Generalized approach for estimating and forecasting of dynamical VaR and CVaR based on Metalog distribution

Grigoriy Zrazhevsky

Department of Theoretical Mechanics

National Taras Shevchenko University of Kiev

64, Volodymyrs'ka St.,01033 Kyiv, Ukraine

zrazhevsky@aorda.com

Vera Zrazhevska

Department of Differential Equations

National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute

: 37, Peremogi st., , 03056, Kiev, Ukraine

vera.zrazhevska@gmail.com

Abstract. The work is devoted to the development of methods for dynamic risk measures VaR and CVaR estimating. As a basic model, a heteroscedastic time series model is considered. The methods proposed in the article are designed for obtaining the forecast estimates of risk measures for volatile time series taking into account the long-range dependence presence. The method of smoothing of the autocorrelation function based on an optimization procedure is used for variance modeling. A metalog distribution is proposed to use for risk measures model residuals estimating. This distribution allows to describe the behavior of the tail part of the distribution with different characteristics. The paper proposes two methods of metalog distribution estimating. The first method is based on an empirical distribution function and the second one on its approximation by sample quantiles. For VaR and CVaR modeling and forecasting, explicit analytical formulas were obtained with different numbers of members of the metalog distribution. The procedure for obtaining of the forecast values of dynamic risk measures VaR and CVaR is formulated as an algorithm. The proposed approach is applied to the time series of the "Russian Trading System" index for the period 14/10/2005 – 10/02/2020. For comparison, the forecast of dynamic risk measures is built using well known methods of risk estimation based on the GEV distribution, GPD and historical modeling. Quantitative and qualitative analyzes of the obtained estimates confirmed the high quality of the obtained estimates.

Keywords: dynamic risk measures VaR and CVaR forecast, FIGARCH model, metalog distribution, long-range dependence, Method of smoothing of the autocorrelation function.

Introduction and literature review. The problem of financial risk forecasting is one of the most actual problems that attract the attention of financiers and mathematicians around the world. VaR (Value-at-Risk) and CVaR (Conditional Value-at-Risk) are among the risk measures that help to estimate the potential losses and as a result to take timely action to prevent excessive financial losses. There are a large number of methods and algorithms for their evaluation and forecasting. The article [1] provides a big list of existing VaR and CVaR estimation methods for one random variable. An analysis of existing models for estimating and forecasting dynamic VaR and CVaR is performed in the article [2]. One of the main approaches for dynamic risks measures modeling is based on a stochastic time series model ([3]). The problem of VaR and CVaR modeling in this case is to build a model for variance and to estimate risk measures model residuals. Examples of practical risk estimating and forecasting of stock indices using GARCH models, see [4,5].

At the same time the problem of VaR and CVaR estimating and forecasting remains relevant our days. In particular, this concerns the problem of modeling risk measures for volatile financial series with long-range dependence. Such specificity of time series complicates the application of classical methods. The new types of distributions, for example, the metalog distribution ([3]), allow to describe the tail distribution of a random value more precisely and thus improve the predictive ability of the model for dynamic risks. This type of distribution is also suitable for risk measures GARCH model residuals estimating.

The aim of this research is to develop new methods for dynamic risk measures VaR and CVaR modeling and forecasting. This new methods must more полно обрабатывать исходные данные, отображать их особенности, тем самым улучшая качество прогнозных оценок. В период кризисных явлений в экономике и финансовой системе качественное прогнозированием мер рисков сможет способствовать стабильности финансовой и, в частности, банковской, сфер.

1. Materials and Methods.

We consider the continuously distributed time series $\{X_t, t \in \mathrm{T}\}$ defined on the probability space $(\Omega, \Phi_t, \mathrm{P}_t)$ with a set of observations $X_1, ..., X_N$ (Φ_t is the information set containing all available at the time t information about the time series). For a fixed confidence level α dynamic risk measure VaR_α^t is a conditional quantile of the probability distribution function for X_t: $\mathrm{P}_t\left[X_t < -VaR_\alpha^t \middle| \Phi_t\right] = \alpha$. Risk measure $CVaR_\alpha^t$ is defined accordingly to [6] as: $CVaR_\alpha^t = E_t[X_{\tilde{t}} \left| X_{\tilde{t}} < -VaR_\alpha^t(t)\right] = -\frac{1}{\alpha}\int_0^\alpha VaR_\gamma^t \, d\gamma$, ($\alpha < 0.5$), where $E_t[\cdot]$ is a conditional expectation defined on Φ_t .

The squared time series is considered for variance modulation. Series $\{X_t^2, t \in \mathrm{T}\}$ is assumed to be stationary and long-range dependent: $\exists\, \kappa > 0, 0 < \omega < 1$: $\lim\limits_{k\to\infty} \rho_k / (\kappa k^{-\omega}) = 1$, where $\rho_k = corr(X_t^2, X_{t+k}^2)$ is an autocorrelation function, $k \in N \cup \{0\}$.

A model FIGARCH(p,d,q) is used to take into account the long-range dependence in the modeling of a series ([2]):

$$X_t = \mu + \varepsilon_t = \mu + \sigma_t \nu_t \tag{1}$$

$$(1 - b(L))\sigma_t^2 = b_0 + (1 - b(L) - \varphi(L)(1-L)^d)\varepsilon_t^2 \tag{2}$$

where μ denotes mean, σ_t^2 is conditional variance, definedf7 on Φ_t, $\nu_t \sim iid(0,1)$, $LX_t = X_{t-1}$, $a(L) = \sum_{i=1}^q a_i L^i$, $b(L) = \sum_{j=1}^p b_j L^j$, $\varphi(L) \equiv (1 - a(L) - b(L))(1-L)^{-1}$, $(1-L)^d = \sum_{j=1}^\infty \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)}(-L)^j$ is a fractional difference operator, $\Gamma(\cdot)$ is a gamma function, $d \in (0, 1/2)$; $b_0 > 0$; $a_i \geq 0, i = 1,2,\ldots,q$; $b_j \geq 0$, $j = 1,2,\ldots,p$; $\sum_{i=1}^q a_i + \sum_{j=1}^p b_j < 1$.

After removing the trend μ from the time series the model for risk measures based on the FIGARCH(p, d, q) model can be written as ([3]):

$$VaR_\alpha\left(X_t\right) = VaR_\alpha(\nu)\sigma_t,\ CVaR_\alpha\left(X_t\right) = CVaR_\alpha(\nu)\sigma_t, \tag{3}$$

where σ_t,and ν are defined in (1), (2). $VaR_\alpha(\nu)$ and $CVaR_\alpha(\nu)$ are risk measures model residuals, on the condition $F(\nu_t) \equiv F(\nu)$. Then the forecasting values for S steps ahead for dynamic risk measures can be found as:

$$VaR_\alpha\left(X_{t+S}\right) = VaR_\alpha(\nu)\sigma_{t+S},\ CVaR_\alpha\left(X_{t+S}\right) = CVaR_\alpha(\nu)\sigma_{t+S}. \tag{4}$$

The following methods are used to estimate $VaR_\alpha(\nu)$ and $CVaR_\alpha(\nu)$.

Historical Simulation Method. According to this method, estimates for risk measures for a random variable X with sample $(X_1, X_2, \ldots, X_N)$ are found as follow:

$$VaR_\alpha\left(X\right)=-X_{([N\alpha])}\ ,\ CVaR_\alpha\left(X\right)=-\left(\sum_{i=1}^{[N\alpha]}X_{(i)}\right)/\left([N\alpha]\right)\ , \tag{5}$$

where $X_{(1)}\le X_{(2)}\le\ldots\le X_{(N)}$ - ordered sample.

Using Generalized Extreme Value (GEV) distribution. Suppose, that random variable X follows GEV distribution with distribution function $F(x)=\begin{cases}\exp\left(-\left(1+\xi\dfrac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right), \xi\ne 0\\ \exp\left(-\exp\left(-\dfrac{x-\mu}{\sigma}\right)\right), \ \xi=0\end{cases}$. Then the risk measures ([7]):

$$VaR_\alpha(X)=\begin{cases}-\mu-\dfrac{\sigma}{\xi}\left(\left(-\ln\alpha\right)^{-\xi}-1\right), \xi\ne 0\\ -\mu+\sigma\ln(-\ln\alpha),\ \xi=0\end{cases}$$

$$CVaR_\alpha(X)=\begin{cases}-\mu-\dfrac{\sigma}{\alpha\xi}\left(\Gamma\left(1-\xi,-\ln\alpha\right)-\alpha\right), \xi\ne 0\\ -\mu-\dfrac{\sigma}{\alpha}\left(li(\alpha)-\alpha\ln(-\ln\alpha)\right),\ \xi=0\end{cases}, \tag{6}$$

where $\Gamma(\cdot,\cdot)$ is upper incomplete gamma function, $li(\alpha)=\int\dfrac{dx}{\ln x}$ is logarithmic integral function.

Using Generalized Pareto Distribution (**GPD**). Suppose that a random variable X follows GPD with distribution function [8]: $F(x)=\begin{cases}1-\left(1+\xi\dfrac{x-\mu}{s}\right)^{-\frac{1}{\xi}}, \xi\ne 0\\ 1-\exp\left(-\dfrac{x-\mu}{s}\right),\ \xi=0\end{cases}$

Then risk measures for losses: $L=-X$, $p=1-\alpha$:

$$VaR_p(L)=\begin{cases}\mu+s\dfrac{(1-p)^{-\xi}-1}{\xi}, \xi\ne 0\\ \mu-s\ln(1-p),\ \xi=0\end{cases}$$

$$CVaR_p(L)=\begin{cases}\mu+s\left(\dfrac{(1-p)^{-\xi}}{1-\xi}+\dfrac{(1-p)^{-\xi}-1}{\xi}\right), \xi\ne 0\\ \mu+s\left(1-\ln(1-p)\right),\ \xi=0\end{cases}. \tag{7}$$

Using metalog distribution. Suppose that a random variable X with a distribution function $F_X(x)$ follows the metalog distribution [9]. The metalog distribution is determined by the quantile function $M_n(\alpha, \mathbf{x}, \mathbf{a})$:

$$M_n(\alpha,\mathbf{x},\mathbf{a}) = \begin{cases} a_1 + a_2 \ln\frac{\alpha}{1+\alpha}, & n=2 \\ a_1 + a_2 \ln\frac{\alpha}{1+\alpha} + a_3(\alpha-0.5)\ln\frac{\alpha}{1+\alpha}, & n=3 \\ a_1 + a_2 \ln\frac{\alpha}{1+\alpha} + a_3(\alpha-0.5)\ln\frac{\alpha}{1+\alpha} + a_4(\alpha-0.5), & n=4 \\ M_{n-1} + a_n(\alpha-0.5)^{\frac{n-1}{2}}, & \text{for odd } n \geq 5 \\ M_{n-1} + a_n(\alpha-0.5)^{\frac{n}{2}-1} \ln\frac{\alpha}{1+\alpha}, & \text{for even } n \geq 6 \end{cases}$$

where $\alpha \in (0,1)$ is probability, $\mathbf{a} = (a_1, a_2, ..., a_n)^T$ is a parameter vector defined by:

$$\mathbf{a} = [\mathbf{Y}_n^T \mathbf{Y}_n]^{-1} \mathbf{Y}_n^T \mathbf{x}. \tag{8}$$

Column vectors $\mathbf{x} = (x_1, x_2, ..., x_m)^T$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m)^T$ are coordinates of the distribution function $F_X(x)$, $m \geq n$, $\alpha_i \in (0,1)$, $i = 1, ..., m$, at least n of α_i are distinct. The matrix $\mathbf{Y}_n$ is defined as follow:

$$\mathbf{Y}_n = \begin{cases} \begin{vmatrix} 1 & \ln\dfrac{\alpha_1}{1+\alpha_1} \\ \cdots & \cdots \\ 1 & \ln\dfrac{\alpha_N}{1+\alpha_N} \end{vmatrix}, & n=2 \\ \begin{vmatrix} 1 & \ln\dfrac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5)\ln\dfrac{\alpha_1}{1+\alpha_1} \\ \cdots & \cdots & \cdots \\ 1 & \ln\dfrac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5)\ln\dfrac{\alpha_N}{1+\alpha_N} \end{vmatrix}, & n=3 \\ \begin{vmatrix} 1 & \ln\dfrac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5)\ln\dfrac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \ln\dfrac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5)\ln\dfrac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5) \end{vmatrix}, & n=4 \\ \left|\begin{array}{c|c} & (\alpha_1-0.5)^{\frac{n-1}{2}} \\ \mathbf{Y}_{n-1} & \cdots \\ & (\alpha_N-0.5)^{\frac{n-1}{2}} \end{array}\right|, & \text{for odd } n \geq 5 \\ \left|\begin{array}{c|c} & (\alpha_1-0.5)^{\frac{n}{2}-1}\ln\dfrac{\alpha_1}{1+\alpha_1} \\ \mathbf{Y}_{n-1} & \cdots \\ & (\alpha_N-0.5)^{\frac{n}{2}-1}\ln\dfrac{\alpha_N}{1+\alpha_N} \end{array}\right|, & \text{for even } n \geq 6 \end{cases} \tag{9}$$

Based on the definition of risk measures we have:

$$VaR_{\tilde{\alpha},n}(X) = -M_n(\tilde{\alpha},\mathbf{x},\boldsymbol{\alpha}),. \tag{10}$$

$$CVaR_{\tilde{\alpha},n}(X) = -\frac{1}{\tilde{\alpha}}\int_0^{\tilde{\alpha}} M_n(y,\mathbf{x},\boldsymbol{\alpha})\,dy.$$

Explicit analytical formulas for finding $CVaR_{\tilde{\alpha},n}(X)$ are obtained for the different values of n in this paper:

$$CVaR_{\tilde{\alpha},n}(X)=\begin{cases} a_1+a_2\left(\dfrac{\ln(1-\tilde{\alpha})}{\tilde{\alpha}}+\ln\dfrac{\tilde{\alpha}}{1-\tilde{\alpha}}\right), & n=2 \\ CVaR_{\tilde{\alpha},2}(X)+\dfrac{a_3}{2}\left(1+(\tilde{\alpha}-1)\ln\dfrac{\tilde{\alpha}}{1-\tilde{\alpha}}\right), & n=3 \\ CVaR_{\tilde{\alpha},3}(X)+\dfrac{a_4(\tilde{\alpha}-1)}{2}, & n=4 \\ CVaR_{\tilde{\alpha},n-1}(X)+\dfrac{2a_n}{\tilde{\alpha}(1+n)}\left((-1)^{\frac{n-1}{2}}(0.5)^{\frac{n+1}{2}}+(\tilde{\alpha}-0.5)^{\frac{n+1}{2}}\right), & \text{for odd}\, n\ge 5 \\ CVaR_{\tilde{\alpha},n-1}(X)+\dfrac{2^{\frac{2-n}{2}}a_n}{n\tilde{\alpha}}2(2\tilde{\alpha}-1)^{\frac{n}{2}}arctg(2\tilde{\alpha}-1)+nG+ & \\ +\ln(1-\tilde{\alpha})+(-1)^{\frac{n+2}{2}}\ln\tilde{\alpha}, & \text{for even }\, n\ge 6 \end{cases}$$

(11)

where $G={}_3F_2\left[1,1,1-{}^{n}\!/\!_{2};\,2,2;\;2\right]+(\tilde{\alpha}-1)\,{}_3F_2\left[1,1,1-{}^{n}\!/\!_{2};\;2,2;\;2-2\tilde{\alpha}\right]+$ $+(-1)^{n/2}\tilde{\alpha}\;{}_3F_2\left[1,1,1-{}^{n}\!/\!_{2};\,2,2;\,2\tilde{\alpha}\right]$, ${}_3F_2\left[c_1,c_2,c_3;\,d_1,d_2;\;z\right]$ is generalized hypergeometric function. Coefficients $\mathbf{a}=(a_1,a_2,...,a_n)^T$ are defined from (8).

As we see from (11), the explicit formula for finding $CVaR_{\tilde{\alpha},n}(X)$ for even $n\ge 6$ is quite complex. Therefore, for convenience of practical application, approximate formulas were obtained on the basis of the expansion of $CVaR_{\tilde{\alpha},n}(X)$ in powers of $\tilde{\alpha}$. In particular:

$$\begin{aligned} &CVaR_{\tilde{\alpha},6}(X)=CVaR_{\tilde{\alpha},5}(X)+\\ &a_6\left({}^{1}\!/\!_{12\tilde{\alpha}}\ln(1-\tilde{\alpha})+{}^{1}\!/\!_{6}(\tilde{\alpha}-1)+\left({}^{1}\!/\!_{4}-{}^{\tilde{\alpha}}\!/\!_{2}+{}^{\tilde{\alpha}^2}\!/\!_{3}\right)\ln\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)+o\left(\tilde{\alpha}^2\right)\\ &CVaR_{\tilde{\alpha},8}(X)=CVaR_{\tilde{\alpha},7}(X)+\\ &a_8\left({}^{1}\!/\!_{8}-{}^{1}\!/\!_{8}\tilde{\alpha}+{}^{1}\!/\!_{12}\tilde{\alpha}^2+\left(-{}^{1}\!/\!_{8}+{}^{3\tilde{\alpha}}\!/\!_{8}-{}^{\tilde{\alpha}^2}\!/\!_{2}+{}^{\tilde{\alpha}^3}\!/\!_{4}\right)\ln\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right)+o\left(\tilde{\alpha}^3\right) \end{aligned} \qquad (12)$$

The proposed theoretical approach allowed to formulate the algorithm for VaR and CVaR forecasting.

1. Statistical data analysis. Test the time series on volatility and ARCH effect. Test the time series of squares on the long-range dependence: the Hurst parameter estimating with standard methods and obtaining the estimations $\hat{H}_i, i=1,...,I$.

2. Variance modeling and prediction of σ_t^2 with the method of smoothing of the autocorrelation function (MSAF) ([10]). According to the MSAF, a regression model is used to find estimates of the autocorrelation function (ACF), which follows from the definition of a long-range dependence:

$$\rho_k = \beta_1 H(2H-1)k^{2H-2} + \beta_2 + \upsilon_k,\ \upsilon_k \text{ - iid.},\ k_0 \le k \le N \ . \qquad (13)$$

According to (13), ACF estimates are: $\tilde{\rho}_k = \tilde{\beta}_1 \hat{H}_m (2\hat{H}_m - 1)k^{2\hat{H}_m - 2} + \tilde{\beta}_2$, where $\widehat{H}_m = \frac{1}{I}\sum_{i=1}^{I} \widehat{H}_i$, coefficients estimates $\tilde{\beta}_1, \tilde{\beta}_2$ are found with the method of least squares. The estimates $\tilde{\rho}_k$ and the value of the Hurst parameter are corrected with the help of the optimization procedure:

$$\lambda q + (1-\lambda)\frac{1}{k_2 - k_1}\sum_{k=k_1+1}^{k_2} (\rho_k - \tilde{\rho}_k)^2 \to \min$$

$$(\rho_k - \tilde{\rho}_k)^2 \le \varepsilon^2 + q,\ k = k_0, \ldots, k_1\ ,\ q \ge 0\ ,$$

where the parameters of the optimization procedure k_1 and $0 \le \lambda \le 1$ are chosen to describe the behavior of the ACF for large values of the argument in the best way. are found using the optimization procedure ([10]). The obtained estimates $\hat{H}_{opt}$, $\hat{\rho}_k = \hat{\beta}_1 \hat{H}_{opt}(2\hat{H}_{opt} - 1)k^{2\hat{H}_{opt} - 2} + \hat{\beta}_2$, $k_0 \le k \le N$ and the forecasting value $\hat{\rho}_{N+1} = \hat{\beta}_1 \hat{H}_{opt}(2\hat{H}_{opt} - 1)(N+1)^{2\hat{H}_{opt} - 2} + \hat{\beta}_2$ are used for building the variance model.

To predict the variance, the FIGARCH(p,d,q) model (2) is rewritten as an AR (∞) model for squared process ε_t^2:

$$\varepsilon_t^{\ 2} = \gamma_0 + \sum_{i=1}^{\infty} \gamma_i \varepsilon_{t-i}^{\ \ 2} + \zeta_t\ , \qquad (14)$$

where $\zeta_t = \varepsilon_t^{\ 2} - \sigma_t^{\ 2}$ are uncorrelated with mean 0. To find estimates of the autoregressive coefficients $\hat{\gamma}_i$, a reduced system of normal equations is used: $\sum_{j=1}^{k} \hat{\rho}_{|i-j|}\gamma_j = \hat{\rho}_i$, $i = 1, \ldots, k$. The value k ($k \le N$) is determined by the condition of practical convergence of solutions. The coefficients of the system are taken as estimates $\hat{\rho}_i$ for $i = 1, \ldots, k_1$, evaluated using observed values of the time series. $\hat{\rho}_i$ for $i = k_1 + 1, \ldots, k-1$ are found by the optimization procedure. $\hat{\rho}_k$ is evaluated by model extrapolation. The order of the reduced autoregressive model (14) $m \le k$ is determined using information criteria. The built model is used to find variance estimates: $\hat{\sigma}_t^2 = \hat{\gamma}_0 + \sum_{i=1}^{m} \hat{\gamma}_i \hat{\sigma}_{t-i}^2$, $t = 1, \ldots, N$ and to predict variance:

$\hat{\sigma}_{t+s}^2 = \hat{\gamma}_0 + \sum_{i=s}^{m+s} \hat{\gamma}_i \hat{\sigma}_{t-i+1}^2$, $t = N,...,$ $s = 1,...,S$. The estimated variance $\hat{\sigma}_t^2$ is used to find the residuals of model (1): $\hat{v}_t = X_t / \hat{\sigma}_t$. The residuals $\hat{\mathbf{v}} = \{\hat{v}_t, t = 1,...,N\}$ are checked to be iid.

3. Risk measures model residuals $VaR_\alpha(v)$ and $CVaR_\alpha(v)$ estimating. Methods based on the metalog distribution (analytical formulas (10) - (12)) are proposed to estimate the measures $VaR_\alpha(v)$ and $CVaR_\alpha(v)$.

The first method, the Classical Metalog Method (Metalog full), fulfill the parameter estimation of the metalog distribution using the empirical distribution function: $\hat{\mathbf{v}}^* = (\hat{v}_{(2)},...,\hat{v}_{(N)})^T$, $\hat{\boldsymbol{\alpha}} = (\frac{1}{N}, \frac{2}{N},..., \frac{N-1}{N})^T$. Evaluated by (8) and (9) coefficients $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2,...,\hat{a}_n)^T$ are used to estimate $M_n(\tilde{\alpha}, \hat{\mathbf{v}}^*, \hat{\mathbf{a}})$ and find $VaR_{\tilde{\alpha},n}(\hat{\mathbf{v}})$ by (10), $CVaR_{\tilde{\alpha},n}(\hat{\mathbf{v}})$ by (11) or (12) for large even n.

The second method, the Quantile Metalog Method (Metalog quant), fulfill the parameter estimation of the metalog distribution using the empirical quantiles. In this case: $\hat{\mathbf{v}}^* = (\tau_{\alpha_1},...,\tau_{\alpha_p})^T$, $p < N$. The empirical quantile τ_{α_i} can be found under the following formula:

$$\tau_{\alpha_i} = \begin{cases} \hat{v}_{([N\alpha_i]+1)}, & N\alpha_i \notin Z \\ \dfrac{\bar{\alpha} - \alpha_i}{\bar{\alpha} - \underline{\alpha}} \hat{v}_{(\underline{\alpha})} + \dfrac{\alpha_i - \underline{\alpha}}{\bar{\alpha} - \underline{\alpha}} \hat{v}_{(\bar{\alpha})}, & N\alpha_i \in Z \end{cases},$$

where: $\underline{\alpha} = \dfrac{[N\alpha_i]}{N}$, $\bar{\alpha} = \dfrac{[N\alpha_i + 1]}{N}$, $\alpha_i \in (0,1)$.

Similar to the previous approach, the coefficients $\mathbf{a}$ are estimated by (8), (9). Risk measures are obtained by (10), (11) or (12). Since the values of the quantiles τ_{α_i} for the tail part of the distribution are the most important to evaluate $VaR_{\tilde{\alpha},n}(\hat{\mathbf{v}})$ and $CVaR_{\tilde{\alpha},n}(\hat{\mathbf{v}})$, it is advisable to choose the value of α_i over a non-uniform grid, for example, $\alpha_i = ih$, $h = 0.005$ for $i = \overline{1,10}$, and $\alpha_i = ih$, $h = 0.01$ for $i = \overline{11,100}$. Thus, the second proposed method more fully uses the available data to model the tail of the distribution function, which can improve the quality of risk measures estimates.

4. Estimation and forecasting of dynamic risk measures. Obtained estimates of variance and risk measures model residuals are used to find estimates of dynamic risk measures $VaR_\alpha(\hat{X}_t)$, $CVaR_\alpha(\hat{X}_t)$ by (3). Extrapolation of the model allows to obtain predictive values $VaR_\alpha(\hat{X}_{t+S})$, $CVaR_\alpha(\hat{X}_{t+S})$ by (4).

5. Analysis of estimates. To test the quality of the methods proposed in the work, the predicted values are compared with the estimates based on the observed data. The values of mean error (ME), mean absolute error (MAE), and root mean square error (MSE) are used for quantitative analysis of the estimates. Qualitative analysis is performed using backtesting. For the VaR estimates, the Kupiec test (LRuc statistics), the Kristoffersen independence test (LRind statistics), the Kupiec and Kristoffersen combined test (LRss statistics) are used. For CVaR estimates, the V - test with statistics V_1, V_2, V, and statistic $bPoE(\alpha)$: $bPoE(\alpha) = 1 - CVaR_{\alpha}^{-1}(X)$ are used ([8]).

Experiment, Results and Discussions. To validate the proposed approach to VaR and CVaR forecasting, time series of returns on a daily basis of the official indicator of the Russian Stock Exchange System (RTS index) is considered. The length of the total sample is 3386 values for the period from 10/14/2005 to 10/02/2020. The confidence level for the risk measures is $\alpha = 0.1$. Method of forecasting is straight, multi-step (S = 5 steps forward), window with accumulation. The length of the initial window is half of the sample. A total number of windows is 337. At each step, the predicted values of the risk measures are compared with the real ones. To obtain the final characteristics of the forecast quality, all S-step forecasts are combined into one.

Five methods are used to estimate the Hurst parameter: the R/S method, the aggregated variance method, the method of absolute values of the aggregated series, the periodogram method, the method of residuals of regression. The analysis of the obtained values confirmes the presence of the long-range dependence for all windows. The average value is $\widehat{H}_m = 0,8639$. The optimal autoregressive order is determined by AIC and HQC information criteria: $m = 26$. Risk measures estimates for model residuels (1) $\hat{\mathbf{v}}$ are obtained: using formulas (5) (historical method), formulas (6) (GEV method), formulas (7) (GPD method), using the Classical Metalog Method (Metalog full) and Quantile Metalog Method (Metalog quant). Metalog distribution uses $n = 5$. The results of estimation of probability densities for ν and its tail part using the above methods are shown in Figure 1 and Figure 2 respectively. Note that further for convenience the sample values are multiplied by (-1), $p = 1 - \alpha$.

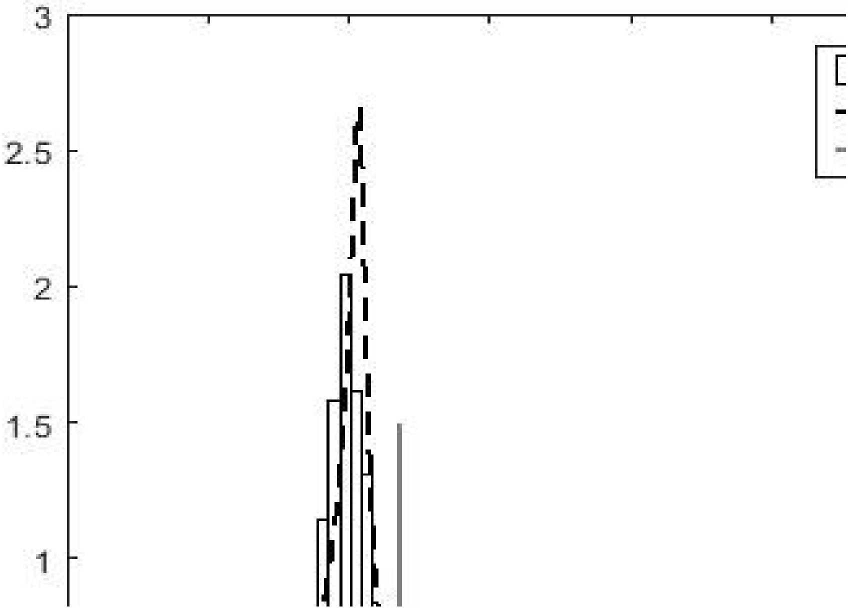

Figure. 1. Empirical probability density function (histogram) and estimated probability density function for ν using Metalog quant

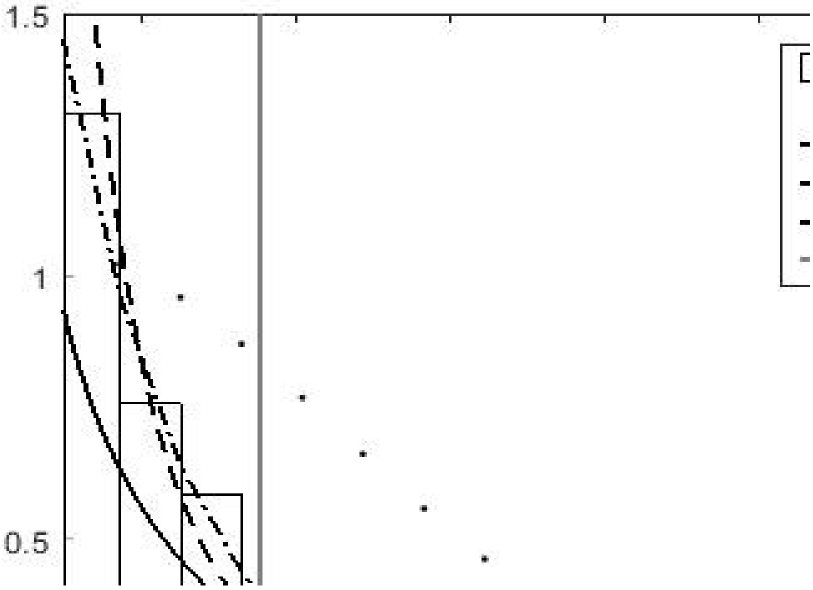

Figure. 2. Tail part for empirical probability density function (histogram) and estimated tail part for probability density function for ν using GEV, GPD, Metalog full, Metalog quant and historical $VaR_{0.9,5}(\hat{\mathbf{v}})$.

Figures 1 and 2 show. that the estimates obtained by the Quantile Metalog Method allowes to describe the behavior of the tail part of the function better than the other methods. This leads to the improvement of estimates $VaR_{0.9,5}(\hat{\mathbf{v}})$ and $CVaR_{0.9,5}(\hat{\mathbf{v}})$. Table 1 demonstrates minimum (min), maximum (max) and average (mn) values of the risk measures for 337 windows.

Table 1

Risk measures model residuals estimates for different windows for RTS index time series

Міра /Метод	historical	GEV	GPD	Metalog full	Metalog quant
$VaR_{0.9,5}(\hat{\mathbf{v}})\,\mathrm{min}$	0,050522	0,091294	0,047275	0,050834	0,046954
$VaR_{0.9,5}(\hat{\mathbf{v}})\,\mathrm{max}$	0,603436	2,676512	0,555759	0,635795	0,546643
$VaR_{0.9,5}(\hat{\mathbf{v}})\,mn$	0,495985	0,7103547	0,4639331	0,473871	0,456269
$CVaR_{0.9,5}(\hat{\mathbf{v}})\,\mathrm{min}$	0,093860	0,143814	0,093859	0,107196	0,085901
$CVaR_{0.9,5}(\hat{\mathbf{v}})\,\mathrm{max}$	1,434677	11,32127	1,180898	1,984921	1,075892
$CVaR_{0.9,5}(\hat{\mathbf{v}})\,mn$	0,965006	1,179965	0,938788	1,115082	0,880454

The dynamic risk model is used to find forecast stimates $VaR_p\left(\hat{X}_{t+S}\right)$ and $CVaR_p\left(\hat{X}_{t+S}\right)$. As an example Figure 3 shows the historical data of the RTS time series and the estimated values of the dynamic risk measures obtained for 337 windows using Quantile Metalog Method.

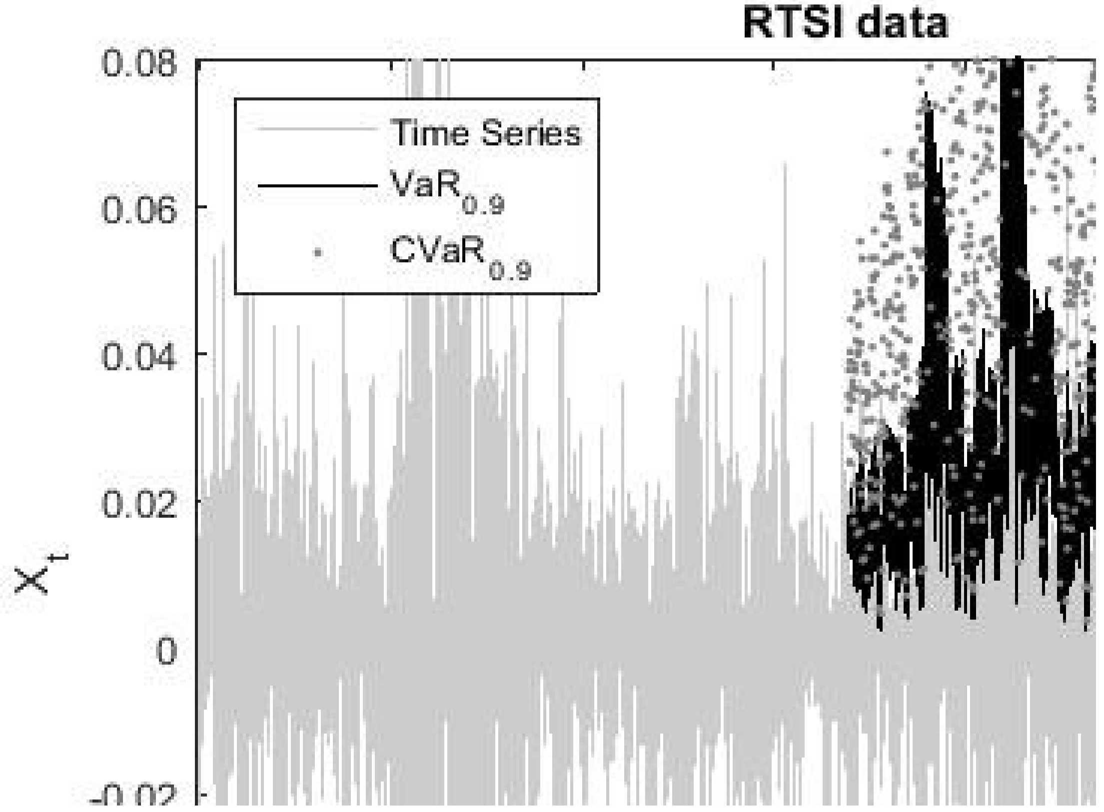

Figure. 3. Historical RTS time series and dynamic risk estimates using Quantile Metalog Method

A visual comparison of the estimated values with the historical values shows a sufficiently high quality of the obtained estimates: the VaR and CVaR forecast values qualitatively describe the dynamics of the time series and show only a slight delay in the determination of extreme outliers. Prediction errors of risk measures for different methods are given in Table 2.

Table 2

The prediction errors of risk measures $VaR_{0.9}\left(\hat{X}_{t+S}\right)$ and $CVaR_{0.9}\left(\hat{X}_{t+S}\right)$ for RTS index time series

Метод/ Оцінка	$VaR_{0.9}\left(\hat{X}_{t+S}\right)$			$CVaR_{0.9}\left(\hat{X}_{t+S}\right)$		
	ME $(\times 10^{-3})$	MAE $(\times 10^{-3})$	MSE $(\times 10^{-4})$	ME $(\times 10^{-3})$	MAE $(\times 10^{-3})$	MSE $(\times 10^{-3})$
Historical	3	11.7	3	15	24	1.3
GEV	12	19.4	26	27	38	37.8
GPD	3.2	11.1	2.7	12	23	1.2
Metalog full	3.4	11.3	2.9	21	29	2.0
Metalog quantl	2.5	10.9	2.6	11	20	1.0

Table 2 shows that the estimates obtained using the GEV function have the biggest errors, the estimates obtained using the Quantile Metalog Method show the best results. This proves the advantage of the proposed method.

The results of the qualitative analysis are shown in Table 3. For the Kupiec and the Kristoffersen tests the significance level is 0.05. Table 3 shows p-values of statistics. If the forecast estimates for $CVaR_{0.9}\left(\hat{X}_{t+S}\right)$ are good the statistics V, V_1, V_2 are close to zero.

Table 3

The results of the qualitative analysis of predictive dynamics measures $VaR_{0.9}\left(\hat{X}_{t+S}\right)$ and $CVaR_{0.9}\left(\hat{X}_{t+S}\right)$ for RTS index time series

Метод/ Оцінка	LRuc	LRind	LRcc	V	V_1	V_2
Historical	0,0418	0,4743	0,0976	0,024673	-0,00612	-0,04323
GEV	0,0000	0,7330	0,0000	0,031603	-0,00650	-0,06196

GPD	0,7137	0,4123	0,6680	0,022887	-0,00407	-0,04171
Metalog full	0,1738	0,1462	0,1380	0,031372	-0,01151	-0,05184
Metalog quant	0,8388	0,3555	0,6392	0,021145	-0,00393	-0,03836

Table 2 shows the correctness of the estimates obtained by all methods, except the estimates obtained with the GEV function, which corresponds to the results of visual and quantitative analyzes. This once again confirmes the superiority of the estimates found using the proposed Quantile Metalog Method.

Conclusion. The paper considers the problem of dynamic risks measures VaR and CVaR forecast based on the heteroskedastic time series model. Squared time series is considered to be long-range dependent. This specificity of the input data is typical for time series, which describe returns of the different stock exchanges indices, especially for developing countries exchanges. Application of the heteroskedastic time series models reduces the problem of dynamic risks modeling to the problem of variance modeling and risk measures model residuals estimating. The method of smoothing of the autocorrelation function is used to model the variance of the time series. For risk measures model residuals estimating two new parametric methods based on the metalog distribution are proposed: Full Metalog Method and Quantile Metalog Method. The approach based on metalog distribution allows to estimate the tail part of the distribution more precisely which is especially important for qualitative risk measures estimation. The methods give explicit analytical formulas for CVaR estimating with a different number of members in the metalog distribution n. The choice of parameter n allows to choose formulas for modeling risk measures that are most appropriate for the input data.

To test the proposed methods, the risk measures forecast for the time series of daily return of the Russian Stock Exchange Trading System is built. For comparison, similar model was built and forecast estimates were obtained using widely known methods based on historical modeling, methods using GEV, GPD functions. Conducted quantitative and qualitative analyzes of the obtained estimates confirmed the effectiveness of the new methods proposed in the work on the basis of the metalog distribution.

The proposed approach for dynamic VaR and CVaR modeling and forecasting using metalog distribution can be extended to a multidimensional case and can be applied in portfolio optimization theory.

[1] N. G. Zrazhevskaja, and A. G. Zrazhevskij, “Classification of methods for risk measures VaR and CVaR calculation and estimation”, *Системні дослідження та*

інформаційні технології, no.3. c. 118-125, 2016. DOI: 10.20535/SRIT.2308-8893.2016.3.11

[2] N. Zrazhevska, “Construction and application of the classification scheme of dynamic risk measures estimating”, *Eureka: Physics and Engineering*, no.5, pp. 67-80, 2016. DOI: 10.21303/2461-4262.2016.00162.

[3] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton (NJ), USA: PrincetonUniversityPress, 2005.

[4] Echaust, K.; Just, M. Value at Risk Estimation Using the GARCH-EVT Approach with Optimal Tail Selection. *Mathematics* 2020, *8*, 114.

[5] Omari, C.; Mwita, P.; Waititu, A. Using Conditional Extreme Value Theory to Estimate Value-at-Risk for Daily Currency Exchange Rates. J. Math. Financ. 2017, 7, 846–870

[6] Carlo Acerbi; Dirk Tasche (2002). "Expected Shortfall: a natural coherent alternative to Value at Risk"(PDF). Economic Notes.31(2): 379–388. arXiv:cond-mat/0105191. doi:10.1111/1468-0300.00091. RetrievedApril 25,2012.

[7] Khokhlov, Valentyn (2018-06-21). "Conditional Value-at-Risk for Uncommon Distributions". SSRN 3200629

[8] Norton, Matthew; Khokhlov, Valentyn; Uryasev, Stan (2018-11-27). "Calculating CVaR and bPOE for Common Probability Distributions With Application to Portfolio Optimization and Density Estimation". arXiv:1811.11301 [q-fin.RM].

[9] Thomas W. Keelin The Metalog Distributions Decision Analysis, Volume 13, Issue 4 December 2016 Pages 223-293, https://doi.org/10.1287/deca.2016.0338

[10] N. D. Pankratova, N. G. Zrazhevskaja, ”Method of dynamic VaR and CVaR risk measures forecasting for long range dependent time series on the base of the heteroscedastic model”, *Intelligent Control and Automation Journal,*vol. 8, no. 2, 2017, DOI: 10.4236/ica.2017.82010.