

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE
THE NATIONAL TECHNICAL UNIVERSITY OF UKRAINE
“Igor Sikorsky Kyiv Polytechnic Institute”

HIGHER MATHEMATICS

Series

Elements of Theory

Kyiv

Igor Sikorsky Kyiv Polytechnic Institute

2021

Zhuravska Ganna. Higher Mathematics. Series. / Zhuravska Ganna – Kyiv, “Igor Sikorsky Kyiv Polytechnic Institute”, 2021 – 67 p.

Approval stamp is provided by the Methodical Council of the Igor Sikorsky Kyiv Polytechnic Institute (protocol №7 from 13.05.21) on the submission of the Academic Council of Faculty of Physics and Mathematics (protocol №3 from 29.03.21)

Electronic online educational edition

HIGHER MATHEMATICS

Series

Elements of Theory.

Compiler: Zhuravska Ganna – docent, PhD

Responsible editor: Stepanenko Natalia – docent, PhD

Reviewer: Samoilenko Tetiana – PhD

V. M. Glushkov Institute of Cybernetics

Laboratory of Methods of Mathematical modeling of

Processes of Environment and Energy № 141

Timoshenko Oleksandr – PhD

assistant professor at Igor Sikorsky Kyiv Polytechnic Institute

This textbook is designed for students of the second year of technical university. It covers one of content areas to be studied in the third semester: Series.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of the theory of series are explained and illustrated by figures and examples.

© Igor Sikorsky Kyiv Polytechnic Institute, 2021

CONTENTS

Introduction.....	4
1. Numerical Series.....	5
1.1 Basic Concepts and Definitions of Numerical Series.	5
1.2 Necessary Condition for Convergence of a Series.....	10
1.3 Comparing Series with Positive Terms.....	11
1.4 Tests of Convergence.....	14
1.5 Alternating Series.....	20
2. Functional Series.....	24
2.1 The Concept of a Functional Series.....	24
2.2 Uniform Convergence of a Functional Series.....	26
3. Power Series.....	28
3.1 The Concept of a Power Series.....	28
3.2 Taylor's and Maclaurin's Series.....	34
3.3 Application of Power Series.....	42
4. Fourier Series.....	50
4.1 The Concept of a Fourier Series.....	50
4.2 Fourier Series for Even and Odd Functions.....	57
4.3 Fourier Series for 2ℓ -periodic Functions.....	60
4.4 Fourier Series for Functions Defined on an Arbitrary Interval.....	63
References.....	67

Introduction

This textbook is designed for students of the second year of technical university. It covers three content areas to be studied in the third semester: Numerical Series, Functional Series and Fourier Series.

The manual can be helpful for students who want to study the concepts of series and apply them for solving some tasks.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts are explained and illustrated by figures and examples.

The first part deal with numerical series: concept of series, its sum and convergence. Here students could find different tests for studying convergence of series.

The second and third sections are devoted to functional series. The second part includes general theory for functional series. In the third chapter it is introduced the concept of power series, Taylor's and Maclaurin's series expansions and their applications for solving differential equations and in approximation theory.

The last part concerned with the concept of Fourier series expansions for a function.

1. Numerical Series

1.1 Basic Concepts and Definitions of Numerical Series

I. Consider the infinite sequence of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

Definition. The expression

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is called a *numerical series* and is denoted by

$$\sum_{n=1}^{\infty} a_n \quad (1.1)$$

The numbers $a_1, a_2, a_3, \dots, a_n, \dots$ are called the *terms of the series* and the number n is called the *index of summation*.

Since we can find the sum of only **finite** amount of numbers, how should we understand the meaning of infinite sum?

Let us define the sequence

$$S_1 = a_1;$$

$$S_2 = a_1 + a_2;$$

...

$$S_n = a_1 + a_2 + a_3 + \dots + a_n;$$

...

Definition. The sum of the first n terms of a series is called the *n th partial sum of the series*.

Hence, we can define the sum of series as

$$\lim_{n \rightarrow \infty} S_n = \sum_{n=1}^{\infty} a_n.$$

Definition. If there exist a finite limit of the sequence $\{S_n\}$

$$\lim_{n \rightarrow \infty} S_n = S$$

it is called the *sum of the series* (1) and we say the series *converges* (is convergent).

If the limit does not exist (for example, $\{S_n\}$ tends to infinity), then the series (1.1) *has no sum and diverges (is divergent)*.

Examples. The following series are useful in later development.

1. Consider the series

$$\sum_{n=1}^{\infty} a^{n-1} = 1 + a + a^2 + a^3 + \cdots + a^n + \cdots$$

This is a geometric progression with the first term 1 and ratio a , ($a \neq 0$).

The sum of the first n terms of the geometric progression is

$$S_n = \frac{1-a^n}{1-a}, \text{ when } a \neq 1.$$

Let us consider several cases.

Case 1. If $|a| > 1$, then $|a|^n \rightarrow \infty, n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1-a^n}{1-a} = \pm \infty.$$

Therefore, the limit of $\{S_n\}$ does not exist and, according to definition, the series is divergent.

Case 2. If $|a| < 1$, then $|a|^n \rightarrow 0, n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1-a^n}{1-a} = \lim_{n \rightarrow \infty} \left(\frac{1}{1-a} - \frac{a^n}{1-a} \right) = \frac{1}{1-a}.$$

In this case the series is convergent and

$$\sum_{n=1}^{\infty} a^{n-1} = \frac{1}{1-a}.$$

Case 3. If $a = 1$, then

$$\sum_{n=1}^{\infty} a^{n-1} = 1 + 1 + \cdots$$

and

$$S_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n.$$

Thus

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} n = \infty$$

and the series is divergent.

Case 4. If $a = -1$, then

$$\sum_{n=1}^{\infty} a^{n-1} = 1 - 1 + 1 - 1 + \dots$$

In this case

$$S_n = \begin{cases} 0, & \text{when } n \text{ is even,} \\ 1, & \text{when } n \text{ is odd.} \end{cases}$$

Therefore, the series is divergent.

2. Prove that the following series (**harmonic series**) is divergent

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

Let us group the terms of series

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \left(\frac{1}{17} + \dots + \frac{1}{32}\right) + \dots$$

Each group contains twice the number of terms as the preceding group. Since increasing the denominator decreases the value of a fraction, we have

$$\begin{aligned} \frac{1}{3} + \frac{1}{4} &> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}; \\ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}; \\ \frac{1}{9} + \dots + \frac{1}{16} &> \frac{1}{16} + \frac{1}{16} + \dots + \frac{1}{16} = \frac{1}{2}; \\ \frac{1}{17} + \dots + \frac{1}{32} &> \frac{1}{32} + \dots + \frac{1}{32} = \frac{1}{2}; \dots \end{aligned}$$

Since the sum of the terms within each set of parentheses is greater than $\frac{1}{2}$, we obtain the following inequalities:

$$\begin{aligned} S_4 &> 1 + \frac{1}{2} + \frac{1}{2} = 2 > 1,5 = 3 \cdot \frac{1}{2}; \\ S_8 &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 4 \cdot \frac{1}{2}; \\ S_{16} &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 5 \cdot \frac{1}{2}; \\ S_{32} &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 6 \cdot \frac{1}{2}; \dots \end{aligned}$$

Using mathematical induction, it can be proved that for every positive integer m

$$S_{2^m} > (m + 1) \cdot \frac{1}{2}.$$

It follows that S_n can be made as large as desired by taking n sufficiently large and

$$\lim_{n \rightarrow \infty} S_n = \infty.$$

Since $\{S_n\}$ diverges, the given series is divergent.

Properties of Numerical Series.

I. If a series obtained from a given series (1.1) by suppression of some amount of its terms converges, then the given series itself is convergent. Conversely, if a given series converges, then a series obtained from a given series (1.1) by suppression of some amount of its terms also converges.

Proof. Rewrite the series (1.1) as follows

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_k}_{C_k} + \sum_{n=k}^{\infty} a_n.$$

Let S_n be the sum of the first n terms, C_k ($k < n$) – the sum of k suppressed terms (it is constant value independent of n). Define $\sigma_{n-k} = S_n - C_k$, the sum of terms that enter into the sum S_n but do not enter into C_k . Then

$$S_n = C_k + \sigma_{n-k}.$$

From this relation it follows that if $\lim_{n \rightarrow \infty} \sigma_{n-k}$ exists, then exists $\lim_{n \rightarrow \infty} S_n$ as well.

Conversely, if $\lim_{n \rightarrow \infty} S_n$ exists, then $\lim_{n \rightarrow \infty} \sigma_{n-k}$ also exists, which proves property.

II. If a series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

converges and its sum is S , then the series

$$Ca_1 + Ca_2 + Ca_3 + \cdots + Ca_n + \cdots,$$

where C is some fixed number, also converges, and its sum is CS .

Proof. Since the original series is convergent and

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (a_1 + \cdots + a_n) = S,$$

we have

$$\lim_{n \rightarrow \infty} \sigma_n = \lim_{n \rightarrow \infty} (Ca_1 + \cdots + Ca_n) = \lim_{n \rightarrow \infty} C(a_1 + \cdots + a_n) = C \lim_{n \rightarrow \infty} S_n = CS.$$

Thus, the series $Ca_1 + Ca_2 + \cdots + Ca_n + \cdots$ is convergent and its sum is CS .

III. If the series

$$a_1 + a_2 + \cdots + a_n + \cdots$$

and

$$b_1 + b_2 + \cdots + b_n + \cdots$$

converge and their sums, respectively, are S_a and S_b , then the series

$$(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) + \cdots$$

and

$$(a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n) + \cdots$$

are also convergent and their sums are $S_a + S_b$ and $S_a - S_b$, respectively.

Proof. Let us prove the property for the first new series.

Since the original series are convergent and

$$\lim_{n \rightarrow \infty} S_{an} = \lim_{n \rightarrow \infty} (a_1 + \cdots + a_n) = S_a$$

and

$$\lim_{n \rightarrow \infty} S_{bn} = \lim_{n \rightarrow \infty} (b_1 + \cdots + b_n) = S_b,$$

we get for the new series

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} ((a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)) = \\ &= \lim_{n \rightarrow \infty} ((a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)) = \\ &= \lim_{n \rightarrow \infty} (a_1 + \cdots + a_n) + \lim_{n \rightarrow \infty} (b_1 + \cdots + b_n) = \\ &= \lim_{n \rightarrow \infty} S_{an} + \lim_{n \rightarrow \infty} S_{bn} = S_a + S_b. \end{aligned}$$

Therefore, the series $(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) + \cdots$ is convergent.

For the second series the proof is similar.

One of the main questions about series is that of whether the given series converges or diverges. But it is complicated to find out by using limit of partial sums, because we should know the general formula for partial sum. But finding general formula is complicated process in the most of cases. That's why we need sufficient conditions to decide the problem of convergence. We shall also establish the necessary condition for convergence.

1.2 Necessary Condition for Convergence of a Series

Theorem 1.1

If a series converges, then its n th term tends to zero as $n \rightarrow \infty$ ($\lim_{n \rightarrow \infty} a_n = 0$).

Proof.

Let us consider the convergent series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

According to definition, we have for the sequence of partial sums $\{S_n\}$

$$\lim_{n \rightarrow \infty} S_n = S,$$

where S is a finite fixed number - the sum of series.

It is obvious that

$$\lim_{(n-1) \rightarrow \infty} S_{n-1} = \lim_{n \rightarrow \infty} S_{n-1} = S.$$

Subtracting the second equation from the first termwise, we obtain

$$\lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = S - S = 0.$$

But

$$\lim_{n \rightarrow \infty} S_n - \lim_{n \rightarrow \infty} S_{n-1} = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = \lim_{n \rightarrow \infty} a_n = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} a_n = 0.$$

Note. Remember this theorem gives us a requirement for convergence, but not a guarantee of convergence. That is, if $\lim_{n \rightarrow \infty} a_n = 0$, the series may be divergent!

For example, let us consider harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

We have proved that it is divergent, but

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Therefore, from the fact that $\lim_{n \rightarrow \infty} a_n = 0$, it does not follow that the series converges.

To examine convergence of a series it is not enough to prove that $\lim_{n \rightarrow \infty} a_n = 0$, since that may be true for divergent as well as for convergent series.

As a corollary for this theorem we have a test for divergence of the series.

Divergence test.

If the n th term of a series **does not** tends to zero as $n \rightarrow \infty$ ($\lim_{n \rightarrow \infty} a_n \neq 0$), then the series **diverges**.

Example.

Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{5n+1} = \frac{1}{6} + \frac{2}{11} + \frac{3}{16} + \cdots + \frac{n}{5n+1} + \cdots$$

Since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{5n+1} = \frac{1}{5} \neq 0,$$

the given series is divergent.

1.3 Comparing Series with Positive Terms

Consider two series with positive terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \quad (1.1)$$

and

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots + b_n + \cdots \quad (1.2)$$

Comparison Test I.

If $\forall n \geq 1, a_n \leq b_n$ and the series (1.2) converges, then the series (1.1) is also convergent.

Proof. Let s_n and σ_n be, respectively, the partial sums of the first and the second series:

$$s_n = a_1 + a_2 + \cdots + a_n,$$

$$\sigma_n = b_1 + b_2 + \cdots + b_n.$$

The series (1.2) converges and

$$\lim_{n \rightarrow \infty} \sigma_n = \sigma.$$

Since $\forall n \geq 1, 0 < a_n \leq b_n$,

$$s_n \leq \sigma_n < \sigma.$$

Thus, we have that the sequence of partial sums $\{s_n\}$ increases as n increases and it is bounded. It follows that it has a limit

$$\lim_{n \rightarrow \infty} s_n = s$$

and

$$s < \sigma.$$

Example.

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots$$

Since

$$\frac{1}{3^3} < \frac{1}{2^2}, \quad \frac{1}{4^4} < \frac{1}{2^4}, \quad \cdots \quad \frac{1}{n^n} < \frac{1}{2^n}, \quad \cdots$$

The series $1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots$ is converges because its terms, beginning with the second, form a geometric progression with common ratio $\frac{1}{2}$. The sum of this series is equal to $1\frac{1}{2}$.

Hence, according to Comparison Test I, the given series also converges, and its sum does not exceed $1\frac{1}{2}$.

Comparison Test II.

If $\forall n \geq 1, a_n \geq b_n$ and the series (1.2) diverges, then the series (1.1) is also divergent.

Proof. Since $\forall n \geq 1, a_n \geq b_n > 0$,

$$s_n \geq \sigma_n.$$

The series (1.2) is diverges and its partial sums increases with increasing n . It follows that

$$\lim_{n \rightarrow \infty} \sigma_n = \infty.$$

As a result

$$\lim_{n \rightarrow \infty} s_n = \infty.$$

That is the series (1.1) is divergent.

Example.

The series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n+1}} + \cdots$$

is divergent because its terms are greater than the corresponding terms of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} + \cdots$$

Limit Comparison Test.

Consider

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c.$$

If c is positive finite number then either both series are convergent or both series are divergent.

Proof. Since $0 < c < \infty$, there exist two positive finite numbers, m and M , such that

$$m < c < M.$$

Hence, for large enough n the quotient $\frac{a_n}{b_n}$ is close to c and there exists a positive integer N such that if $n > N$ we have

$$m < \frac{a_n}{b_n} < M.$$

Thus, multiplying the inequality by b_n

$$mb_n < a_n < Mb_n, \quad n > N.$$

If $\sum b_n$ is divergent then $\sum mb_n$ diverges and so since $mb_n < a_n$ for all sufficiently large n by the Comparison Test II the series $\sum a_n$ also diverges.

Similarly, if $\sum b_n$ converges then $\sum Mb_n$ is convergent and so since $a_n < Mb_n$ for all sufficiently large n by the Comparison Test I the series $\sum a_n$ is also convergent.

Example. Consider the series

$$\sum_{n=1}^{\infty} \sin \frac{1}{2^n} = \sin \frac{1}{2} + \sin \frac{1}{4} + \cdots + \sin \frac{1}{2^n} + \cdots$$

Let us compare it to convergent series (geometric progression)

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots$$

Since

$$\lim_{n \rightarrow \infty} \frac{\sin \frac{1}{2^n}}{\frac{1}{2^n}} = 1,$$

according to Limit Comparison Test, the given series converges.

1.4 Tests of Convergence

There are a lot of series involving quite complicated expressions that makes it challenging the application of comparison tests. We introduce sufficient tests that are useful while determining convergence or divergence of series.

I. Ratio Test (D'Alembert's Test)

Consider the positive-term series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Suppose

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = D, \tag{1.3}$$

then

- the series is convergent for $D < 1$;
- the series is divergent for $D > 1$;
- for $D = 1$ apply a different test (the series may be convergent or divergent).

Proof.

1) Let $D < 1$. Consider a number $\varepsilon > 0$ such that $D + \varepsilon < 1$.

According to (1.3) and the definition of a limit, there exists the number $N(\varepsilon) = N$ that for all values of $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} - D \right| < \varepsilon,$$

that is

$$D - \varepsilon < \frac{a_{n+1}}{a_n} < D + \varepsilon.$$

Hence, $\forall n \geq N$,

$$a_{n+1} < (D + \varepsilon)a_n$$

and

$$\begin{aligned} a_{N+1} &< (D + \varepsilon)a_N \\ a_{N+2} &< (D + \varepsilon)a_{N+1} < (D + \varepsilon)^2 a_N \\ a_{N+3} &< (D + \varepsilon)a_{N+2} < (D + \varepsilon)^3 a_N \\ &\dots \end{aligned}$$

Now we get two series

$$a_1 + a_2 + \dots + a_N + a_{N+1} + a_{N+2} + \dots$$

and

$$a_N + (D + \varepsilon)a_N + (D + \varepsilon)^2 a_N + \dots$$

The second series converges as the geometric progression and, according the Comparison Test I and Property I, the original series is convergent.

2) Let $D > 1$.

According to (1.3) and the definition of a limit, there exists the number $N(\varepsilon) = N$ that for all values of $n \geq N$,

$$\frac{a_{n+1}}{a_n} > 1.$$

Hence,

$$a_{n+1} > a_n.$$

Therefore, the sequence $\{a_n, n \geq N\}$ increases and $\lim_{n \rightarrow \infty} a_n \neq 0$. But this means that the original series diverges.

Example. Test the following series for convergence

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!} + \cdots$$

Let us apply the Ratio Test.

Here,

$$a_n = \frac{1}{n!}, \quad a_{n+1} = \frac{1}{(n+1)!} = \frac{1}{\underbrace{1 \cdot 2 \cdot \dots \cdot n}_{n!} \cdot (n+1)} = \frac{1}{n! (n+1)}.$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n!}{n! (n+1)} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1.$$

The given series is convergent.

II. Root Test (Cauchy's Test)

Consider the positive-term series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Suppose

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = C, \tag{1.4}$$

then

- the series is convergent for $C < 1$;
- the series is divergent for $C > 1$;
- for $C = 1$ apply a different test (the series may be convergent or divergent).

Proof.

1) Let $C < 1$. Consider a number $\varepsilon > 0$ such that $C + \varepsilon < 1$.

According to (1.4) and the definition of a limit, there exists the number $N(\varepsilon) = N$ that for all values of $n \geq N$,

$$|\sqrt[n]{a_n} - C| < \varepsilon,$$

that is

$$C - \varepsilon < \sqrt[n]{a_n} < C + \varepsilon.$$

Hence, $\forall n \geq N$,

$$a_n < (C + \varepsilon)^n.$$

Since the series $\sum (C + \varepsilon)^n$ converges as the geometric progression, the original series is convergent according the Comparison Test I and Property I.

2) Let $C > 1$.

According to (1.3) and the definition of a limit, there exists the number $N(\varepsilon) = N$ that for all values of $n \geq N$,

$$\sqrt[n]{a_n} > 1.$$

Hence, $a_n > 1$ and

$$\lim_{n \rightarrow \infty} a_n \neq 0.$$

Therefore, the original series diverges.

Example. Test the following series for convergence

$$\sum_{n=1}^{\infty} \left(\frac{n}{3n-1} \right)^n = \frac{1}{2} + \frac{4}{25} + \frac{9}{64} + \cdots + \left(\frac{n}{3n-1} \right)^n + \cdots$$

According the Root Test

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{3n-1} \right)^n} = \lim_{n \rightarrow \infty} \frac{n}{3n-1} = \frac{1}{3} < 1.$$

The series is convergent.

III. Integral Test

Consider the positive-term series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots$$

and let $f(n) = a_n$, where f is the function obtained by replacing x with n .

If for every real number $x \geq 1$ $f(x)$ is a positive-valued, continuous and nonincreasing function, then the series

- converges if $\int_1^{\infty} f(x) dx$ is convergent;
- diverges if $\int_1^{\infty} f(x) dx$ is divergent.

Proof.

Consider the positive-valued, continuous and nonincreasing for every real number $x \geq 1$ function $y = f(x)$, such that $f(n) = a_n$ (Fig. 1, Fig.2).

For positive integer $n \geq 1$ the area of the circumscribed rectangular polygon (Fig 1.) is

$$\sum_{k=1}^{n-1} f(k) = f(1) + f(2) + \dots + f(n-1).$$

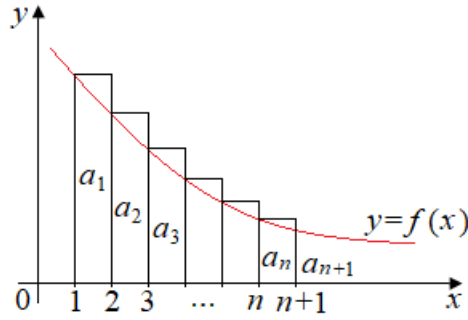


Figure 1.

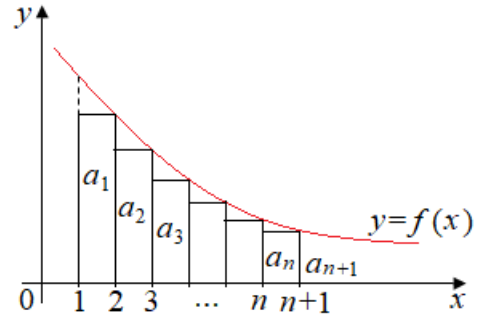


Figure 2.

In the same fashion we get the area of the inscribed rectangular polygon (Fig 2.)

$$\sum_{k=2}^n f(k) = f(2) + f(3) + \dots + f(n).$$

Since $\int_1^n f(x) dx$ is the area of the curvilinear trapezoid created by the function $y = f(x)$, for $1 \leq x \leq n$,

$$\sum_{k=2}^n f(k) \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} f(k).$$

Denote the n th partial sum of the series $\sum_{k=1}^{\infty} f(k)$ as S_n , then previous inequality may be written

$$S_n - f(1) \leq \int_1^n f(x) dx \leq S_{n-1}.$$

If the improper integral $\int_1^{\infty} f(x) dx$ is convergent and equals $K > 0$, then, according to preceding inequalities,

$$S_n - f(1) \leq K, \text{ or } S_n \leq f(1) + K$$

for every positive integer $n \geq 1$.

Hence, the sequence $\{S_n\}$ is monotone increasing and bounded for $n \geq 1$. Therefore, it exists the finite limit $\lim_{n \rightarrow \infty} S_n = S$ and the series $\sum_{n=1}^{\infty} a_n$ converges.

If the improper integral is divergent, then

$$\lim_{n \rightarrow \infty} \int_1^n f(x) dx = \infty,$$

and since $\int_1^n f(x) dx \leq S_{n-1}$, we have $\lim_{n \rightarrow \infty} S_{n-1} = \infty$. Thus, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example. Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

This series is called *hyperharmonic series*

Let us apply the integral test, putting

$$f(x) = \frac{1}{x^p}.$$

This function satisfy all the conditions of theorem. Consider the improper integrals for various values of p .

For $p > 1$,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left. \frac{1}{(1-p)x^{p-1}} \right|_1^b = \lim_{b \rightarrow \infty} \frac{1}{(1-p)b^{p-1}} - \frac{1}{1-p} = \frac{1}{p-1},$$

the integral is convergent and, hence, the series converges.

For $p = 1$,

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln|x| \Big|_1^b = \lim_{b \rightarrow \infty} \ln b - \ln 1 = \infty,$$

the integral is divergent and, hence, the series diverges.

For $p < 1$,

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx = \lim_{b \rightarrow \infty} \left. \frac{x^{1-p}}{1-p} \right|_1^b = \lim_{b \rightarrow \infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p} = \infty,$$

the integral is divergent and, hence, the series diverges.

The summing-up is

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergent for } p > 1, \\ \text{divergent for } p \leq 1. \end{cases}$$

1.5 Alternating Series

I. Previously, we have considered positive-term series. But there are many series out of there that have both positive and negative terms.

One of the simplest, and the most useful, series of this type is an *alternating series*, whose terms have alternating signs. It could be written in one of the forms

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad (1.5)$$

or

$$-a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots = \sum_{n=1}^{\infty} (-1)^n a_n = - \sum_{n=1}^{\infty} (-1)^{n-1} a_n,$$

where $a_n > 0$ for every n .

Since tests that we looked at for series convergence have required that all the terms in the series be positive, we cannot apply them to alternating series. The test that we are going to consider now will be a test for alternating series.

II. Alternating Series Test (Leibniz's Theorem)

Consider the alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n, \quad (a_n > 0).$$

If the following two conditions are satisfied:

- $a_1 > a_2 > a_3 > \cdots > a_n > a_{n+1} > \cdots$;
- $\lim_{n \rightarrow \infty} a_n = 0$;

then the series is convergent, its sum is positive and does not exceed the first term.

Proof.

1) Let us consider the sum of the first $n = 2m$ terms:

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2m-1} - a_{2m}).$$

Since $a_n > a_{n+1}$ for every n , each of the expressions in parentheses is positive and the sum $S_{2m} > 0$. Moreover, it increases with increasing m .

Rewrite this sum as follows

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2m-2} - a_{2m-1}) - a_{2m}.$$

According to the condition $a_n > a_{n+1}$, each of the parentheses is positive and, subtracting these parentheses from a_1 we get

$$S_{2m} < a_1.$$

Hence, the sequence $\{S_{2m}\}$ is increasing and bounded above. Consequently, $\{S_{2m}\}$ has the limit:

$$\lim_{m \rightarrow \infty} S_{2m} = S \text{ and } 0 < S < a_1.$$

2) Now let us consider the sum of the first $n = 2m + 1$ terms:

$$S_{2m+1} = a_1 - a_2 + \cdots - a_{2m} + a_{2m+1} = S_{2m} + a_{2m+1}.$$

Since $\lim_{n \rightarrow \infty} a_n = 0$, it follows that

$$\lim_{m \rightarrow \infty} S_{2m+1} = \lim_{m \rightarrow \infty} (S_{2m} + a_{2m+1}) = \lim_{m \rightarrow \infty} S_{2m} + \lim_{m \rightarrow \infty} a_{2m+1} = S.$$

Therefore, we have proved that $\lim_{n \rightarrow \infty} S_n = S$ both for even n and for odd n . The original series is convergent.

Corollary. If the alternating series is convergent, then the error obtained when replacing S by S_n does not exceed in absolute value the first of the suppressed terms, that is

$$|S - S_n| = |r_n| \leq a_{n+1}.$$

Example. Test the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

for convergence.

Since

- $1 > \frac{1}{2} > \frac{1}{3} > \cdots > \frac{1}{n} > \frac{1}{n+1} > \cdots;$
- $\lim_{n \rightarrow \infty} \frac{1}{n} = 0;$

this series converges.

III. Absolute and Conditional Convergence.

The alternating series is a special case of so called *plus-minus series*. Plus-minus series has both positive and negative terms. For example

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{4}}{n^2} &= \sin \frac{\pi}{4} + \frac{\sin \frac{\pi}{2}}{4} + \frac{\sin \frac{3\pi}{4}}{9} + \frac{\sin \pi}{16} + \frac{\sin \frac{5\pi}{4}}{25} + \frac{\sin \frac{3\pi}{2}}{36} + \frac{\sin \frac{7\pi}{4}}{49} + \dots = \\ &= \frac{\sqrt{2}}{2} + \frac{1}{4} + \frac{\sqrt{2}}{18} - \frac{\sqrt{2}}{50} - \frac{1}{36} - \frac{\sqrt{2}}{98} + \dots\end{aligned}$$

Let us give an important sufficient condition for the convergence of a plus-minus series.

Definition. The plus-minus series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (1.6)$$

is called *absolutely convergent* if a series made up of the absolute values of its terms

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots + |a_n| + \dots \quad (1.7)$$

converges.

If the series (1.6) converges, while the series (1.7) diverges, then the given plus-minus series is called a *conditionally convergent series*.

Theorem 1.2 (Sufficient condition)

If the plus-minus series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (1.6)$$

is such that a series made up of the absolute values of its terms

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots + |a_n| + \dots \quad (1.7)$$

converges, then the given series is also convergent.

Proof. Assume, that S_n and σ_n are the n th partial sums of the series (1.6) and (1.7). Let s_{1n} be the sum of all positive terms and s_{2n} the sum of absolute values of all negative terms of the sum S_n , then

$$S_n = s_{1n} - s_{2n} \text{ and } \sigma_n = s_{1n} + s_{2n}.$$

Since the series (1.7) converges, $\lim_{n \rightarrow \infty} \sigma_n = \sigma$ and the values of s_{1n} and s_{2n} are less than σ . Moreover, the sequences $\{s_{1n}\}$ and $\{s_{2n}\}$ have limits s_1 and s_2 respectively.

Hence,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (s_{1n} - s_{2n}) = s_1 - s_2.$$

Thus, the plus-minus series (1.6) converges.

Examples.

1. Consider the series

$$\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{4}}{n^2} = \sin \frac{\pi}{4} + \frac{\sin \frac{\pi}{2}}{4} + \frac{\sin \frac{3\pi}{4}}{9} + \frac{\sin \pi}{16} + \frac{\sin \frac{5\pi}{4}}{25} + \frac{\sin \frac{3\pi}{2}}{36} + \frac{\sin \frac{7\pi}{4}}{49} + \dots$$

Test the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin \frac{\pi n}{4}}{n^2} \right| = \left| \sin \frac{\pi}{4} \right| + \left| \frac{\sin \frac{\pi}{2}}{4} \right| + \left| \frac{\sin \frac{3\pi}{4}}{9} \right| + \left| \frac{\sin \pi}{16} \right| + \dots + \left| \frac{\sin \frac{\pi n}{4}}{n^2} \right| + \dots$$

for convergence.

Since $\left| \frac{\sin \frac{\pi n}{4}}{n^2} \right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as hyperharmonic series ($p = 2 > 1$),

according to the Comparison Test I, the series $\sum_{n=1}^{\infty} \left| \frac{\sin \frac{\pi n}{4}}{n^2} \right|$ is convergent. Hence, the series

$\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{4}}{n^2}$ also converges. Moreover, it is absolutely convergent.

2. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{n} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is conditionally convergent, since it is convergent, but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Properties of absolutely convergent and conditionally convergent series

I. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and its sum is S then any rearrangement of the terms of $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and its sum is S .

II. If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent and σ is any real number then there is a rearrangement of the terms of $\sum_{n=1}^{\infty} a_n$ whose sum will be σ .

2. Functional Series

2.1 The Concept of a Functional Series

Consider the series

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots \quad (2.1)$$

Definition. The series whose terms are functions of x is called *functional series*.

For example,

$$1) \sum_{n=1}^{\infty} \frac{\ln^n x}{n^3} = \ln x + \frac{\ln^2 x}{2^3} + \frac{\ln^3 x}{3^3} + \cdots + \frac{\ln^n x}{n^3} + \cdots;$$

$$2) \sum_{n=1}^{\infty} (x+3)^n = (x+3) + (x+3)^2 + (x+3)^3 + \cdots + (x+3)^n + \cdots$$

For each real number x (definite numerical value) we get different numerical series, which may be prove to be convergent or divergent.

Definition. The set of all those values of x for which the functional series converges is called *the domain of convergence* of the series.

The method of determining the domain of convergence is based on investigating the series made up of the absolute values of its terms $(\sum_{n=1}^{\infty} |u_n(x)|)$. To determine the convergence of these positive-term series we apply Ratio or Root Tests.

Example.

Find the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{\ln^n x}{n^3} = \ln x + \frac{\ln^2 x}{2^3} + \frac{\ln^3 x}{3^3} + \cdots + \frac{\ln^n x}{n^3} + \cdots$$

Applying the Root Test, we get

$$\lim_{n \rightarrow \infty} \sqrt[n]{|u_n(x)|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{\ln^n x}{n^3} \right|} = \lim_{n \rightarrow \infty} \frac{|\ln x|}{\sqrt[n]{n^3}} = |\ln x|.$$

Thus, the series converges when $|\ln x| < 1$ and diverges when $|\ln x| > 1$.

Let us solve the inequality

$$|\ln x| < 1,$$

$$\begin{aligned} -1 < \ln x < 1, \\ \frac{1}{e} < x < e. \end{aligned}$$

Hence, the interval of convergence is $\left(\frac{1}{e}, e\right)$. At the extremities of the interval it is impossible to investigate the series by means of Root Test. But we can do it separately by investigation of numerical series obtained in the points $x = e$ and $x = \frac{1}{e}$.

For $x = e$ we have

$$\sum_{n=1}^{\infty} \frac{\ln^n e}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Since it is hyperharmonic series and $p = 3 > 1$, it converges.

For $x = \frac{1}{e}$ we get

$$\sum_{n=1}^{\infty} \frac{\ln^n \frac{1}{e}}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.$$

It is absolutely convergent alternating series, since the series made up of the absolute values of its terms $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Finally, the domain of convergence of $\sum_{n=1}^{\infty} \frac{\ln^n x}{n^3}$ is

$$x \in \left[\frac{1}{e}, e\right].$$

Definition. The sum of the functional series in the domain of convergence is some function of x : $S(x)$.

Denote by $S_n(x)$ the n th partial sum of the series (2.1). If the series converges then its sum can be written in a form

$$S(x) = S_n(x) + r_n(x),$$

where $r_n(x)$ is called the remainder of the series (2.1) and

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \dots$$

Since

$$\lim_{n \rightarrow \infty} r_n(x) = \lim_{n \rightarrow \infty} (S(x) - S_n(x)) = 0,$$

we get, that the remainder of the convergent series approaches to zero as n tends to infinity.

2.2 Uniform Convergence of a Functional Series

Definition. The functional series

$$\sum_{n=1}^{\infty} u_n(x)$$

is called *dominated* in some range of x if there exists a convergent numerical positive-term series

$$\sum_{n=1}^{\infty} \alpha_n$$

such that for all values of x from this range the following relations are fulfilled:

$$|u_1(x)| \leq \alpha_1, \quad |u_2(x)| \leq \alpha_2, \dots, \quad |u_n(x)| \leq \alpha_n, \dots$$

In other words, the series is *dominated* if each of its terms does not exceed, in absolute value, the corresponding term of some convergent numerical series with positive terms.

It follows from the definition that a series dominated in some range converges absolutely at all points of this range.

Definition. The functional series $\sum_{n=1}^{\infty} u_n(x)$ is called a *uniformly convergent* on the interval $[a, b]$ if for any arbitrary small $\varepsilon > 0$ there is an integer N such that for all $n \geq N$ the inequality

$$|S(x) - S_n(x)| < \varepsilon$$

will be fulfilled for any x of the interval $[a, b]$.

Properties of uniformly convergent series

I. A dominated series is a series that uniformly converges.

II. The sum of a series of continuous functions $\sum_{n=1}^{\infty} u_n(x)$ dominated on some interval $[a, b]$ is a function continuous on that interval.

Note. This property holds true only for dominated series. Some nondominated functional series with continuous terms have for the sum a discontinuous function.

III. Let there be a series of continuous functions $\sum_{n=1}^{\infty} u_n(x)$ dominated on some interval $[a, b]$ and let $S(x)$ be the sum of the series. Then the integral of $S(x)$

from α to x , which limits belong to the interval $[a, b]$, is equal to the sum of integrals of the terms of the given series:

$$\int_{\alpha}^x S(t) dt = \int_{\alpha}^x \sum_{n=1}^{\infty} u_n(t) dt = \int_{\alpha}^x u_1(t) dt + \int_{\alpha}^x u_2(t) dt + \cdots + \int_{\alpha}^x u_n(t) dt + \cdots$$

Note. If a series is not dominated, term-by-term integration of it is not always possible.

IV. If the series $\sum_{n=1}^{\infty} u_n(x)$ made up of the functions having continuous derivatives on the interval $[a, b]$ converges to the sum $S(x)$ and the series

$$u'_1(x) + u'_2(x) + \cdots + u'_n(x) + \cdots$$

made up of the derivatives of its terms is dominated on the same interval, then the sum of the series of derivatives is equal to the derivative of the sum of the original series; that is,

$$S'(x) = \left(\sum_{n=1}^{\infty} u_n(x) \right)' = u'_1(x) + u'_2(x) + \cdots + u'_n(x) + \cdots$$

Note. The requirement of dominance of a series of derivatives is extremely essential, and if not fulfilled it can make term-by-term differentiation of the series impossible

The proofs of these properties are beyond of our course, but reader could find them in [1].

3. Power Series

3.1 The Concept of a Power Series

I. Let us consider the most useful case of a functional series – power series.

Definition. A power series about a (or just power series) is any functional series that can be written in the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \quad (3.1)$$

where $a, a_0, a_1, a_2, \dots, a_n$ are real numbers and a_n are called *coefficients of the series*.

When $a = 0$, we have special case of power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad (3.2)$$

Substituting $y = x - a$, we obtain the series of type (3.2) from the series of type (3.1)

$$\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} a_n y^n.$$

We are going to look through the properties of the power series for the type (3.2). Due to the substitution, all the properties are retained completely for a series of the type (3.1).

II. Interval of Convergence.

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

The question of convergence is very important for power series. The domain of convergence is always some interval, which can be degenerate into a point or spread over the real axis (in particular cases).

Theorem 3.1 (Abel's Theorem).

1. If a power series converges for a nonzero value x_1 , then it converges absolutely whenever $|x| < |x_1|$.

2. If a power series diverges for a nonzero value x_2 , then it diverges whenever $|x| > |x_2|$.

Proof. 1). Since the numerical series

$$\sum_{n=0}^{\infty} a_n x_1^n = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n + \cdots$$

converges, it follows that $a_n x_1^n \rightarrow 0$, $n \rightarrow \infty$. Thus, there exists a number $M > 0$ such that $\forall n \geq 0$, $|a_n x_1^n| < M$.

Rewrite the common term of the series (3.2)

$$|a_n x^n| = |a_n x_1^n| \cdot \left| \frac{x}{x_1} \right|^n < M \cdot \left| \frac{x}{x_1} \right|^n.$$

For $|x| < |x_1|$ the series $\sum_{n=0}^{\infty} M \cdot \left| \frac{x}{x_1} \right|^n$ is convergent as a geometric progression with common ratio $\left| \frac{x}{x_1} \right| < 1$. According to the Comparison Test, we have, the series $\sum_{n=0}^{\infty} |a_n x^n|$ converges and the series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

2). Let the series (3.2) is divergent at some point x_2 . Assume, that at the point x_3 , $|x_3| > |x_2|$, the series $\sum_{n=0}^{\infty} a_n x_3^n$ converges. By virtue of the first part (just proved), it should be convergent at the point x_2 as well, since $|x_2| < |x_3|$. But this contradicts the condition that at the point x_2 the series diverges. Hence, the series diverges at the point x as well.

From Abel's Theorem we have the structure of the domain of convergence (Fig. 3).

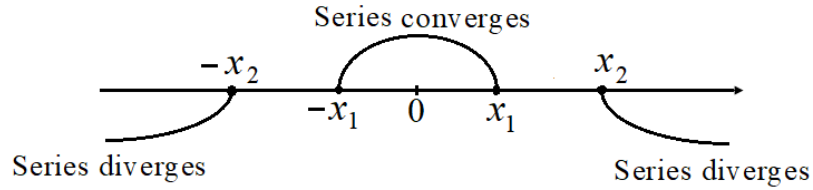


Figure 3.

Thus, the domain of convergence of a power series is an interval with center at the coordinate origin. It is obvious, that there exists number R , $|x_1| < R < |x_2|$, such that for any point x lying inside this interval, the series converges and converges absolutely, while for points lying outside it, the series diverges (Fig. 4).

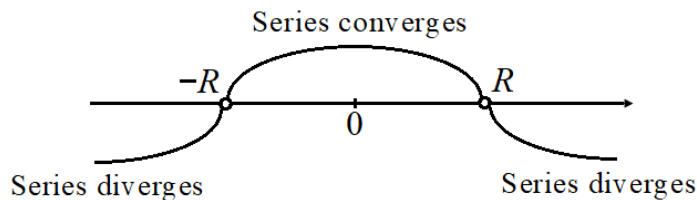


Figure 4.

Definition. The number R is called the *radius of convergence* of the power series. The interval $(-R, R)$ is called the *interval of convergence*.

At the end points of the interval convergence and divergence of a given series should be investigated separately. To determine the convergence of these positive-term series we apply Ratio or Root Tests.

Example. Determine the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{2^{n+2}} = \frac{1}{2^2} + \frac{x}{2^3} + \frac{x^2}{2^4} + \cdots + \frac{x^n}{2^{n+2}} + \cdots$$

Applying Root Test directly, we obtain

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{x^n}{2^{n+2}} \right|} = \frac{|x|}{2}.$$

Hence, the series converges when $\frac{|x|}{2} < 1$, that is, if $|x| < 2$. The series diverges when $|x| > 2$. The radius of convergence is $R = 2$.

If $x = 2$, then $\sum_{n=0}^{\infty} \frac{2^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{1}{4}$ and it diverges since $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{4} = \frac{1}{4} \neq 0$.

If $x = -2$, then $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4}$ and it diverges since $\lim_{n \rightarrow \infty} \frac{1}{4} = \frac{1}{4} \neq 0$.

Finally, the domain of convergence is $(-2, 2)$.

For the power series about a

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

we have that it is converges for values of x that satisfy the inequality (Fig. 5)

$$|x-a| < R,$$

$$-R < x-a < R,$$

$$a-R < x < a+R.$$

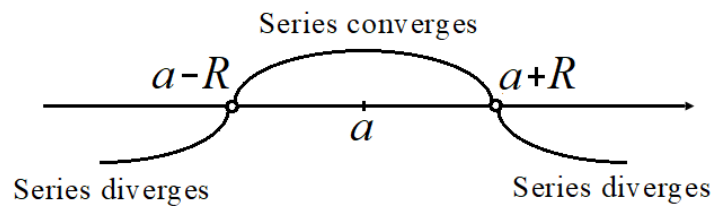


Figure 5.

Example. Determine the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2} = (x-1) + \frac{(x-1)^2}{4} + \frac{(x-1)^3}{9} + \cdots + \frac{(x-1)^n}{n^2} + \cdots$$

Applying Ratio Test, we obtain

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{n+1}}{(n+1)^2}}{\frac{(x-1)^n}{n^2}} \right| = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} |x-1| = |x-1|.$$

The series is convergent when

$$\begin{aligned} |x-1| &< 1, \\ -1 &< x-1 < 1, \\ 0 &< x < 2. \end{aligned}$$

If $x = 2$, then

$$\sum_{n=0}^{\infty} \frac{(2-1)^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

and it converges as hyperharmonic series.

If $x = 0$, then

$$\sum_{n=0}^{\infty} \frac{(0-1)^n}{n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

and it is absolutely convergent diverges since $\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=0}^{\infty} \frac{1}{n^2}$ is convergent.

Therefore, the domain of convergence is $[0, 2]$.

III. Properties of Power Series.

Theorem 3.2

The power series $\sum_{n=0}^{\infty} a_n x^n$ is dominated on any interval $[-r, r]$ that lies completely inside the interval of convergence ($[-r, r] \subset (-R, R)$).

Proof. Since $r < R$, the number positive-term series

$$\sum_{n=0}^{\infty} |a_n| r^n = |a_0| + |a_1| r + |a_2| r^2 + \cdots + |a_n| r^n$$

converges. But when $|x| < r$, the term of the series $\sum_{n=0}^{\infty} a_n x^n$ do not exceed, in absolute value, the corresponding terms of series $\sum_{n=0}^{\infty} |a_n| r^n$. Therefore, the series $\sum_{n=0}^{\infty} a_n x^n$ is dominated on the interval $[-r, r]$.

Theorem 3.3

The sum of a power series $\sum_{n=0}^{\infty} a_n x^n$ is a continuous function on every interval lying entirely within the interval of convergence.

Proof. Since the series on this interval is dominated and its terms are continuous functions of x , according to the Property II. of the dominated functional series, we have, that the sum of the series $\sum_{n=0}^{\infty} a_n x^n$ is a continuous function.

Theorem 3.4

If the limits of integration α and β lie within the interval of convergence of a power series ($[\alpha, \beta] \subset (-R, R)$), then the integral of the sum of the series is equal to the sum of the integrals of the terms of the series:

$$\begin{aligned} \int_{\alpha}^{\beta} S(x) dx &= \int_{\alpha}^{\beta} \sum_{n=0}^{\infty} a_n x^n dx = \\ &= \int_{\alpha}^{\beta} a_0 dx + \int_{\alpha}^{\beta} a_1 x dx + \int_{\alpha}^{\beta} a_2 x^2 dx + \cdots + \int_{\alpha}^{\beta} a_n x^n dx + \cdots \end{aligned} \quad (3.3)$$

Proof. Since limits of integration may be taken in the interval the series on this interval $[-r, r]$, where series is dominated, and according to the Property III. of the dominated functional series, we can fulfill the term-by-term integration.

Theorem 3.5

Inside the interval of convergence $(-R, R)$ of the power series $\sum_{n=0}^{\infty} a_n x^n$ (3.2) the derivative of the sum is equal to the sum of the series obtained by termwise differentiation of the original series, that is for $x \in (-R, R)$

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} (a_n x^n)' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots \quad (3.4)$$

Proof. First, let us prove that the series

$$\sum_{n=0}^{\infty} (a_n x^n)' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots$$

is dominated on any interval $[-r, r] \subset (-R, R)$.

Consider the point r such that $r < \rho < R$. Since the series (3.2) converges at this point, we have $\lim_{n \rightarrow \infty} a_n \rho^n = 0$ and there exist positive number M such that for any $n \in \mathbb{N}$

$$|a_n \rho^n| < M.$$

Then for $x \in [-r, r]$

$$|na_n x^{n-1}| \leq |na_n r^{n-1}| = n|a_n \rho^{n-1}| \left| \frac{r}{\rho} \right|^{n-1} < nMq^{n-1},$$

where $q = \frac{r}{\rho} < 1$.

Now investigate for convergence the series

$$\frac{M}{\rho} (1 + 2q + 3q^2 + \cdots + nq^{n-1} + \cdots).$$

Let us apply the Root Test

$$\lim_{n \rightarrow \infty} \sqrt[n]{nq^{n-1}} = q < 1.$$

Thus, the terms of the series $\sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} na_n x^{n-1}$, for $x \in [-r, r]$, are less, in absolute value, than the terms of the convergent numerical positive-term series $\frac{M}{\rho} \sum_{n=1}^{\infty} nq^{n-1}$. Therefore, the series (3.4) is dominated on the interval $[-r, r]$, and, since every interior point of the interval $(-R, R)$ may be included in some interval $[-r, r]$, it follows that the series (3.4) converges for any $x \in (-R, R)$.

According to the Property IV. of the dominated functional series, we have, that

$$S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)' = \sum_{n=0}^{\infty} (a_n x^n)' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots$$

Note. Series (3.4) may be differentiated again term by term, and this may be continued as many times as you need. The interval of convergence of each series obtained by differentiation is the same interval $(-R, R)$.

3.2 Taylor's and Maclaurin's Series

I. Power Series Representation for Function

Since the convergent power series has a sum, we say, that a power series determines a function $f(x)$ whose domain is the interval of converges of the series. That is, for any $x \in (-R, R)$

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

If the function $f(x)$ is defined in this way, then $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series representation for $f(x)$ (f is represented by the power series)

Power series representation for functions is useful in advanced mathematics and applications, numerical computation using power series provide the basis for the construction of mathematical tables, the same ideas are used in calculators.

II. Taylor's and Maclaurin's Series

Consider the function $f(x)$ has derivatives of all orders in the neighborhood of the point $x = a$. Assume that the function has a power series representation about $x = a$

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \quad (3.5)$$

Obviously, we have to determine the coefficients of the series representation.

First, according to the equality (3.5), we have for $x = a$

$$f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + \cdots + a_n(a-a)^n + \cdots = a_0.$$

Thus,

$$a_0 = f(a).$$

Second, let us find first derivative of the function and its power series:

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots,$$

and put $x = a$ into the equality obtained

$$f'(a) = a_1 + 2a_2(a-a) + \cdots + na_n(a-a)^{n-1} + \cdots = a_1.$$

Hence,

$$f'(a) = a_1.$$

Continue applying differentiation and find the second derivative

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + \cdots + n(n-1)a_n(x-a)^{n-2} + \cdots$$

Then for $x = a$

$$f''(a) = 2a_2$$

and

$$a_2 = \frac{f''(a)}{2} = \frac{f''(a)}{2!}.$$

Using the third derivative gives

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x - a) + \dots + n(n - 1)(n - 2)a_n(x - a)^{n-3} + \dots$$

and for $x = a$

$$a_3 = \frac{f'''(a)}{3 \cdot 2} = \frac{f'''(a)}{3!}.$$

From the fourth derivative as $x = a$

$$a_4 = \frac{f^{IV}(a)}{4 \cdot 3 \cdot 2} = \frac{f^{IV}(a)}{4!}.$$

Now we are ready to write the formula for general term. Since

$$f^n(x) = n(n - 1)(n - 2) \cdot \dots \cdot 2a_n + (n + 1)(n - 1) \cdot \dots \cdot 2a_{n-1}(x - a) + \dots$$

we obtain

$$a_n = \frac{f^n(a)}{n!}.$$

Therefore, the power series representation for the function $f(x)$ about $x = a$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n = \\ &= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^n(a)}{n!} (x - a)^n + \dots \end{aligned} \quad (3.6)$$

It is called the *Taylor Series for $f(x)$ about $x = a$* .

Let us rewrite equality (3.6) as

$$f(x) = P_n(x) + R_n(x), \quad (3.7)$$

where $P_n(x)$ is the *n th degree Taylor polynomial for $f(x)$*

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^n(a)}{n!} (x - a)^n$$

and $R_n(x)$ is the *remainder term* (from Single Variable Calculus)

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - a)^{n+1}, \quad \xi \in (a, x). \quad (3.8)$$

Since $P_n(x)$ is the n th partial sum of the series (3.6), its limit as $n \rightarrow \infty$ is equal to the sum of the series. That is

$$f(x) = \lim_{n \rightarrow \infty} P_n(x).$$

Consider equality (3.7) as $n \rightarrow \infty$, then

$$f(x) = \lim_{n \rightarrow \infty} P_n(x) + \lim_{n \rightarrow \infty} R_n(x) = f(x) + \lim_{n \rightarrow \infty} R_n(x).$$

Therefore,

$$\lim_{n \rightarrow \infty} R_n(x) = 0.$$

From the foregoing it follows that *the Taylor series (3.6) is a given function $f(x)$ only when $\lim_{n \rightarrow \infty} R_n(x) = 0$.*

If $\lim_{n \rightarrow \infty} R_n(x) \neq 0$, then the series is not the given function, although it may converge to a different function.

Since formula (3.8) is quite complicated to prove that the reminder term approaches to zero, it is better to use another sufficient condition.

Theorem 3.6

If the function $f(x)$ has derivatives of all orders in the neighborhood of the point $x = a$ and there exists such number M that for any x from this neighborhood

$$|f^{(n)}(x)| \leq M,$$

then the Taylor series (3.6) converges to the function $f(x)$.

Proof. Let us consider the reminder term of (3.6) given by (3.8)

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}, \quad \xi \in (a, x).$$

Hence,

$$|R_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x-a|^{n+1} \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

Since $\lim_{n \rightarrow \infty} \frac{|x-a|^{n+1}}{(n+1)!} = 0$, we obtain

$$\lim_{n \rightarrow \infty} R_n(x) = 0.$$

Thus, the Taylor series (3.6) converges to the function $f(x)$.

If $a = 0$, then the Taylor Series for $f(x)$ about $x = 0$ is called the *Maclaurin Series* for $f(x)$ and

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n + \dots \end{aligned} \quad (3.9)$$

The Maclaurin series (3.7) is a given function $f(x)$ only when $\lim_{n \rightarrow \infty} R_n(x) = 0$, where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}, \quad \xi \in (0, x).$$

III. Examples of Expansion of Functions in Maclaurin Series

1. Expansion of the function $f(x) = e^x$.

The function $f(x) = e^x$ has a derivative of any order

$$f^{(n)}(x) = e^x.$$

Since the values of the function and its first n derivatives at the point $x = 0$

$$f(0) = f'(0) = \dots = f^{(n)}(0) = \dots = 1,$$

we obtain

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots, \quad (3.10)$$

Let us prove that this series presentation is true for every real number x .

Consider the neighborhood $(-h, h)$ of the point $x = 0$. Here for any x from $(-h, h)$

$$|f^{(n)}(x)| = |e^x| \leq e^h = M.$$

Hence, $\lim_{n \rightarrow \infty} R_n(x) = 0$.

2. Expansion of the function $f(x) = \sin x$.

Let us find the successive derivatives of $f(x) = \sin x$.

$$f(x) = \sin x, \quad f(0) = 0;$$

$$f'(x) = \cos x = \sin\left(x + \frac{\pi}{2}\right), \quad f'(0) = 1;$$

$$f''(x) = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right), \quad f''(0) = 0;$$

$$f'''(x) = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right), \quad f'''(0) = -1;$$

$$f^{IV}(x) = \sin x = \sin\left(x + 4\frac{\pi}{2}\right), \quad f^{IV}(0) = 0;$$

$$\dots\dots\dots$$

$$f^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right), \quad f^{(n)}(0) = \sin n\frac{\pi}{2};$$

$$\dots\dots\dots$$

Substituting the values into (3.9) we get an expansion:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \quad (3.11)$$

Since $|f^{(n)}(x)| = \left|\sin\left(x + n\frac{\pi}{2}\right)\right| \leq 1$, we have $\lim_{n \rightarrow \infty} R_n(x) = 0$.

3. Expansion of the function $f(x) = \cos x$.

Finding the values of the derivatives for $x = 0$ of the function $f(x) = \cos x$ and substituting them into (3.9), we obtain the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \quad (3.12)$$

Since $|f^{(n)}(x)| = \left|\cos\left(x + n\frac{\pi}{2}\right)\right| \leq 1$, we have $\lim_{n \rightarrow \infty} R_n(x) = 0$.

4. Expansion of the function $f(x) = (1+x)^m$ (the Binomial Series).

Consider the function

$$f(x) = (1+x)^m,$$

where m is an arbitrary constant number.

Since $f'(x) = m(1+x)^{m-1} = \frac{m(1+x)^m}{(1+x)}$, the function $f(x)$ satisfies the differential equation

$$(1+x)f'(x) = mf(x) \quad (3.13)$$

and the condition

$$f(0) = 1. \quad (3.14)$$

Let us find a power series

$$f(x) = 1 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

that satisfies the equation (3.13) and condition (3.14).

Putting this series into (3.13), we obtain

$$(1+x)(a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots) = \\ = m(1 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots)$$

Collecting the coefficients of identical powers of x in different parts of the equation, we find

$$a_1 = m, \quad a_1 + 2a_2 = ma_1, \quad \cdots \quad na_n + (n+1)a_{n-1} = ma_n, \cdots$$

Therefore, we get

$$\begin{aligned} a_0 &= 1, \\ a_1 &= m, \\ a_2 &= \frac{(m-1)a_1}{2} =, \\ a_3 &= \frac{(m-2)a_2}{3} = \frac{(m-2)(m-1)a_1}{2 \cdot 3}, \\ &\cdots \\ a_n &= \frac{m(m-1) \cdots (m-n+1)}{2 \cdot 3 \cdots n}. \end{aligned}$$

These are *binomial coefficients*.

Hence, we obtain

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1) \cdots (m-n+1)}{n!}x^n + \cdots \quad (3.15)$$

If m is a positive integer, then $a_{m+1} = a_{m+2} = \cdots = 0$ and the series converted into a polynomial.

For m fractional or negative integer, we have an infinite series.

Let us determine the interval of convergence of series (3.15) by applying the Ratio Test:

$$u_{n+1} = \frac{m(m-1) \cdots (m-n+1)}{n!}x^n, \quad u_n = \frac{m(m-1) \cdots (m-n+2)}{(n-1)!}x^{n-1}$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{m(m-1) \cdots (m-n+1)(n-1)!}{m(m-1) \cdots (m-n+2)n!}x \right| = \lim_{n \rightarrow \infty} \left| \frac{m-n+1}{n} \right| |x| = |x|$$

Therefore, series (3.15) converges for $x \in (-1, 1)$.

At the end points $x = -1$ and $x = 1$ the question of convergence or divergence of a series (3.15) is decided separately for each specific series.

Finally,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots, \\ x \in (-1,1). \quad (3.16)$$

Let us look through some particular cases.

1. For $m = -1$, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots, \quad x \in (-1,1). \quad (3.17)$$

Changing x by $-x$, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + \dots, \quad x \in (-1,1). \quad (3.18)$$

2. For $m = \frac{1}{2}$, we have

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \dots + (-1)^{n-1} \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \dots \cdot 2n}x^n + \dots, \\ x \in [-1,1]. \quad (3.19)$$

3. For $m = -\frac{1}{2}$, we have

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1}{2 \cdot 4}x^2 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 + \dots + (-1)^n \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \dots \cdot 2n}x^n + \dots, \\ x \in (-1,1]. \quad (3.20)$$

The binomial expansion is very useful for obtaining the expansion of other functions.

5. Expansion of the function $f(x) = \ln(1+x)$.

Since $\int_0^x \frac{1}{1+x} dx = \ln(1+x)|_0^x = \ln(1+x)$, let us apply Theorem 3.4 about integration of power series and expansion (3.16).

Hence,

$$\ln(1+x) = \int_0^x \frac{1}{1+x} dx = \int_0^x (1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots) dx = \\ = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots, \quad x \in (-1,1).$$

Series is divergent for $|x| > 1$ and $x = -1$, but as $x = 1$ we get

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

Therefore,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots, \quad x \in (-1, 1]. \quad (3.21)$$

6. Expansion of the function $f(x) = \arctan x$.

Here we are going to use the same idea as for logarithmic function.

Since $\int_0^x \frac{1}{1+x^2} dx = \arctan x|_0^x = \arctan x$, let us find the expansion for $\frac{1}{1+x^2}$ using formula (3.17). Changing x by x^2 , we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots, \quad x \in (-1, 1).$$

Applying Theorem 3.4

$$\begin{aligned} \arctan x &= \int_0^x \frac{1}{1+x^2} dx = \int_0^x (1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots) dx = \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \quad x \in (-1, 1). \end{aligned}$$

Series is divergent for $|x| > 1$ and $x = -1$, but for $x = 1$ and $x = -1$ we get

$$\begin{aligned} \arctan 1 &= \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^n}{2n+1} + \cdots, \\ \arctan(-1) &= -\frac{\pi}{4} = -1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots \end{aligned}$$

Thus,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \quad x \in [-1, 1]. \quad (3.22)$$

7. Expansion of the function $f(x) = \arcsin x$.

Similarly, we get the expansion

$$\begin{aligned} \arcsin x &= x + \frac{x^3}{2 \cdot 3} + \frac{3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{3 \cdot 5 \cdot \cdots (2n-1)x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2n+1)} + \cdots, \\ &\quad x \in [-1, 1]. \quad (3.23) \end{aligned}$$

3.3 Application of Power Series.

I. Approximation of Values of Functions

We can use Maclaurin or Taylor series to approximate values of functions at the specific point x . Let us illustrate this application by examples.

Examples.

1. Approximate the number e with an error less than 10^{-3} .

Let us use the expansion (3.10) for $x = 1$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

Here we should choose appropriate n to approximate the function e^x within the desired accuracy.

Since for any $x \in [-1, 1]$: $|e^x| \leq 3$, we can estimate the residual term as follows

$$|R_n(x)| = \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \leq \frac{3}{(n+1)!}.$$

If we need the error to be less than 10^{-3} , it is enough to solve inequality

$$\frac{3}{(n+1)!} \leq 10^{-3},$$

Solving the inequality by trying various n , we obtain, that for $n = 6$

$$\frac{3}{(n+1)!} = \frac{3}{7!} \approx 0.0006 < 10^{-3} = 0.001.$$

Thus, we have

$$\begin{aligned} e &= 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \approx \\ &\approx 2 + 0.5 + 0.167 + 0.042 + 0.008 + 0.001 = 2.718. \end{aligned}$$

2. Use the first two nonzero terms of the Maclaurin series to evaluate $\sin 0.1$ approximately and estimate the error in this approximation.

Consider the Maclaurin series (3.11) for $\sin x$ as $x = 0.1$

$$\sin 0.1 = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \cdots = 0.1 - \frac{0.001}{3!} + \frac{0.00001}{5!} - \cdots.$$

Since the series obtained is alternating, according to Corollary to Alternating Series Test the error does not exceed in absolute value the third term

$$\frac{0.00001}{5!} = \frac{0.00001}{120} \approx 0.00000008.$$

Hence, to six decimal places

$$\sin 0.1 = 0.1 - \frac{0.001}{3!} \approx 0.099833$$

II. Evaluating Indeterminate Forms

Sometimes we can evaluate indeterminate forms by expressing the functions involved as Taylor series.

Examples.

1. Evaluate

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{\arctan x}.$$

We represent $\ln(1+x)$ and $\arctan x$ as a corresponding Maclaurin series

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} - x + \ln(1+x)}{x - \arctan x} &= \left\{ \frac{0}{0} \right\} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} - x + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right)}{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \right)} = \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} - \frac{x^4}{4} + \dots}{\frac{x^3}{3} - \frac{x^5}{5} + \dots} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3} \left(1 - \frac{3x}{4} + \dots \right)}{\frac{x^3}{3} \left(1 - \frac{3x^2}{5} + \dots \right)} = 1. \end{aligned}$$

2. Evaluate

$$\lim_{x \rightarrow 1} \frac{2 - 2e^{x-1}}{\sin(x-1)}.$$

Here we are going to use a Taylor series representation in powers $(x-1)$ for e^{x-1} and $\sin(x-1)$. This can be obtained by replacing x by $(x-1)$ in the series (3.10) and (3.11).

Thus,

$$\begin{aligned} e^{x-1} &= 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots, \\ \sin(x-1) &= (x-1) - \frac{(x-1)^3}{3!} + \frac{(x-1)^5}{5!} - \dots. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \lim_{x \rightarrow 1} \frac{2 - 2 \left(1 + (x-1) + \frac{(x-1)^2}{2!} + \dots \right)}{(x-1) - \frac{(x-1)^3}{3!} + \dots} = \\
& = \lim_{x \rightarrow 1} \frac{-2(x-1) - 2 \frac{(x-1)^2}{2!} + \dots}{(x-1) - \frac{(x-1)^3}{3!} + \dots} = \lim_{x \rightarrow 1} \frac{(x-1)(-2 - (x-1) + \dots)}{(x-1) \left(1 - \frac{(x-1)^2}{3!} + \dots \right)} = -2.
\end{aligned}$$

III. Evaluating Nonelementary Integrals

The approximation can then be used for integration of the functions. Of course, we are going to use this technique with functions for which the traditional calculus methods do not work. For example, we may need to compute $\int \frac{\sin x}{x} dx$. Since a primitive for $\frac{\sin x}{x}$ cannot be expressed in terms of elementary functions, this integral cannot be done by the classical integration techniques.

Examples.

1. Express $\int \frac{\sin x}{x} dx$ as a power series

Expand the integrand in a series using (3.11)

$$\begin{aligned}
\frac{\sin x}{x} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots}{x} = \\
&= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \dots.
\end{aligned}$$

This series is convergent for all values of x . Integrating termwise, we get

$$\begin{aligned}
\int \frac{\sin x}{x} dx &= \int 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \dots dx = \\
&= x - \frac{x^3}{3! \cdot 3} + \frac{x^5}{5! \cdot 5} - \frac{x^7}{7! \cdot 7} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)! (2n+1)} + \dots.
\end{aligned}$$

2. Evaluate the integral $\int_0^1 \frac{\sin x}{x} dx$ with an error of less than 10^{-3} .

Let us use the result of previous example

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3! \cdot 3} + \frac{1}{5! \cdot 5} - \frac{1}{7! \cdot 7} + \frac{1}{9! \cdot 9} - \frac{1}{11! \cdot 11} + \dots.$$

The series on the right side is alternating, and we find by numerical evaluation that

$$\frac{1}{7!7} \approx 0.0002$$

is the first term to be numerically less than $10^{-3} = 0.001$. Therefore? for evaluating this integral with an error of less than 10^{-3} we should calculate first three terms of the series

$$\int_0^1 \frac{\sin x}{x} dx \approx 1 - \frac{1}{3!3} + \frac{1}{5!5} \approx 0.946.$$

IV. Series Solutions to Differential Equations

There are a lot of differential equations that can't be solved explicitly in terms of finite combinations of elementary functions. However, it is important to be able to solve such equations because they arise from the problems of physics or mechanics.

Let us consider the method of power series solution to Cauchy problem for differential equation. There are two main cases of applying this method.

First Case.

We represent the solution as a Taylor series, the sum of finite number of terms of this series will be approximately equal to the desired particular solution.

Consider, for example, the Cauchy problem for second order differential equation

$$y'' = F(x, y, y'), \quad (3.24)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1. \quad (3.25)$$

Assume that the solution $y = y(x)$ exists and may be represented in the form of a Taylor series

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \dots \quad (3.26)$$

Since we know $y(x_0)$ and $y'(x_0)$ from (3.25), we have to find the values of $y''(x_0)$, $y'''(x_0)$, ... $y^n(x_0)$, ... We can do it by means of equation (3.24) and conditions (3.25).

From the equation (3.24) we obtain

$$y''(x_0) = F(x_0, y(x_0), y'(x_0)).$$

Differentiate both sides of (3.24) with respect to x

$$y''' = (F(x, y, y'))' = F'_y x(x, y, y') + F'_y(x, y, y')y' + F'_{y'}(x, y, y')y''. \quad (3.37)$$

Substituting x_0 , we get $y'''(x_0)$.

Differentiating (3.37) once again, we obtain $y^{IV}(x_0)$ and so on.

Then, we put these values of derivatives into (3.26). This series represents the solution of the Cauchy problem (3.24), (3.25) in the interval of convergence.

Note. The same idea could be applied to the equations of any order.

Example 1. Find the first four nonzero terms of the series representation for the solution of the Cauchy problem

$$y' = xy^2 - 1, \quad y(1) = 1.$$

Let us consider a Taylor series representation

$$y(x) = y(1) + y'(1)(x - 1) + \frac{y''(1)}{2!}(x - 1)^2 + \frac{y'''(1)}{3!}(x - 1)^3 + \dots$$

Here we have from the condition

$$y(1) = 1$$

and from the equation

$$y'(1) = 1 \cdot (y(1))^2 - 1 = 0.$$

Differentiating the equation, we get

$$y'' = (xy^2 + 1)'_x = y^2 + 2xyy'.$$

Then,

$$y''(1) = (y(1))^2 + 2y(1)y'(1) = 1.$$

Differentiate again

$$y''' = (y^2 + 2xyy')'_x = 4yy' + 2xy'^2 + 2xyy''$$

and calculate at $x = 1$

$$y'''(1) = 4y(1)y'(1) + 2(y'(1))^2 + 2y(1)y''(1) = 2.$$

Next,

$$y^{IV} = 6y'^2 + 6yy'' + 6xy'y'' + 2xyy'''$$

and at $x = 1$

$$y^{IV}(1) = 10.$$

Finally, putting these values back into the solution we obtain

$$\begin{aligned} y(x) &= 1 + \frac{1}{2!}(x - 1)^2 + \frac{2}{3!}(x - 1)^3 + \frac{10}{4!}(x - 1)^4 + \dots = \\ &= 1 + \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} + \frac{5(x - 1)^4}{12} + \dots \end{aligned}$$

Second Case.

We suppose that there exists a power series solution of the form

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n + \cdots$$

and the task is to determine the coefficients $a_0, a_1, a_2, \dots$. This task is accomplished by differentiating the power series term by term, substituting into the differential equation and equating coefficients of the same powers of x .

Example 2. Find the series representation for the solution of the Cauchy problem

$$y'' - xy = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

We are looking for a solution in the form,

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

Let us differentiate this series twice

$$\begin{aligned} y'(x) &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots, \\ y''(x) &= 2a_2 + 3 \cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots \end{aligned}$$

From the initial conditions we can find

$$y(0) = a_0 = 0 \text{ and } y'(0) = a_1 = 1.$$

Plugging the series expressions into the equation, we get the identity

$$2a_2 + 3 \cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots = a_1x^2 + a_2x^3 + \cdots + a_{n-3}x^{n-2} + \cdots$$

Collecting coefficients of the same powers of x , we obtain

$$a_2 = 0, \quad a_3 = 0, \quad 4 \cdot 3a_4 = a_1 = 1, \quad \cdots \quad n(n-1)a_n = a_{n-3}.$$

Hence,

$$\begin{aligned} a_4 &= \frac{1}{3 \cdot 4}, & a_5 &= 0, & a_6 &= 0, & a_7 &= \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}, \\ a_8 &= 0, & a_9 &= 0, & a_{10} &= \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} \end{aligned}$$

.....

$$a_{3m-1} = 0, \quad a_{3m} = 0, \quad a_{3m+1} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3m \cdot (3m+1)}.$$

Substituting the coefficients obtained into the series, we get the solution

$$y(x) = x + \frac{1}{3 \cdot 4}x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 + \cdots + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3m \cdot (3m+1)}x^{3m+1} + \cdots$$

This series converges for all values of x .

Example 3. Find the series solution of the equation

$$y'' + y = 0.$$

Assume, that the solution is in the form,

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

Differentiate this series termwise

$$y''(x) = 2a_2 + 3 \cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}.$$

and plug into the equation

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=0}^{\infty} a_nx^n = 0.$$

Let us combine everything into a single series. To do this we rewrite the first series as follows

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

It helps us to collect the coefficients of the same powers of x easier.

Therefore,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_nx^n = 0.$$

And

$$\sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} + a_n)x^n = 0,$$

Hence,

$$(n+2)(n+1)a_{n+2} + a_n = 0.$$

As a result, we obtain the *recursion formula* for calculating coefficients

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n = 0, 1, 2, 3, \dots$$

If a_0 and a_1 are known, we can determine the remaining coefficients recursively.

For example,

$$n = 0: a_2 = \frac{-a_0}{1 \cdot 2} = -\frac{a_0}{2!};$$

$$n = 1: a_3 = \frac{-a_1}{2 \cdot 3} = -\frac{a_1}{3!};$$

$$n = 2: a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{a_0}{4!};$$

$$n = 3: a_5 = \frac{-a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{a_1}{5!};$$

$$n = 4: a_6 = \frac{-a_4}{5 \cdot 6} = \frac{-a_0}{4! \cdot 5 \cdot 6} = -\frac{a_0}{6!};$$

$$n = 5: a_7 = \frac{-a_5}{6 \cdot 7} = \frac{-a_1}{5! \cdot 6 \cdot 7} = -\frac{a_1}{7!};$$

.....

$$n = 2k: a_{2k} = (-1)^k \frac{a_0}{(2k)!};$$

$$n = 2k + 1: a_{2k+1} = (-1)^k \frac{a_1}{(2k + 1)!};$$

Putting the coefficients obtained into the series, we get

$$\begin{aligned} y(x) &= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \dots \\ &\quad + (-1)^k \frac{a_0}{(2k)!} x^{2k} + (-1)^k \frac{a_1}{(2k + 1)!} x^{2k+1} + \dots \end{aligned}$$

or

$$\begin{aligned} y(x) &= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 \dots + \frac{(-1)^k}{(2k)!} x^{2k} + \dots \right) + \\ &\quad + a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots + \frac{(-1)^k}{(2k + 1)!} x^{2k+1} + \dots \right) = \\ &= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}. \end{aligned}$$

Do not worry about the fact that we don't know what a_0 and a_1 are. We actually need these to be in the problem to get the correct solution as arbitrary constants.

Since the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ and $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ are well known Maclaurin series for $\cos x$ and $\sin x$, we could write the solution as

$$y(x) = a_0 \cos x + a_1 \sin x.$$

Note. Usually we are not able to express power series solutions of differential equations in terms of known functions.

4. Fourier Series

4.1 The Concept of a Fourier Series

I. Harmonic Oscillations and Periodic Functions

There are many processes in nature and technic that are repeated at regular intervals of time. Such the processes are called periodic. For example mechanical oscillations of a string, electromagnetic vibrations and so on.

Mathematically, we can describe periodic processes by using periodic functions. The periodic function is a function that repeats its values at regular intervals or periods. That is, $f(x)$ is periodic if there exists positive number T such that for any x from the domain of function the equality $f(x + T) = f(x)$ holds.

The simplest example of an oscillations is a harmonic oscillation provided by the function

$$f(x) = a \sin(\omega x + \varphi_0), \quad x \geq 0,$$

where a is an *amplitude* (maximum displacement from the equilibrium position), ω – *angular frequency* and φ_0 – *initial phase*. General period for this function is $T = \frac{2\pi}{\omega}$.

This function is called *the simple harmonic* and can also be written in the form

$$f(x) = a \sin(\omega x + \varphi_0) = A \cos \omega x + B \sin \omega x,$$

Where $a = \sqrt{A^2 + B^2}$, $\tan \varphi_0 = \frac{B}{A}$.

Let us look through the oscillations obtained by adding several simple harmonics. For example, function

$$f(x) = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x$$

is a sum of three simple harmonics $\sin x$, $\frac{1}{3} \sin 3x$ and $\frac{1}{5} \sin 5x$.

The graphs of these functions and their sum are shown on the Figure 6.

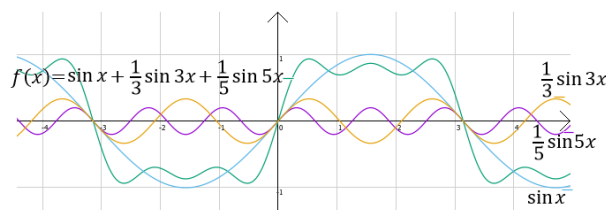


Figure 6.

Let us add a lot more simple harmonics (Fig.7)

$$f(x) = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots + \frac{1}{49} \sin 49x + \frac{1}{51} \sin 51x.$$



Figure 7.

Thus, the sum of simple harmonics give us new much more complicated periodic oscillations.

Naturally, the question arises whether it is possible to determine complicated periodic function as a sum of simple harmonics. Generally, it is impossible if the sum is finite, but infinite sums of simple harmonics (trigonometric series) could represent other periodic functions, even if they are a bit weird.

A functional series of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called a *trigonometric series*. The constants a_0 , a_n and b_n are called *coefficients of the trigonometric series*.

If such a series is convergent, then its sum is periodic function with period 2π , since $\cos nx$ and $\sin nx$ are periodic functions with period 2π .

II. Expansion of 2π -periodic Functions in Fourier Series

Consider the trigonometric series convergent to 2π -periodic function in interval $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (4.1)$$

Suppose that the series is termwise integrable. This is possible if the series (4.1) is convergent absolutely at each point of interval $[-\pi, \pi]$.

Let us evaluate the coefficients of trigonometric series (4.1).

Integrate both sides of equality (4.1) from $-\pi$ to π :

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos nx dx + \int_{-\pi}^{\pi} b_n \sin nx dx \right),$$

Compute separately each integral on the right side:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{a_0}{2} dx &= \pi a_0 \\ \int_{-\pi}^{\pi} a_n \cos nx dx &= a_n \frac{\sin nx}{n} \Big|_{-\pi}^{\pi} = 0 \\ \int_{-\pi}^{\pi} b_n \sin nx dx &= -b_n \frac{\cos nx}{n} \Big|_{-\pi}^{\pi} = 0 \end{aligned}$$

Whence

$$\int_{-\pi}^{\pi} f(x) dx = \pi a_0$$

and

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

To evaluate coefficient a_m , $m \neq 0$, multiply both sides of (4.1) by $\cos mx$ and integrate termwise from $-\pi$ to π :

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx + \\ &+ \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos nx \cos mx dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx dx \right). \end{aligned}$$

Since,

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x \, dx = 0,$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n+m)x + \cos(n-m)x \, dx = 0, \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2mx) \, dx = \frac{x}{2} \Big|_{-\pi}^{\pi} = \pi$$

we obtain

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi a_m$$

and

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx.$$

Multiplying both sides of (4.1) by $\sin mx$, integrating termwise from $-\pi$ to π and remembering, that

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin(n+m)x + \sin(n-m)x \, dx = 0,$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos(n-m)x - \cos(n+m)x \, dx = 0, \quad m \neq n$$

$$\int_{-\pi}^{\pi} \sin^2 mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2mx) \, dx = \frac{x}{2} \Big|_{-\pi}^{\pi} = \pi$$

we find

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_m \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi b_m$$

and

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx.$$

Definition. The trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (4.2)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad (4.3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (4.4)$$

is called a *Fourier series* of the function $f(x)$. The coefficients a_0 , a_n and b_n are called *Fourier coefficients* of the function $f(x)$.

II. Sufficient Conditions for Expandability

To find the Fourier coefficients, it was assumed that trigonometrical series (4.1) is convergent. But what properties should the function $f(x)$ possess for the Fourier series (4.2) to converge and so that the sum of the Fourier series is equal to the values of the function $f(x)$ at the corresponding points?

Before we state the theorem about sufficient conditions for representing a function by Fourier series, let us look through some definitions.

Definition. The function is called piecewise smooth if it can be divided into finite number of distinct parts and on each part both the function and its first derivative are continuous.

A piecewise smooth function may not be continuous everywhere (Fig. 8). However the only discontinuities that are allowed are a finite number of *jump* and *removable* discontinuities of the first kind (for the point $x = c$ there exist the one-sided limits

$$\lim_{x \rightarrow c+} f(x) = f(c+0) \text{ and } \lim_{x \rightarrow c-} f(x) = f(c-0)).$$

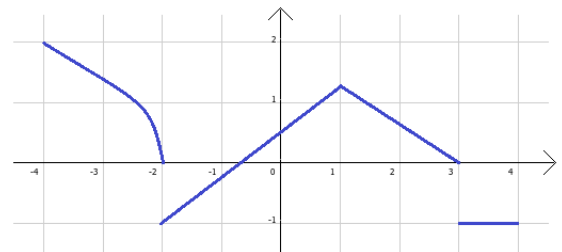


Figure 8.

Theorem 4.1. (Sufficient conditions for representing a function by Fourier series)

If the 2π -periodic function $f(x)$ is a piecewise smooth on the interval $[-\pi, \pi]$, then the Fourier series for $f(x)$ converges at all points.

At the points of its continuity the values of the sum of series $S(x)$ is equal to the values of the function $f(x)$.

At the points of discontinuities of $f(x)$ the sum is the average of the two one-sided limits; that is for the point $x = c$:

$$S(c) = \frac{f(c+0) + f(c-0)}{2}.$$

The proof of this theorem is out of our course.

Since the set of functions that may be represented by Fourier series is broad, the Fourier series have a lot of applications in various sections of mathematics, physics and mechanics.

Example.

Find the Fourier series expansion for the 2π -periodic function

$$f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 3, & 0 < x < \pi. \end{cases}$$

Function $f(x)$ is piecewise smooth and has a jump discontinuity at $x = 0$ (Fig. 9). Hence, it admits expansion in a Fourier series.

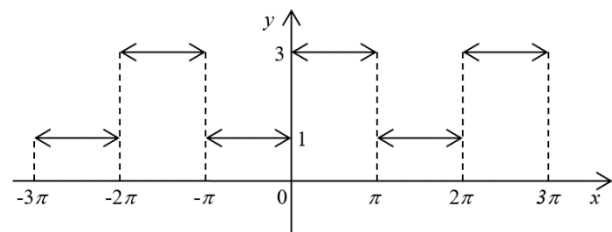


Figure 9.

Let us calculate the coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left(\int_{-\pi}^0 dx + 3 \int_0^{\pi} dx \right) = \frac{1}{\pi} (x|_{-\pi}^0 + 3x|_0^{\pi}) = 4;$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left(\int_{-\pi}^0 \cos nx dx + 3 \int_0^{\pi} \cos nx dx \right) = \\ &= \frac{1}{\pi} \left(\frac{\sin nx}{n} \Big|_{-\pi}^0 + 3 \frac{\sin nx}{n} \Big|_0^{\pi} \right) = 0; \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \left(\int_{-\pi}^0 \sin nx dx + 3 \int_0^{\pi} \sin nx dx \right) = \\
 &= \frac{1}{\pi} \left(-\frac{\cos nx}{n} \Big|_{-\pi}^0 - 3 \frac{\cos nx}{n} \Big|_0^{\pi} \right) = \frac{1}{n\pi} (-1 + \cos n\pi - 3 \cos n\pi + 3) = \\
 &= \frac{2}{n\pi} (1 - \cos n\pi) = \frac{2}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & \text{for } n \text{ even number;} \\ \frac{4}{n\pi}, & \text{for } n \text{ odd number.} \end{cases}
 \end{aligned}$$

Thus, we obtain the series

$$f(x) = 2 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

At the points of discontinuities this series converges to value 2 – the average of the two one-sided limits.

Figure 10 shows how the partial sums S_n (S_3 , S_{11} , S_{29} (*purpur lines*)) of the series more and more accurately represent the function $f(x)$ (*red line*) as $n \rightarrow \infty$.

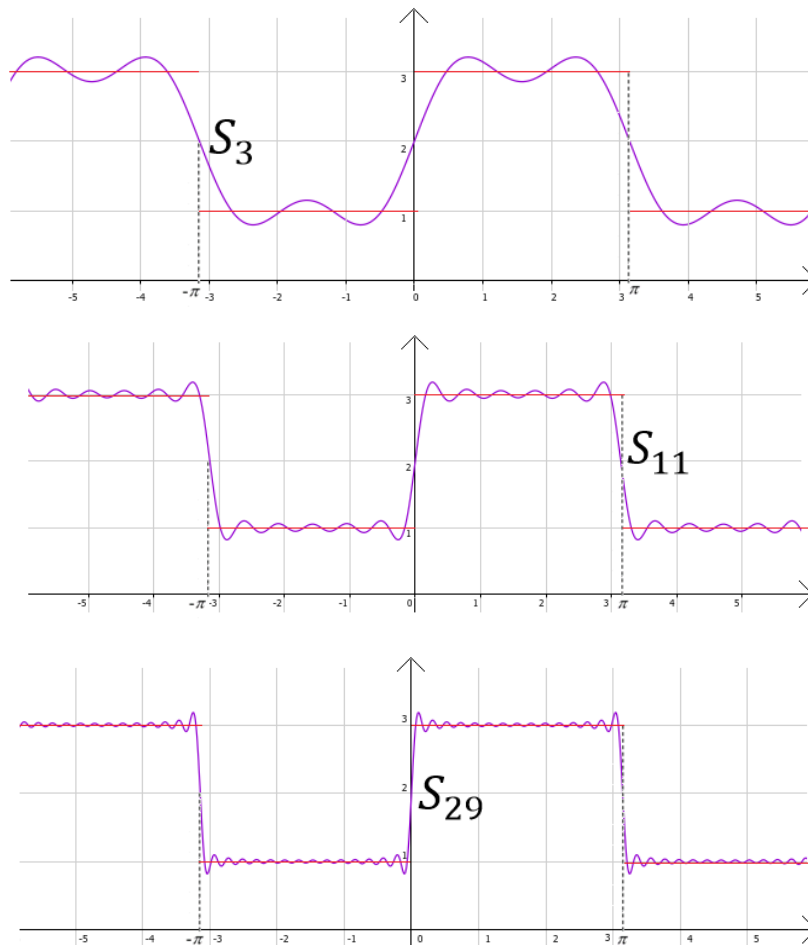


Figure 10.

Let us look through some very important and useful properties of Fourier series.

Theorem 4.2 (for termwise differentiation of a Fourier series)

Let the function $f(x)$ such that its derivative is piecewise smooth and its Fourier series is continuous, then the Fourier series can be termwise differentiated and the result of the differentiation is the Fourier series of the derivative.

Note. The Fourier series is continuous everywhere in the interval $[-\pi, \pi]$ if it have no discontinuities in $[-\pi, \pi]$.

Theorem 4.3 (for termwise integration of Fourier series)

Let the function $f(x)$ is piecewise smooth then the Fourier series of the function can be termwise integrated and the result is a series (may not be the Fourier series for the integral of the function) that converges to the integral of $f(x)$.

4.2 Fourier Series for Even and Odd Functions

I. Expansion of 2π -periodic Even Functions in Fourier Series

Let the function $f(x)$ is integrable in the interval $[-\pi, \pi]$.

If the function is even ($f(-x) = f(x)$), then

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^0 f(x)dx + \int_0^{\pi} f(x)dx = \int_0^{\pi} f(-x)dx + \int_0^{\pi} f(x)dx = \\ &= \int_0^{\pi} f(x)dx + \int_0^{\pi} f(x)dx = 2 \int_0^{\pi} f(x)dx.\end{aligned}$$

Similarly, if the function is odd ($f(-x) = -f(x)$), then

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)dx &= \int_{-\pi}^0 f(x)dx + \int_0^{\pi} f(x)dx = \int_0^{\pi} f(-x)dx + \int_0^{\pi} f(x)dx = \\ &= - \int_0^{\pi} f(x)dx + \int_0^{\pi} f(x)dx = 0.\end{aligned}$$

Note. Product of even and odd functions is odd function, product of two even functions is even function and product of two odd functions is even function.

Let the function $f(x)$ is 2π -periodic even function and admits expansion in a Fourier series.

Hence,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cos nx}_{\text{even function}} dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad n \in \{1, 2, \dots\};$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \sin nx}_{\text{odd function}} dx = 0, \quad n \in \{1, 2, \dots\}.$$

Thus, the Fourier series of an even function contains “only cosines” and has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad (4.5)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx, \quad n \in \{0, 1, 2, \dots\}; \quad (4.5)$$

Example.

Find the Fourier series expansion for the 2π -periodic function

$$f(x) = |x|,$$

determined on the interval $[-\pi, \pi]$.

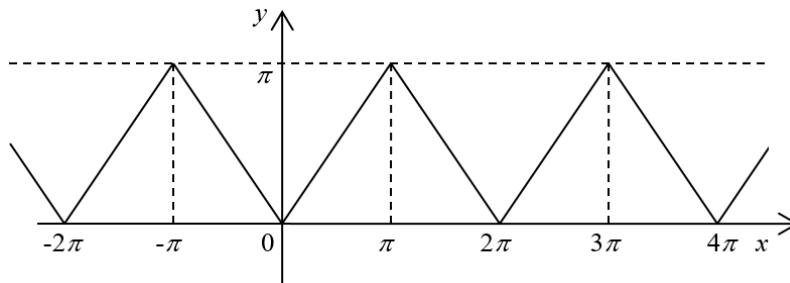


Figure 11.

Function $f(x)$ is piecewise smooth and even (Fig. 11). Thus, the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Let us calculate the coefficients:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \cdot \frac{x^2}{2} \Big|_0^{\pi} = \pi;$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx = \frac{2}{\pi} \left(\frac{x \sin nx}{n} \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx dx \right) = \frac{1}{\pi} \cdot \frac{\cos nx}{n^2} \Big|_0^{\pi} =$$

$$= \frac{1}{n^2 \pi} (\cos n\pi - 1) = \frac{2}{n\pi} ((-1)^n - 1) = \begin{cases} 0, & \text{for } n \text{ even;} \\ -\frac{4}{n^2 \pi}, & \text{for } n \text{ odd.} \end{cases}$$

Hence,

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

Consider 2π -periodic odd function the function $f(x)$ and admits expansion in a Fourier series.

Hence,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \cos nx}_{\text{odd function}} dx = 0, \quad n \in \{0, 1, 2, \dots\};$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(x) \sin nx}_{\text{even function}} dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \quad n \in \{1, 2, \dots\}.$$

Thus, the Fourier series of an even function contains “only sines” and has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{4.7}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \quad n \in \{1, 2, \dots\}. \tag{4.8}$$

Example.

Find the Fourier series expansion for the 2π -periodic function

$$f(x) = \begin{cases} -2, & -\pi < x < 0; \\ 2, & 0 < x < \pi. \end{cases}$$

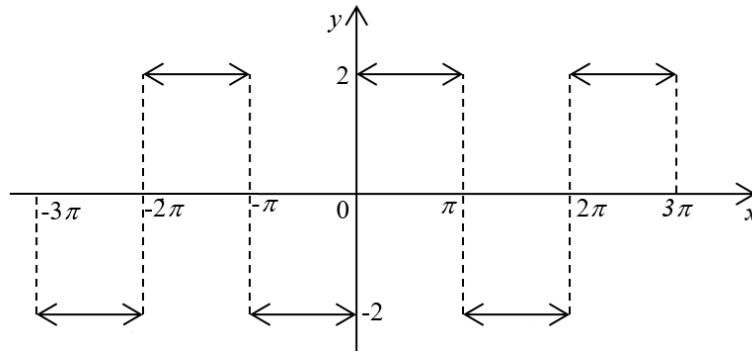


Figure 12.

Function $f(x)$ is piecewise smooth and odd (Fig. 12). Thus, the Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n x.$$

Let us calculate the coefficients:

$$a_n = 0, \quad n \in \{0, 1, 2, \dots\};$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin n x dx = \frac{4}{\pi} \int_0^{\pi} \sin n x dx = -\frac{4}{\pi} \frac{\cos n x}{n} \Big|_0^{\pi} = \\ &= \frac{4}{n\pi} (1 - \cos n \pi) = \frac{4}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & \text{for } n \text{ even;} \\ \frac{8}{n\pi}, & \text{for } n \text{ odd.} \end{cases} \end{aligned}$$

Thus,

$$f(x) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}.$$

At the points of discontinuities this series converges to value 0 – the average of the two one-sided limits.

4.3 Fourier Series for 2ℓ -periodic Functions

Consider 2ℓ -periodic function (x) , that is piecewise smooth on the interval $[-\ell, \ell]$.

Let us expand this function in a Fourier series.

Make the substitution

$$x = \frac{\ell t}{\pi}.$$

Then the function $f\left(\frac{\ell t}{\pi}\right)$ is 2π -periodic function of t .

Thus, it may be represented as a Fourier series on the interval $[-\pi, \pi]$.

$$f\left(\frac{\ell t}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \cos nt dt,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{\ell t}{\pi}\right) \sin nt dt.$$

Return to the original variable

$$x = \frac{\ell}{\pi} t \Rightarrow t = \frac{\pi}{\ell} x \Rightarrow dt = \frac{\pi}{\ell} dx.$$

$$\begin{array}{c|c|c} t & \ell & -\ell \\ \hline x & \pi & -\pi \end{array}$$

Hence, we obtain the Fourier series for 2ℓ -periodic function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right), \quad (4.9)$$

where

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) dx;$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx, \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \quad (4.10)$$

Theorem 4.2. (Sufficient conditions for representing a function by Fourier series)

If the 2ℓ -periodic function $f(x)$ is a piecewise smooth on the interval $[-\ell, \ell]$, then the Fourier series for $f(x)$ converges at all points. At the points of its continuity the values of the sum of series $S(x)$ is equal to the values of the function $f(x)$. At the points of discontinuities of $f(x)$ the sum is the average of the two one-sided limits.

The Fourier series of an even 2ℓ -periodic function $f(x)$ has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \quad (4.11)$$

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx; \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx. \quad (4.12)$$

If 2ℓ -periodic function $f(x)$ is odd then the Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}, \quad (4.13)$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \quad (4.14)$$

Example.

Find the Fourier series expansion for the function

$$f(x) = \begin{cases} 0, & -2 < x \leq 0; \\ \frac{1}{2}x, & 0 < x < 2, \end{cases}$$

defined on the interval $[-2, 2]$. Period is 4 (Fig. 13).

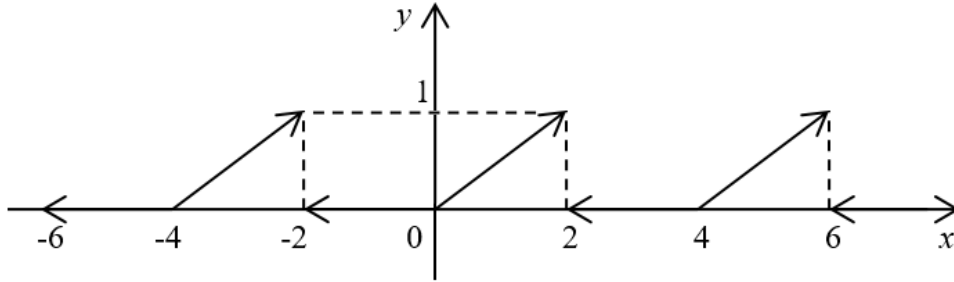


Figure 13.

Function $f(x)$ is piecewise smooth and $\ell=2$. Hence, the Fourier series has a form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right).$$

Let us calculate the coefficients:

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_0^2 \frac{1}{2} x dx = \frac{1}{4} \cdot \frac{x^2}{2} \Big|_0^2 = \frac{1}{2};$$

$$\begin{aligned}
a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \int_0^2 \frac{x}{2} \cos \frac{n\pi x}{2} dx = \frac{1}{4} \int_0^2 x \cos \frac{n\pi x}{2} dx = \\
&= \frac{1}{4} \left(\frac{2}{n\pi} \cdot x \sin \frac{n\pi x}{2} \Big|_0^2 - \frac{2}{n\pi} \int_0^2 \sin \frac{n\pi x}{2} dx \right) = \frac{1}{n^2 \pi^2} \cdot \cos \frac{n\pi x}{2} \Big|_0^2 = \\
&= \frac{1}{n^2 \pi^2} (\cos n\pi - 1) = \frac{1}{n^2 \pi^2} ((-1)^n - 1) = \begin{cases} 0, & \text{for } n \text{ even;} \\ -\frac{2}{n^2 \pi^2}, & \text{for } n \text{ odd.} \end{cases} \\
b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \int_0^2 \frac{x}{2} \sin \frac{n\pi x}{2} dx = \frac{1}{4} \int_0^2 x \sin \frac{n\pi x}{2} dx = \\
&= \frac{1}{4} \left(-\frac{2}{n\pi} \cdot x \cos \frac{n\pi x}{2} \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} dx \right) = \\
&= \frac{1}{4} \left(-\frac{4}{n\pi} \cdot \cos n\pi + \frac{4}{n^2 \pi^2} \cdot \sin \frac{n\pi x}{2} \Big|_0^2 \right) = \frac{(-1)^{n+1}}{n\pi}.
\end{aligned}$$

At the points of continuity of function

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left(-\frac{2}{(2n-1)^2 \pi^2} \cos \frac{(2n-1)\pi x}{2} + \frac{(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{2} \right).$$

At the points of discontinuities of $f(x)$ this series converges to value $\frac{1}{2}$ – the average of the two one-sided limits.

4.4 Fourier Series for Functions Defined on an Arbitrary Interval

I. Let there be given, on some interval $[a, b]$, a piecewise smooth function $f(x)$ (Fig. 14). It is possible to represent this function in the form of a sum of a Fourier series at the points of continuity. To do this, we should consider so-called *periodic extension* of the function $f(x)$.

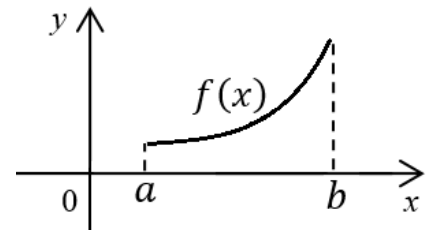


Figure 14.

Periodic extension is an arbitrary periodic piecewise smooth function $g(x)$ with a period $2L \geq b - a$, which coincides with the function $f(x)$ on the interval $[a, b]$ (Fig. 15).

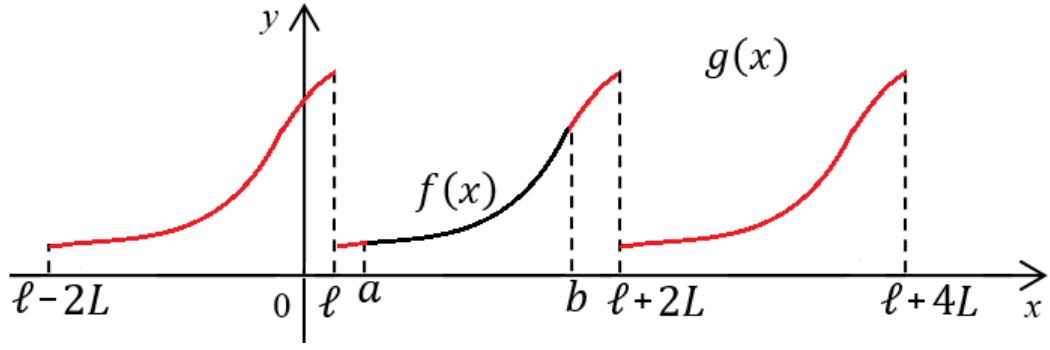


Figure 15.

The simplest periodic extension of the function $f(x)$ is the function $h(x)$ when $L = \frac{b-a}{2}$. We get it by taking the graph of the function $f(x)$ on the interval $[a, b]$ and then repeating that graph to the right and left of the graph of the original function (Fig. 16).

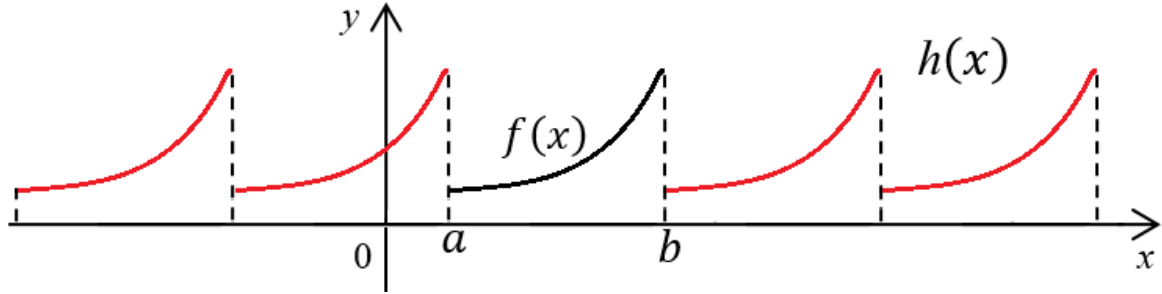


Figure 16.

In any case we can expand periodic extensions in a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \quad (4.11)$$

where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx;$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \quad (4.12)$$

At all points of interval $[a, b]$ (with exception of points of discontinuity) the sum of this series coincides with the given function. That means we have expanded the function $f(x)$ in a Fourier series on the interval $[a, b]$.

II. Let a piecewise smooth function $f(x)$ be given on the interval $[0, \ell]$.

According to **I.**, since $L = \frac{\ell}{2}$, there exists a Fourier series expansion for this function on the interval $[0, \ell]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{\ell} + b_n \sin \frac{2n\pi x}{\ell} \right), \quad (4.13)$$

where

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx;$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{2n\pi x}{\ell} dx, \quad b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{2n\pi x}{\ell} dx. \quad (4.14)$$

Let us consider two important cases. We can construct the odd or the even extension of $f(x)$ on the interval $[-\ell, 0]$.

If we redefine the given function as

$$f_{\text{even}}(x) = \begin{cases} f(-x), & x \in [-\ell, 0], \\ f(x), & x \in [0, \ell], \end{cases}$$

we get an even function (Fig. 17), and the expansion in a Fourier series contains only cosines.

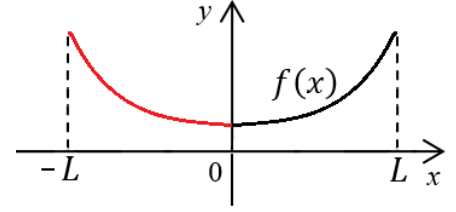


Figure 17.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}, \quad (4.15)$$

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f(x) dx; \quad a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \cos \frac{n\pi x}{\ell} dx. \quad (4.16)$$

Thus, we have **expanded in cosines** the function $f(x)$ on the interval $[0, \ell]$

If we redefine the given function as

$$f_{\text{odd}}(x) = \begin{cases} -f(-x), & x \in [-\ell, 0], \\ f(x), & x \in [0, \ell], \end{cases}$$

we get an odd function (Fig. 18), and the expansion in a Fourier series contains only sines.

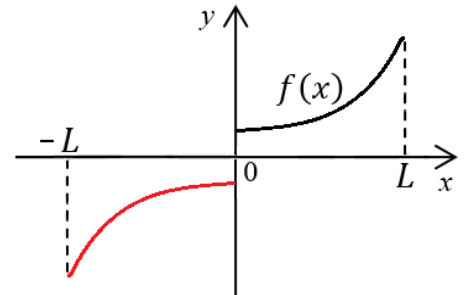


Figure 18.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{\ell}, \quad (4.17)$$

where

$$b_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx. \quad (4.18)$$

Thus, we have ***expanded in sines*** the function $f(x)$ on the interval $[0, \ell]$.

REFERENCES

1. N.Piscunov Differential and Integral Calculus/ N.Piscunov - Mir Publisher, Moscow, 1966 - 895 p.
2. Smimov V. I. A Course of Higher Mathematics, Vo l.: Elementary Calculus/ Smimov V. I. - Oxford, Pergamon Press - 1964 – 558 p.
3. H. Jerome Keisler Elementary Calculus: an Infinitesimal Approach / H. Jerome Keisler - On-line Edition. 2000 - <https://www.math.wisc.edu/~keisler/calc.html>
4. <http://tutorial.math.lamar.edu>
5. Swokowski Earl William Calculus. 5th Edition / Swokowski Earl W., - Brooks/Cole, 1991 - 1152 p.