ON D-CHARACTERISTICS OF ONE-DIMENSIONAL BOUNDARY PROBLEMS IN SPACES OF DIFFERENTIABLE FUNCTIONS

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Let numbers $m, l, n \in \mathbb{N}$ be arbitrarily chosen. On the finite interval (a, b), we present some results on d-characteristics of the following linear boundary-value problem

$$(Ly)(t) := y'(t) + A(t)y(t) = f(t), \quad t \in (a, b), \tag{1}$$

$$By = c. \tag{2}$$

Here the matrix function $A \in (C^{(n-1)})^{m\times m}$, the vector function $f \in (C^{(n-1)})^{m}$, the numerical vector $c \in \mathbb{C}^l$ and the linear continuous operator $B\colon (C^{(n)})^m \to \mathbb{C}^l$ are arbitrarily chosen.

We also let boundary condition (2) to be *overdetermined* ($l > m$) or *undetermined* ($l < m$) with respect to the differential system (1).

Using the linear continuous operator

$$(L, B)\colon (C^{(n)})^m \to (C^{(n-1)})^m \times \mathbb{C}^l, \tag{3}$$

we rewrite the boundary-value problem (1), (2) in the form of the equation

$$(L, B)y = (f, c).$$

Theorem 1. *Operator* (3) *is Fredholm with index* $m - l$.

By $Y \in (C^{(n)})^{m\times m}$ we denote a unique solution of the following matrix Cauchy problem:

$$Y'(t) + A(t)Y(t) = O_m, \quad t \in (a, b), \qquad Y(a) = I_m, \tag{4}$$

where O_m and I_m denote the null and identity $m \times m$ matrices, respectively.

Definition 1. We call a complex numerical $l \times m$ matrix *characteristic* for the boundary-value problem (1), (2) if each column of this matrix is the result of the action of the operator B upon the column with the same number of the matrix function Y. We denote this characteristic matrix by $M(L, B)$.

Theorem 2. *For d-characteristics of the Fredholm operator* (3)*, the relations*

$$\dim\ker(L, B) = m - \operatorname{rank} M(L, B), \tag{5}$$

$$\dim\operatorname{coker}(L, B) = l - \operatorname{rank} M(L, B). \tag{6}$$

are true.

Corollary 1. *Operator* (3) *is invertible (i.e., a topological isomorphism) if and only if* $l = m$ *and the square matrix* $M(L, B)$ *is nonsingular.*

Together with the problem (1), (2), consider a sequence of boundary-value problems of the form

$$(L_k y)(t) := y'(t) + A_k(t)y(t) = f_k(t), \quad t \in (a, b), \tag{7}$$

$$B_k y = c_k. \tag{8}$$

Here, $k \in \mathbb{N}$, $A_k \in (C^{(n-1)})^{m\times m}$, $f_k \in (C^{(n-1)})^m$, $c_k \in \mathbb{C}^l$, and B_k is a linear continuous operator on the pair of the spaces $(C^{(n)})^m$ and $\mathbb{C}^l$.

Let associate the boundary-value problems (7), (8) with the sequence of linear continuous operators $(L_k, B_k)\colon (C^{(n)})^m \to (C^{(n-1)})^m \times \mathbb{C}^l$ and the sequence of their characteristic matrices $M(L_k, B_k) \in \mathbb{C}^{l\times m}$. As usual, $(L_k, B_k) \xrightarrow{s} (L, B)$ denotes the convergence of the sequence of operators (L_k, B_k) to (L, B) in the strong operator topology. Here, all limits are considered as $k \to \infty$.

Theorem 3. *If* $(L_k, B_k) \xrightarrow{s} (L, B)$*, then* $M(L_k, B_k) \to M(L, B)$ *(as* $k \to \infty$*).*

Theorem 4. *Suppose that*

$$(L_k, B_k) \xrightarrow{s} (L, B) \quad \textit{as} \quad k \to \infty. \tag{9}$$

Then

$$\dim\ker(L_k, B_k) \leqslant \dim\ker(L, B) \quad \textit{for all} \quad k \gg 1,$$

$$\dim\operatorname{coker}(L_k, B_k) \leqslant \dim\operatorname{coker}(L, B) \quad \textit{for all} \quad k \gg 1.$$

Assuming that condition (9) is satisfied, we obtain the following.

Corollary 2. *If operator* (3) *is invertible, then operators* (L_k, B_k) *are also invertible for all* $k \gg 1$.

Corollary 3. *If the boundary-value problem* (1), (2) *has a solution for any given right-hand sides, then this is also true for the boundary-value problems* (7), (8) *for all* $k \gg 1$.

Corollary 4. *If the boundary-value problem* (1), (2) *has only a trivial solution in the case of zero right-hand sides, then this is also true for the boundary-value problems* (7), (8) *for all* $k \gg 1$.

This result was published in [1].

References

[1] Soldatov V. *Solvability of Linear Boundary-Value Problems for Ordinary Differential Systems in the Space* C^n // Ukr. Math. J. — 2025. — V. 77, No 3. — P. 360–368. DOI:10.1007/s11253-025-02451-x

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