing element is present, e.g. inductance or capacitor. There must be two types of independent energy-storing elements capable of interchanging energy between one another, for example, inductance and capacitor.

Circuits where the resonance phenomenon appears are called *oscillatory circuits*. The simplest oscillating circuits include inductive and capacitive elements connected in series or parallel.

There are two types of resonances known as voltage resonance in a series loop and current resonance in a parallel loop.

9.1. Series oscillatory circuit. Voltage resonance

Thus far we have analysed the behaviour of a series *RLC* (Fig. 9.1) circuit whose voltage source is a steady state sinusoidal supply with the fixed frequency.

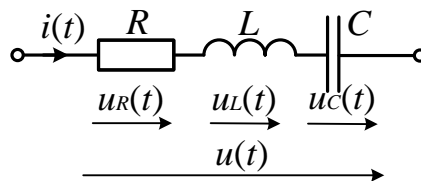


Fig. 9.1

Firstly, let us consider what we already know about series *RLC* circuits.

Inductive reactance:

$$X_L = \omega L = 2\pi fL.$$

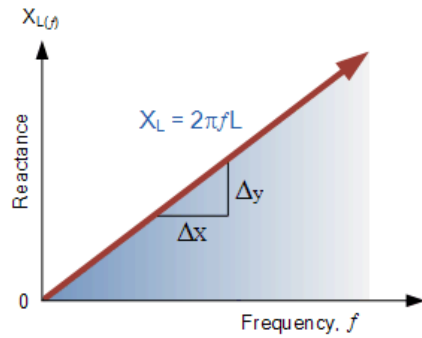


Fig. 9.2

Capacitive reactance:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}.$$

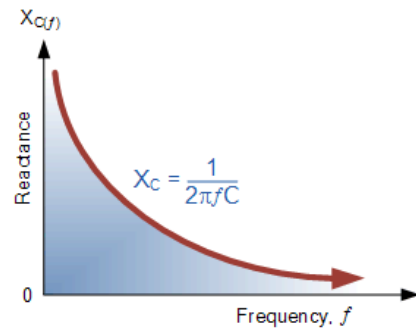


Fig. 9.3

Total circuit reactance:

$$X = X_L - X_C = \omega L - 1/\omega C.$$

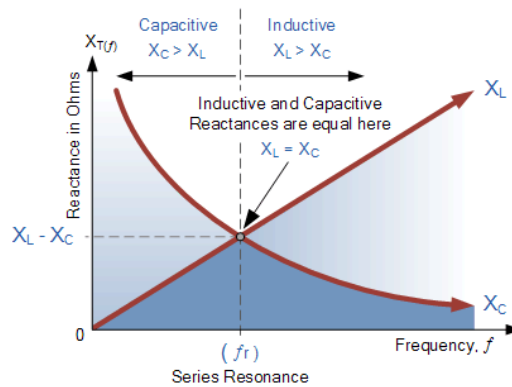


Fig. 9.4

Total circuit complex impedance: $\underline{Z} = R + j(\omega L - 1/\omega C) = \sqrt{R^2 + (\omega L - 1/\omega C)^2} e^{j\varphi},$

where phase shift is $\varphi = \arctg \frac{\omega L - 1/\omega C}{R}.$

When $\omega L > \frac{1}{\omega C}$, the circuit is inductive, and if $\omega L < \frac{1}{\omega C}$, the circuit is capacitive.

When the inductive and the capacitive elements are connected in series, one can obtain the conditions under which voltage resonance (also known as series resonance) will take place.

Electrical resonance occurs in an AC circuit when the two reactance elements, which are opposite and equal, cancel each other out under the condition that:

$$\omega L = \frac{1}{\omega C}.$$

And the point on the graph (Fig. 9.4) at which this happens is formed when the two reactance curves cross each other.

In a series resonant circuit, the resonant frequency, f_R point can be calculated as follows:

$$\omega_R = \frac{1}{\sqrt{LC}}, \text{ rads} \quad \text{or} \quad f_R = \frac{1}{2\pi\sqrt{LC}}, \text{ Hz.} \quad (9.1)$$

We can see then that at resonance the two reactances cancel each other out, thereby making a series LC combination act as a short circuit, with the only opposition to current flow in a series resonance circuit being the resistance, R . In complex form, the resonant frequency is the frequency at which the total impedance of a series RLC circuit becomes purely “*real*”, i.e. no imaginary impedance exists. This is because at resonance they are cancelled out. So the total impedance of the series circuit becomes just the value of the resistance and therefore:

$$\underline{Z} = R + j \left(\omega L - \frac{1}{\omega C} \right) \bigg|_{\omega L = 1/\omega C} = R. \quad (9.2)$$

Then at resonance, the impedance of the series circuit is at its minimum value and is equal only to the resistance R of the circuit. The circuit impedance at resonance is called “dynamic impedance” of the circuit, and depending upon the frequency, X_C (typically at high frequencies) or X_L (typically at low frequencies) will dominate on the respective side of resonance as shown below.

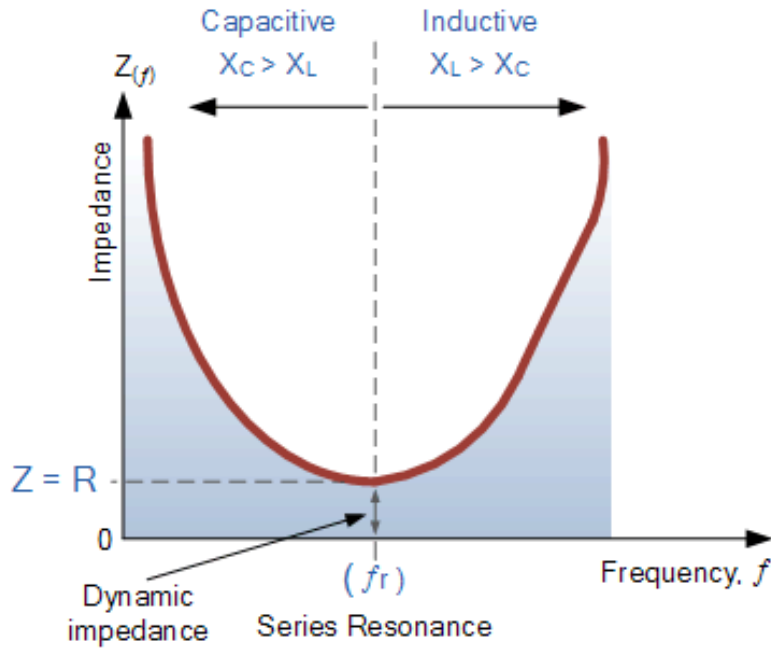


Fig. 9.5

Note that when the capacitive reactance dominates the circuit, the impedance curve has a hyperbolic shape to itself, but when the inductive reactance dominates the circuit, the curve is non-symmetrical due to the linear response of X_L .

In addition, note that if the circuit impedance is at its minimum at resonance, then the resulting current flowing through the circuit may be dangerously high. Since the current flowing through a series resonance circuit is the voltage divided by impedance, at resonance the impedance Z gets its minimum value as R . Therefore, the circuit current at this frequency will be at its maximum value as shown below

$$I = \frac{\dot{U}}{Z} \bigg|_{\omega L = \frac{1}{\omega C}} = \frac{\dot{U}}{\left(R + j \left(\omega L - \frac{1}{\omega C} \right) \right)} \bigg|_{\omega L = \frac{1}{\omega C}} = \frac{\dot{U}}{R}.$$

The voltage across a series combination is the phasor sum of \dot{U}_R , \dot{U}_L and \dot{U}_C . Hence, if at resonance the two reactances are equal and cancelling, the two voltages representing \dot{U}_L and \dot{U}_C must also be opposite and equal in value, thereby cancelling each other out because with pure components the phasor voltages are drawn at $+90^\circ$ and -90° , respectively.

Then in a series resonance circuit, when $\dot{U}_L = \dot{U}_C$, the resulting reactive voltages are zero, and all the supply voltage is dropped across the resistor. Therefore, $\dot{U}_{\text{supply}} = \dot{U}_R$, and it is for this reason that series resonance circuits are known as voltage resonance circuits (as opposed to parallel resonance circuits which are current resonance circuits).

The frequency response curve of a series resonance circuit shows that the magnitude of the current is a function of frequency and plotting this onto a graph shows us that the response starts at near to zero, reaches maximum value at the resonance frequency when $I_{\text{max}} = IR$ and then drops again to nearly zero as f becomes infinite. The result of this is that the magnitudes of the voltages across the inductor, L and the capacitor, C can become many times larger than the supply voltage, even at resonance, but as they are equal and at opposition, they cancel each other out.

You may also notice that as the maximum current through the circuit at resonance is limited only by the value of the resistance (a pure and real value), the source voltage and circuit current must therefore be in phase with each other at this frequency. Then the phase angle between the voltage and current of a series resonance circuit, being also a function of frequency for a fixed supply voltage, is equal to zero at the resonant frequency point when U , I and U_R are all in phase with each other as shown below. Consequently, if the phase angle is zero, then the power factor must therefore be unity.

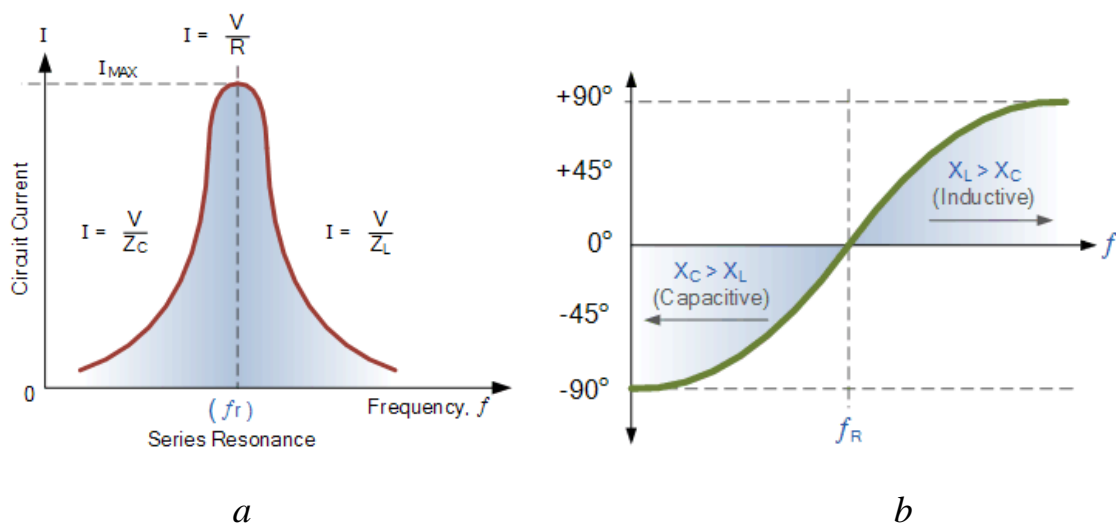


Fig. 9.6

Note that the phase angle is positive for frequencies above f_R and negative for frequencies below f_R , and this can be proven by

$$\varphi = \arctg \frac{\omega L - 1/\omega C}{R} = 0.$$

The frequency response of the circuit current magnitude above relates to the “sharpness” of the resonance in a series resonance circuit. The sharpness of the peak is measured quantitatively and is called *Quality factor*, Q of the circuit. The quality factor relates the maximum or peak energy stored in the circuit (the reactance) to the energy dissipated (the resistance) during each cycle of oscillation meaning that it is a ratio of resonant frequency to bandwidth and the higher the circuit Q , the smaller the bandwidth:

$$Q = \frac{U_{C_R}}{U} = \frac{U_{L_R}}{U} = \frac{\omega_R L}{R} = \frac{1}{\omega_R C R} = \frac{\sqrt{L/C}}{R} = \frac{\rho}{R}.$$

Then the relationship between resonance, bandwidth, selectivity and quality factor for a series resonance circuit is defined as:

1. Resonant frequency, (f_R)

$$X_L = X_C \Rightarrow \omega_R L = \frac{1}{\omega_R C} \Rightarrow \omega_R L - \frac{1}{\omega_R C} = 0.$$

$$\omega_R = \frac{1}{\sqrt{LC}}.$$

2. Current I at ω_R

$$Z \Big|_{\omega L = \frac{1}{\omega C}} = \min, \quad I \Big|_{\omega L = \frac{1}{\omega C}} = \max;$$

$$I_{\max} = \frac{U}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \Big|_{\omega L = 1/\omega C} = \frac{U}{R}.$$

3. Quality factor Q

$$Q = \frac{\omega_R L}{R} = \frac{1}{\omega_R C R} = \frac{1}{R} \sqrt{\frac{L}{C}}.$$

4. Phasor diagram

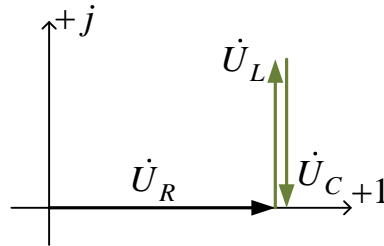


Fig. 9.7

9.2. Parallel oscillatory circuit. Current resonance

Parallel resonance occurs when the supply frequency creates zero phase difference between the supply voltage and current producing a resistive circuit.

A parallel resonance circuit is similar to the series resonance circuit we looked at in the previous topic. Both of them are 3-element networks that contain two reactive components making them a second-order circuit. Both are influenced by variations in the supply frequency, and both have a frequency point where their two reactive components cancel each other out influencing the characteristics of the circuit. Both circuits have a resonant frequency point.

Let us consider parallel oscillatory circuit in Fig. 9.8 that has two parallel branches, where resistance R_1 is in the branch with inductance and resistance R_2 is in the branch with capacitance

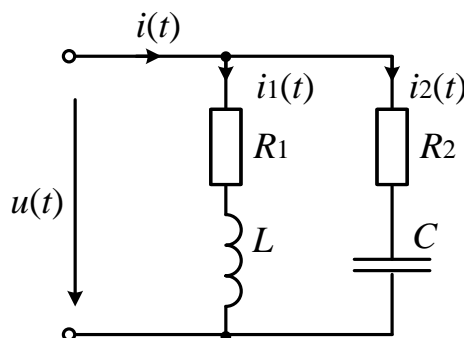


Fig. 9.8

Firstly, let us revise what we already know about series *RLC* circuits.

Inductive susceptance:

$$B_L = \frac{1}{\omega L} = \frac{1}{2\pi f L}$$

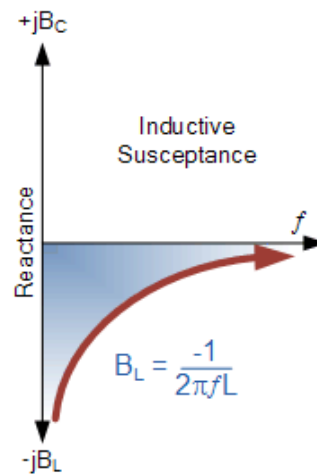


Fig. 9.9

Capacitive susceptance:

$$B_C = \omega C = 2\pi f C$$

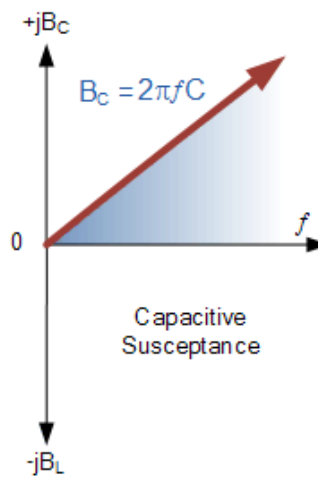


Fig. 9.10

Total circuit susceptance:

$$B = B_C - B_L = \omega C - \frac{1}{\omega L}.$$

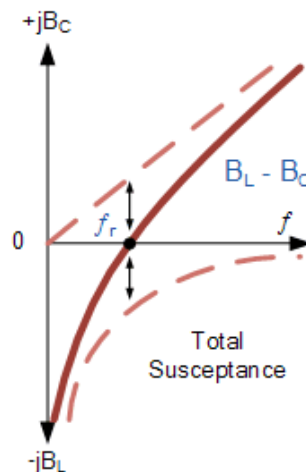


Fig. 9.11

From above, the inductive susceptance, B_L , is inversely proportional to the frequency as represented by the hyperbolic curve. The capacitive susceptance, B_C is directly proportional to the frequency and is therefore represented by a straight line.

The final curve shows the plot of total susceptance of the parallel resonance circuit versus the frequency and is the difference between the two susceptances.

Then we can see that at the resonant frequency point, where it crosses the horizontal axis, the total circuit susceptance is zero. Below the resonant frequency point, the inductive susceptance dominates the circuit producing a “lagging” power factor, whereas above the resonant frequency point the capacitive susceptance dominates producing a “leading” power factor.

Thus, at the resonant frequency the input current and the applied voltage must be in phase; there is only the resistance present in the parallel circuit, so the power factor becomes unity. This is possible when $\varphi = 0$.

Also, as the impedance of a parallel circuit changes with frequency, this makes the circuit impedance “dynamic” with the current at resonance being in-phase with the voltage since the impedance of the circuit acts as a resistance.

The current \dot{I}_1 through the branch with inductance lags in phase with the supply voltage \dot{U} and is expressed by the equation:

$$\begin{aligned}\dot{I}_1 &= \frac{\dot{U}}{\underline{Z}_1} = \dot{U} \underline{Y}_1 = \dot{U} \frac{1}{R_1 + j\omega L} = \\ &= \dot{U} \left(\frac{1}{R_1 + j\omega L} \cdot \frac{R_1 - j\omega L}{R_1 - j\omega L} \right) = \dot{U} \left(\frac{R_1}{R_1^2 + \omega^2 L^2} - j \frac{\omega L}{R_1^2 + \omega^2 L^2} \right) = \dot{U} (G_1 - jB_1).\end{aligned}$$

The current \dot{I}_2 through the branch with capacitance leads in phase with the supply voltage \dot{U} and is expressed by the equation:

$$\begin{aligned}\dot{I}_2 &= \frac{\dot{U}}{\underline{Z}_2} = \dot{U} \underline{Y}_2 = \dot{U} \frac{1}{R_2 - j/\omega C} = \\ &= \dot{U} \left(\frac{1}{R_2 - j/\omega C} \cdot \frac{R_2 + j/\omega C}{R_2 + j/\omega C} \right) = \dot{U} \left(\frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} - j \frac{-\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}} \right) = \dot{U} (G_2 + jB_2),\end{aligned}$$

where

\dot{U} is the applied voltage with frequency ω ;

$G_1 = \frac{R_1}{R_1^2 + \omega^2 L^2}$ is the conductance in the branch with inductance;

$G_2 = \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}}$ is the conductance in the branch with capacitance;

$B_1 = \frac{\omega L}{R_1^2 + \omega^2 L^2}$ is the susceptance in the branch with inductance;

$B_2 = \frac{-\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}$ is the susceptance in the branch with capacitance.

The current through the unbranched part of the circuit is calculated as follows:

$$\dot{I} = \dot{I}_1 + \dot{I}_2 = \frac{\dot{U}}{\underline{Z}} = \dot{U} \underline{Y} = \dot{U} (G_1 + G_2 + j(-B_1 + B_2)) = I_{\text{Re}1} - jI_{\text{Im}1} + I_{\text{Re}2} + jI_{\text{Im}2}.$$

As for the equation above, the applied voltage and input current are in phase when the imaginary parts of the currents magnitudes through the parallel branches are equal to each other:

$$I_{\text{Im}1} = I_{\text{Im}2}.$$

This is possible when the susceptances of parallel branches are equal:

$$\frac{\omega L}{R_1^2 + \omega^2 L^2} = \frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}. \quad (9.3)$$

From Equation (9.3) the resonant frequency of parallel oscillatory circuit with losses is calculated as:

$$\omega_R = \omega_0 \sqrt{\frac{L/C - R_1^2}{L/C - R_2^2}} = \omega_0 \sqrt{\frac{\rho^2 - R_1^2}{\rho^2 - R_2^2}},$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the resonant frequency of lossless parallel oscillatory circuit when $R_1 = R_2 = 0$.

We can see that at resonance the admittance of the circuit is at its minimum and is equal to the conductance of the circuit. Also, at resonance the current drawn

from the supply is also at its minimum and is determined by the value of the parallel resistance.

$$\text{As } \underline{Y} = (G_1 + G_2 + j(-B_1 + B_2)) \Big|_{B_1=B_2} = G_1 + G_2 = \min,$$

$$\text{then } \dot{I} \Big|_{B_1=B_2} = \dot{U} \underline{Y} = \dot{U} (G_1 + G_2) = \min.$$

The phasor diagram is shown in Fig. 9.12.

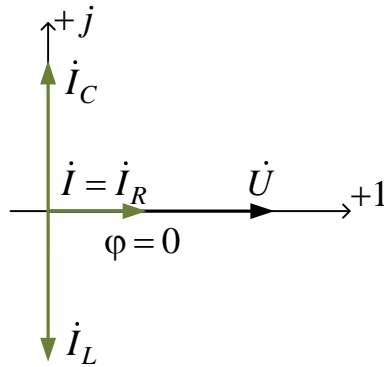


Fig. 9.12

The quality factor is given by the following ratio:

$$Q = \frac{\omega_R L}{R_L} = \frac{1}{\omega_R C R_C}.$$

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