Quintile Regression Based Approach for Dynamical VaR and CVaR Forecasting using Metalog Distribution

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Abstract. The paper proposes a new method of dynamic VaR and CVaR (Expected Shortfall) risk measures forecasting. Quantile linear model GARCH (QLGARCH) is chosen as the main forecasting model for time series quantiles. To build a forecast, the values of quantiles are approximated by the metalog distribution, which makes it possible to use analytical formulas to evaluate risk measures. The method of forecasting of dynamic VaR and CVaR is formulated as a step-by-step algorithm. At the first stage, an initial model is built to obtain variance estimates. The predicted variance values obtained from the constructed model are used at the second stage to find the QLGARCH model coefficients by solving the minimization problem. At the third stage, the QLGARCH models are estimated on a non uniform quantile grid. The obtained predicted values of quantiles are used to estimate the approximating metalog distribution. Estimated metalog distribution allows to use the analytical formulas to find the predicted values of VaR and CVaR. The investigated theory was applied to VaR and CVaR forecasting for time series of daily log-return of the Dow Jones Industrial Average (DJI) index. Using the proposed method, a set of one-step forecasts of risk measures was obtained. Prediction quality was assessed using standard backtesting methods. The results were also compared with the predictions obtained by standard methods based on the LGARCH model with parametric and non-parametric methods for static risk estimation for the model residuals.

Keywords: dynamic risk measures, VaR, CVaR, Expected Shortfall, forecast, Quantile LGARCH model, metalog distribution

1 Introduction

The purpose of this study is to develop the new method of dynamic VaR and CVaR risk measures estimation and forecasting. VaR and CVaR are classic measures that are used in financial risk assessment [9].In the practice of VaR and

CVaR estimating for a random variable that describes the profitability of a financial instrument, two main approaches can be distinguished. The first approach is a nonparametric estimation method that is based on an empirical distribution function. The disadvantage of this estimation method is the critical dependence of the effectiveness of the method on the presence in the initial data of values that arise with low probabilities [9]. The second approach is parametric, based on a priori estimation of the distribution function, which is the main disadvantage of this approach [9].

In time series analysis, in particular, in time series forecasting, in addition to static risks measures, in practice it becomes necessary to build more complex risk models, which take into account the changes of the series over time. In this case, to estimate risk measures, various time series models can be used, such as, for example, ARMA, GARCH. With this approach, the problem of risk measures modeling is reduced to estimate a model for variance and finding static risk measures for its residuals using parametric or nonparametric methods. Examples of this approach are described in [11], [1], [12].

The described above approaches of evaluation of risk measures are based on the construction of a cumulative distribution function on the full space of events. At the same time, to estimate risk measures, it is sufficient to evaluate only the quantile of a given level (for VaR) or the distribution of values exceeding a given level (for CVaR). At the same time, from a practical point of view, the most significant are the quantities and distribution for a relatively small subset of the event space leading to extreme consequences. Accordingly, it is possible to simplify the forecasting task by using the quantile regression model proposed in [15]. [8] contains a detailed description of the theory of quantile regression estimation applicable to standard time series models. Since financial time series, as a rule, are characterized by rather strong volatility, quantile GARCH models are popular for risk analysis. The problem of building quantile models of the GARCH class and their application in VaR forecasting for the series of log returns of stock market indices is considered, for example, in [13], [16].

One of the possible solutions to the problem of approximation to the distribution of the GARCH model residuals is to use the metalog distribution proposed in [6]. This choice is based on the simplicity of quantile formulas and the availability of a sufficient set of parameters of this distribution for an adequate fitting of empirical data of various nature. Thus, in [14], the metalog distribution is used in the development of the extended FAIR-BN combined approach for cyber security risk assessment. In [4], the five-term metalog distribution is used to forecast fertility rates in Canada. SPT (symmetric-percentile triplet) metalog distribution is used in [3] to statistically compare the forecasts of annual production in the oil and gas industry in Norway. In the work [17], the metalog distribution is used for dynamic risk measures VaR and CVaR estimating based on a heteroscedastic time series model, taking into account the strong dependence of the data series. The method of smoothing of the autocorrelation function is used for variance modeling. A metalog distribution is proposed to use for risk measures model residuals estimating. The paper proposes two methods of metalog distribution estimating and explicit analytical formulas for VaR and CVaR modeling and forecasting with different numbers of members in the metalog distribution.

A large number of publications devoted to the risk measures estimation and forecasting testifies to the applied significance of this problem. At the same time, the task of developing the new methods and approaches for risk modeling, which more fully reflect the nature of the modeled series, remains relevant. Most of the forecasting methods are based on the estimation of the entire distribution function. On the one hand, this is an overstated requirement for the model, and on the other hand, it often leads to an incorrect description of tails of distribution. Therefore, in this paper, it is proposed to build volatile models only for the tail parts of the distribution. In this case, the obtained point values of the quantiles can be smoothed, for example, by metalog distribution.

2 Materials and Methods

On the probability space $(\Omega, \Phi_t, \mathrm{P})$ a time series $\{u_t,\ t \in \mathrm{T}\}$ with a finite mean is considered (Φ_t is the information set containing all available at the time t information about the time series). The series is set by its observations at times $u_1, ..., u_N$.

For a fixed confidence level α risk measure VaR_α^t is defined as the conditional α - quantile of the CDF of u_t: $VaR_\alpha^t = F_t^{-1}(\alpha)$. The risk measure $CVaR_\alpha^t$ is defined as the integral: $CVaR_\alpha^t = E_t[u_{\tilde{t}} \,|u_{\tilde{t}} < -VaR_\alpha^t(t)\,] = -\frac{1}{\alpha}\int\limits_0^\alpha VaR_\gamma^t\, d\gamma$, $(\alpha < 0.5)$, where $E_t[\cdot]$ denotes expectation with respect to Φ_t. In this paper, the continuity of the CDF is assumed.

As indicated in the introduction, most methods for dynamic VaR and CVaR risk measures forecasting are based on time series modeling. The GARCH models are among the models that describe volatility of financial time series. In this paper, we consider the Linear GARCH model LGARCH (p, q). This model is frequently used for fitting log-return volatility time series and appropriate for quantile regression because of its linear structure [16].

The time series $\{u_t, t = 0, 1, 2, ...\}$ follows LGARCH (p, q) process if:

$$u_t = \sigma_t \varepsilon_t,\ \ \sigma_t = \beta_0 + \sum\nolimits_{i=1}^{q} \gamma_i \,|u_{t-i}| + \sum\nolimits_{j=1}^{p} \beta_j \sigma_{t-j} \tag{1}$$

where $\{\varepsilon_t\}$ are independent, identically distributed random variables with zero mean and a conditional distribution function $F_\varepsilon(\cdot)$, $\beta_0 > 0$; $(\gamma_1, \gamma_2, ..., \gamma_q)^T \in R_+^q$. Using the heteroscedastic time series model, the dynamic risk measures can be found under the following formulas [9]:

$$VaR_\alpha^t = VaR_\alpha(\varepsilon)\sigma_t, CVaR_\alpha^t = CVaR_\alpha(\varepsilon)\sigma_t, \tag{2}$$

where the model for σ_t is defined in (1), $VaR_\alpha(\varepsilon)$ and $CVaR_\alpha(\varepsilon)$ are static risk measures at time t. Then the P step forecast for dynamic risk measures can be found by model (2) extrapolation:

$$VaR_\alpha^{t+P} = VaR_\alpha(\varepsilon)\sigma_{t+P}, CVaR_\alpha^{t+P} = CVaR_\alpha(\varepsilon)\sigma_{t+P}. \tag{3}$$

In this work, the following methods are used to evaluate static risk measures $VaR_\alpha(\varepsilon)$, $CVaR_\alpha(\varepsilon)$.

Historical simulation method. Let X be a random variable and its sample values are X_1, X_2, ... X_N. In accordance with the historical simulation method, an empirical distribution function is constructed on the sample values. Then according to [18]:

$$\hat{VaR}_\alpha = -X_{([N\alpha])}, C\hat{VaR}_\alpha = -\left(\sum_{i=1}^{[N\,\alpha]} X_{(i)}\right) / ([N\,\alpha])\,, \tag{4}$$

where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(N)}$.

Using Student's t-distribution. If the random variable has the local scale Student's t-distribution with the parameters μ, σ and the degrees of freedom $\upsilon > 2$, then the risk measures can be calculated as (see [18]):

$$\mathrm{VaR}_\alpha = \mu + \sigma t_\upsilon^{-1}(\alpha), \mathrm{CVaR}_\alpha = \mu - \sigma \frac{g_\upsilon(t_\upsilon^{-1}(\alpha))}{\alpha} \cdot \frac{\upsilon + \left(t_\upsilon^{-1}(\alpha)\right)^2}{\upsilon - 1}, \tag{5}$$

where $g_\upsilon(\cdot)$ is the standard PDF and $t_\upsilon{}^{-1}(\cdot)$ is the inverse standard CDF value at α of t-distribution.

Using metalog distribution. Suppose that X has a metalog distribution $F_X(x)$, that is defined by a quantile function $M_n(\alpha, \mathbf{a}(X,\alpha))$ [6]:

$$M_n(\alpha, \mathbf{a}) = \begin{cases} a_1 + a_2 \ln \frac{\alpha}{1+\alpha}, & n = 2 \\ a_1 + a_2 \ln \frac{\alpha}{1+\alpha} + a_3(\alpha - 0.5) \ln \frac{\alpha}{1+\alpha}, & n = 3 \\ a_1 + a_2 \ln \frac{\alpha}{1+\alpha} + a_3(\alpha - 0.5) \ln \frac{\alpha}{1+\alpha} + a_4(\alpha - 0.5), & n = 4 \\ M_{n-1} + a_n(\alpha - 0.5)^{\frac{n-1}{2}}, & \text{for odd}\, n \geq 5 \\ M_{n-1} + a_n(\alpha - 0.5)^{\frac{n}{2}-1} \ln \frac{\alpha}{1+\alpha}, & \text{for even}\, n \geq 5 \end{cases}$$

The coefficients $\mathbf{a} = (a_1, a_2, ..., a_n)^T$ can be found as a solution of the system of equations:

$$\mathbf{a} = [\mathbf{Y}_n{}^T \mathbf{Y}_n]^{-1} \mathbf{Y}_n{}^T \mathbf{X}, \tag{6}$$

where $\mathbf{X} = (X_1, X_2, ..., X_N)^T$, the matrix $\mathbf{Y}_n$ is defined on a vector of cumulative probabilities $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)^T = (F_X(X_1), F_X(X_2), ..., F_X(X_N))^T$ [6]:

$$Y_n = \begin{cases} \begin{bmatrix} 1 & \ln\frac{\alpha_1}{1+\alpha_1} \\ \cdots & \cdots \\ 1 & \ln\frac{\alpha_N}{1+\alpha_N} \end{bmatrix}, & n=2 \\ \begin{bmatrix} 1 & \ln\frac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5)\ln\frac{\alpha_1}{1+\alpha_1} \\ \cdots & \cdots & \cdots \\ 1 & \ln\frac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5)\ln\frac{\alpha_N}{1+\alpha_N} \end{bmatrix}, & n=3 \\ \begin{bmatrix} 1 & \ln\frac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5)\ln\frac{\alpha_1}{1+\alpha_1} & (\alpha_1-0.5) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \ln\frac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5)\ln\frac{\alpha_N}{1+\alpha_N} & (\alpha_N-0.5) \end{bmatrix}, & n=4 \\ \begin{bmatrix} & (\alpha_1-0.5)^{\frac{n-1}{2}} \\ Y_{n-1}| & \cdots \\ & (\alpha_N-0.5)^{\frac{n-1}{2}} \end{bmatrix}, & \text{for odd } n \geqslant 5 \\ \begin{bmatrix} & (\alpha_1-0.5)^{\frac{n}{2}-1}\ln\frac{\alpha_1}{1+\alpha_1} \\ Y_{n-1}| & \cdots \\ & (\alpha_N-0.5)^{\frac{n}{2}-1}\ln\frac{\alpha_N}{1+\alpha_N} \end{bmatrix}, & \text{for even } n \geqslant 6 \end{cases}$$

Following the definitions, risk measures can be found under the formulas:

$$\begin{aligned} VaR_{\tilde{\alpha},n}(X) &= -M_n(\tilde{\alpha}, \mathbf{a}(X)), \\ CVaR_{\tilde{\alpha},n}(X) &= -\frac{1}{\tilde{\alpha}} \int_0^{\tilde{\alpha}} M_n(y, \mathbf{a}(X))\, dy. \end{aligned} \tag{7}$$

Explicit formulas for $CVaR_{\tilde{\alpha},n}(X)$ estimating with different number of members of the metalog distribution were obtained in [17]:

$$CVaR_{\tilde{\alpha},n}(X) = \begin{cases} a_1 + a_2\left(\frac{\ln(1-\tilde{\alpha})}{\tilde{\alpha}} + \ln\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right), & n=2 \\ CVaR_{\tilde{\alpha},2}(X) + \frac{a_3}{2}\left(1 + (\tilde{\alpha}-1)\ln\frac{\tilde{\alpha}}{1-\tilde{\alpha}}\right), & n=3 \\ CVaR_{\tilde{\alpha},3}(X) + \frac{a_4(\tilde{\alpha}-1)}{2}, & n=4 \\ CVaR_{\tilde{\alpha},n-1}(X) + \dfrac{2a_n}{\tilde{\alpha}(1+n)} \times & \\ \times\left((-1)^{\frac{n-1}{2}}(0.5)^{\frac{n+1}{2}} + (\tilde{\alpha}-0.5)^{\frac{n+1}{2}}\right), & \text{odd}\, n \geqslant 5 \\ CVaR_{\tilde{\alpha},n-1}(X) + \frac{2^{\frac{2-n}{2}} a_n}{n\tilde{\alpha}} 2(2\tilde{\alpha}-1)^{\frac{n}{2}} \times & \\ \times arctg(2\tilde{\alpha}-1) + nG + \ln(1-\tilde{\alpha}) + & \\ +(-1)^{\frac{n+2}{2}}\ln\tilde{\alpha}, & \text{even}\, n \geqslant 6 \end{cases} \tag{8}$$

where $G = {}_3F_2\left[1, 1, 1-{}^{n}\!/\!{}_{2};\ 2, 2;\ 2\right] + (\tilde{\alpha}-1){}_3F_2\left[1, 1, 1-{}^{n}\!/\!{}_{2};\ 2, 2;\ 2-2\tilde{\alpha}\right] + +(-1)^{{}^{n}\!/\!{}_{2}} 2\tilde{\alpha}\ {}_3F_2\left[1, 1, 1-{}^{n}\!/\!{}_{2};\ 2, 2;\ 2\tilde{\alpha}\right]$, ${}_3F_2\left[c_1, c_2, c_3;\ d_1, d_2;\ z\right]$ is generalized hypergeometric function.

For determining the parameters of the metalog distribution the Quantile Metalog Method is proposed in [17]. This method uses the approximation of the empirical distribution function by sample quantiles $\mathbf{X} = (\tau_{\alpha_1}, ..., \tau_{\alpha_N})^T$, where

$$\tau_{\alpha_i} = \begin{cases} X_{([N\alpha_i]+1)}, & N\alpha_i \notin \mathrm{Z} \\ \frac{\bar{\alpha}-\alpha_i}{\bar{\alpha}-\underline{\alpha}} X_{(\underline{\alpha})} + \frac{\alpha_i-\underline{\alpha}}{\bar{\alpha}-\underline{\alpha}} X_{(\bar{\alpha})}, & N\alpha_i \in \mathrm{Z} \end{cases},$$

where $\underline{\alpha} = \frac{[N\alpha_i]}{N}$, $\bar{\alpha} = \frac{[N\alpha_i+1]}{N}$, $\alpha_i \in (0,1)$. Thus, the classical approach to risk measures forecasting (3) involves building a model for variance estimating, obtaining model residuals, and using methods for static risk measures $VaR_\alpha(\varepsilon)$, $CVaR_\alpha(\varepsilon)$ estimating.

Another approach is based on estimating of quantile time series models, which makes it possible to directly simulate the time series quantile of a given level. The QLGARCH model is considered (see [8], [16]):

$$Q_{u_t}(\tau\, |\Phi_{t-1}) = \theta_t(\tau)^T z_t, \tag{9}$$

where $Q_{u_t}(\tau\, |\Phi_{t-1})$ is a conditional τ- quantile for $\{u_t\}$,

$$\begin{gathered} z_t = (1, \sigma_{t-1}, ..., \sigma_{t-p}, |u_{t-1}|, ..., |u_{t-q}|)^T, \\ \theta_t(\tau)^T = (\beta_0, \beta_1, ..., \beta_p, \gamma_1, ..., \gamma_q) F^{-1}(\tau) = \\ = (\beta_0(\tau), \beta_1(\tau), ..., \beta_p(\tau), \gamma_1(\tau), ..., \gamma_q(\tau)). \end{gathered}$$

The paper proposes the following methodology. Model (9) is used to construct a set of quantiles. For a more detailed description of the left tail of the CDF, the quantile levels can be found on a non uniform grid. Assuming that the obtained set of quantiles can be fitted by the metalog distribution, the quantile function $M_n(\alpha, \mathbf{a}(X, \alpha))$ is estimated using (6). To obtain forecasts of risk measures, formulas (7), (8) are used. The practical implementation of the method is formulated as a step-by-step algorithm.

An Algorithm for Constructing the Dynamic Risk Measures VaR and CVaR Forecast based on the method QLGARCH - Metalog

1. **Building a variance model to get estimates $\hat{\sigma}_{t-i}, i = \overline{1,p}$.** The model LGARCH(p,q) (1) is written as ARCH(∞): $\sigma_t = \alpha_0 + \sum_{j=1}^{\infty} \alpha_j\, |u_{t-j}|$, where the coefficients α_j satisfy summability conditions implied by the regularity conditions [8]. Due to the assumption of regularity, the coefficients decrease geometrically, therefore, the model can be reduced to ARCH (m). The appropriate lag for reducted ARCH model is chosen on the base of significant values of ACF and PACF for squared values of returns (values that is more than confidence bounds). Estimates of α_j can be obtained in various ways. QMLE is used in this work. The fitted model is used to obtain estimates $\hat{\sigma}_t, ..., \hat{\sigma}_{t-p}$:
$$\hat{\sigma}_{t-i} = \hat{\alpha}_0 + \sum\nolimits_{j=1}^{m} \hat{\alpha}_j\, |u_{t-j-i}|\, , i = \overline{0,p}. \tag{10}$$
2. **Building a set of quantiles predictive estimates** To obtain estimates of the τ-quantile for u_t, the QLGARCH model (9) is used. The orders of

the model p, q can be estimated using Akaike Inform Criteria (AIC) and Hannan-Quinn Inform Criteria (HQIC). It is also possible to use Bayes Inform Criteria (BIC) and Shibata Inform Criteria (SIC). To estimate the vector of parameters $\theta(\tau)^T$, the minimization problem is solved using the quantile regression estimation in the form (see [8]):

$$\min_{\theta} \sum_{t} \rho_{\tau}(u_t - \theta^T \hat{z}_t), \tag{11}$$

where $\rho_{\tau}(u) = u(\tau - I(u < 0))$ is a check function,
$\hat{z}_t = (1, \hat{\sigma}_{t-1}, ..., \hat{\sigma}_{t-p}, |u_{t-1}|, ..., |u_{t-q}|)^T$, taking into account that the estimates $\hat{\sigma}_{t-1}, ..., \hat{\sigma}_{t-p}$ were obtained at previous step (10). The solution of the unconstrained minimization problem (11) makes it possible to estimate the τ-quantile for u_t in the form: $\hat{Q}_{u_t}(\tau\,|\Phi_{t-1}) = \hat{\theta}(\tau)^T \hat{z}_t$. At this step, a grid of τ_i-quantiles, $i = \overline{1, N}$, is constructed and the problem (11) is solved N times. For every quantile regression with $\tau_i, i = \overline{1, N}$ the quantile estimates are $\hat{Q}_{u_t}(\tau_i) = \hat{\theta}_t(\tau_i)^T \hat{z}_t$, $i = \overline{1, N}$. In this case, the predicted values of the conditional quantiles are calculated by extrapolation $\hat{Q}_{u_{t+1}}(\tau_i) = \hat{\theta}(\tau_i)^T \hat{z}_{t+1}$, where $\hat{z}_{t+1} = (1, \hat{\sigma}_t, ..., \hat{\sigma}_{t-p+1}, |u_t|, ..., |u_{t-q+1}|)^T$, and $\hat{\sigma}_{t-i}$, $i = \overline{0, p-1}$ are obtained at the first step of algorithm (10).

3. **Risk measures forecasting** The predictive quantiles $\hat{Q}_{u_{t+1}}(\tau_1), ..., \hat{Q}_{u_{t+1}}(\tau_N)$ (from previous step) are fitted using the metalog distribution. Estimates of the metalog distribution parameters $\hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_n)^T$ are found in accordance with (6), where $\mathbf{X} = (\hat{Q}_{u_{t+1}}(\tau_1), ..., \hat{Q}_{u_{t+1}}(\tau_N))^T$, $\alpha = (\tau_1, \tau_2, ..., \tau_N)^T$. Specifying the quantile function $M_n(\tilde{\alpha}, \hat{\mathbf{a}})$ for a given level $\tilde{\alpha}$ allows the use of analytical expressions (7), (8) to find the predicted values $VaR_{\tilde{\alpha},n}^{t+1}$ and $CVaR_{\tilde{\alpha},n}^{t+1}$.

3 Experiment, Results and Discussions

Proposed algorithm was applied for dynamic risk measures $VaR_{0.1}$, $CVaR_{0.1}$ forecasting for the time series of daily log returns of the Dow Jones Industrial Average index (the DJI time series). The sample length was 3500 values from 2007/02/16 to 2021/01/11. The forecast model was based on 1500 historical values and was extrapolated one value forward. After that, the modeling window was one step shifted, and the model was rebuilt. The procedure was repeated 2000 times (Rolling Forecast Method). The obtained one-step forecasts were compared with real values for the corresponding period of time. To obtain variance estimates (10), the model ARCH(30) was built. Based on the historical values of the time series and the estimated variance values, quantile LGARCH(3,3) models were built for different levels of quantiles. The orders of the models were found using the AIC and HQIC criteria: p = 3, q = 3. For non uniform grid a more detailed description of the left tail of the distribution, an of quantiles was used: $\tau_i = ih$, $h = 0.01$ for $i = \overline{1, 20}$, and $\tau_i = ih$, $h = 0.05$ for $i = \overline{5, 19}$. Estimates of the parameters were obtained using QMLE. The estimated models were used to generate one-step quantile predictions. The predicted quantile values were used to estimate the metalog distribution. Using (6), the estimates for the coefficients of the metalog distributions for $n = 4, 5, 6, 7$ were obtained. Risk measures estimates were calculated using formulas (7) and (8). The results were obtained using R packages *rugarch* [5] and *quantreg* [7].

The results of dynamic VaR and CVaR risk measures forecasting using the QLGARCH - Metalog (rq_met) method are shown in Figure 1. along the historical values (the first 1500) and the real values (1501-3500) of the DJI time series. As can be seen from the graph, the obtained forecast estimates describe the dynamic behavior of the time series quite well.

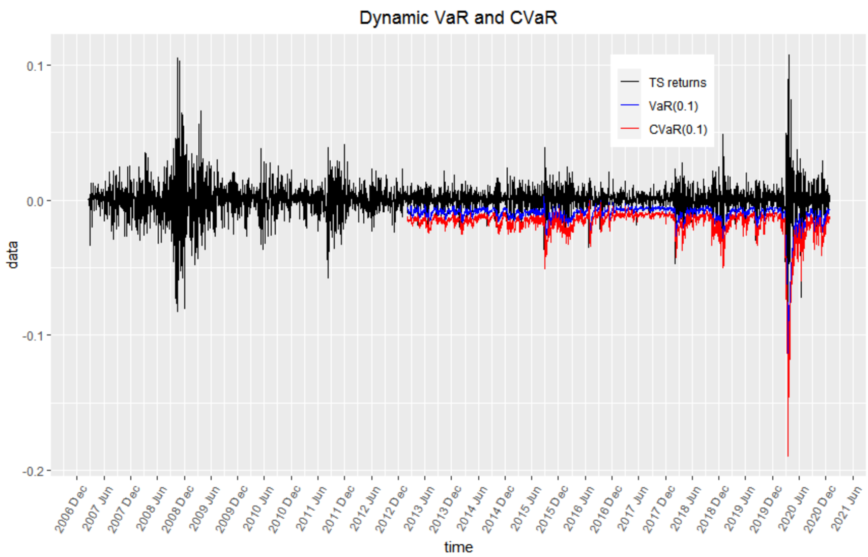

Fig. 1. Historical data of the time series of daily log return of the DJI index (TS returns) from 2007/02/16 to 2021/01/11 and the forecast values for $VaR_{0.1}^{t+1}$ (VaR (0.1)) and $CVaR_{0.1}^{t+1}$ (CVaR (0.1)) obtained by the QLGARCH - Metalog method ($n = 5$) for the period 2013 / 02/01 - 2021/01/11

Historical data and the forecast values of dynamic risk measures over a short period of time is shown on Figure 2. for more convenient visual analysis.

For a comparative analysis of the effectiveness of the proposed method, a forecast of dynamic risk measures for a given time series was built with standard approach on the basis of a heteroscedastic model (3). To estimate the variance, the LGARCH(3,3) model was considered. The AIC and HQIC criteria were used to determine the orders of the model. The QMLE was used to estimate the coefficients. To determine the risk measures for the residuals of the LGARCH(3,3) model, the following methods were used: the Historical simulation method (4) (hist method), explicit formulas (5) under the assumption that the residuals of

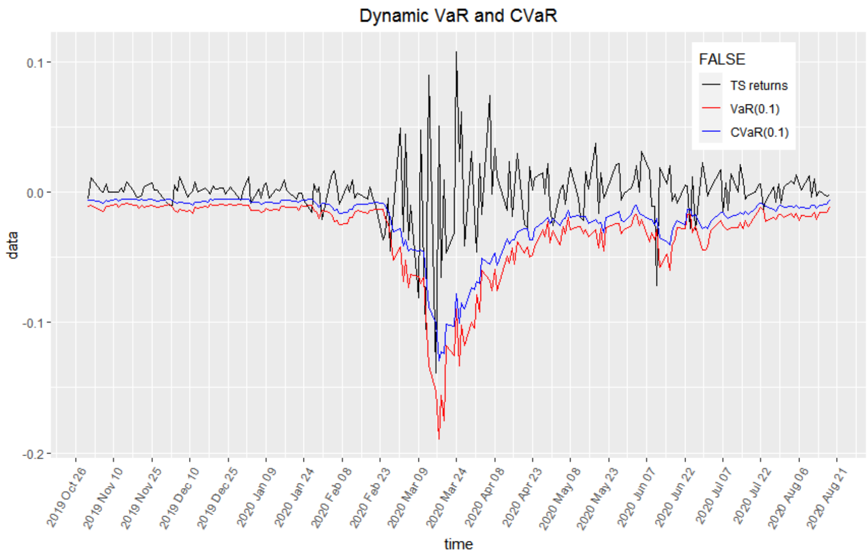

Fig. 2. Historical data of the time series of daily log return of the DJI index (TS returns) from 2007/02/16 to 2021/01/11 and the forecast values for $VaR_{0.1}^{t+1}$ (VaR (0.1)) and $CVaR_{0.1}^{t+1}$ (CVaR (0.1)) obtained by the QLGARCH - Metalog method ($n = 5$) for the period 2019/10/31 - 2020/08/18

the model have the local scale t-distribution (tLS method), explicit formulas (7), (8) based on Quantile Metalog Method for $n = 4, 5, 6, 7$ (metal method).

The analysis of the constructed forecast estimates was carried out using the backtesting procedure. The following tests were used in the work:

- for VaR estimates: the Kupiec test (LRuc), Christoffersen's independence test (LRind), PoE statistics [19];
- for CVaR estimates: two tests proposed in [10] were used: one-sided simple conditional calibration test (scc_1) and two-sided simple conditional calibration test (scc_2); three regression based backtests proposed in [2]: the auxiliary ESR backtest (Aux), the strict ESR backtest (Str), the intercept ESR backtest (Int).

The p-values of these tests are shown in Table 1. Table 2 shows the PoE statistic values for the forecast estimates of dynamic risk measure VaR.

As follows from Table 1, the worst estimates were obtained using the historical simulation method (hist). In particular, as a result of applying the Christoffersen's independence test (LRind) for VaR and the two-sided simple conditional calibration test (scc_2) for CVaR, hypotheses with a significance level of 0.05 were rejected. This indicates the inapplicability of the historical simulation method for predicting the values of the DJI time series. At the same time, all tests showed consistently good quality of forecasts obtained by the QLGARCH - Metalog

Table 1. The results of the qualitative analysis of the forecast estimates of dynamic risk measures VaR and CVaR for DJI time series

Risk meashure	VaR		CVaR				
Method	LRuc	LRind	scc_1	scc_2	Aux	Str	Int
hist	0.0677	0.0091	0.0866	0.0076	0.6453	0.6191	0.5824
tLS	0.4582	0.3359	0.0657	0.1136	0.5364	0.5286	0.4710
metal, n=4	0.1497	0.0802	0.2301	0.0625	0.6980	0.7289	0.6267
metal, n=5	0.1726	0.1497	0.2411	0.0630	0.6883	0.7338	0.6461
metal, n=6	0.0945	0.0393	0.1251	0.0115	0.5707	0.6047	0.5355
metal, n=7	0.0945	0.0474	0.1245	0.0098	0.5261	0.5521	0.5180
rq_met, n=4	0.4983	0.7643	0.1572	0.0869	0.8454	0.8914	0.8072
rq_met, n=5	0.8810	0.5796	0.2741	0.1808	0.8842	0.9164	0.8475
rq_met, n=6	0.4272	0.4072	0.1863	0.1082	0.8566	0.8931	0.8121
rq_met, n=7	0.5476	0.3655	0.2688	0.1757	0.8671	0.8940	0.8450

Table 2. PoE statistic values for the forecast estimates of dynamic risk measure VaR for DJI time series

Method	hist	tLS	metal				rq_met			
			n=4	n=5	n=6	n=7	n=4	n=5	n=6	n=7
PoE	0.0879	0.1051	0.09045	0.09095	0.0889	0.0889	0.0954	0.0989	0.0944	0.0959

method (rq_met) and maximum p-value statistics compared to other methods (tLS and metal).

In the article the metalog distribution with the different number of parameters ($n = 4, 5, 6, 7$) is considered (see Table 1). An increase in the number of parameters potentially increases the accuracy of the estimates, but can lead to the problem of overfitting. As a result of the backtesting for forecasted dynamic risk measures VaR and CVaR obtained with the Quantile Metalog Method (metal), the choose of large n probably leads to overfitting. The estimates of VaR and CVaR obtained using the metalog distribution for $n = 4$ and $n = 5$ are consistently better than the estimates obtained using the same sample for $n = 6$ and $n = 7$. At the same time, the QLGARCH - Metalog (rq_met) method shows less dependence on the number of parameters of the metalog distribution. The results of the qualitative analysis for VaR and CVaR forecasts obtained by this method are relatively uniform for all n. Although it should be noted that according to the results of all tests, the highest quality forecasts for risk measures were obtained at $n = 5$. The best results of VaR forecasting (see Table 2) according to PoE statistics is obtained also by rq_met method for $n = 5$ (has the least deviation from the target value of 0.1).

The paper proposes a method that is a natural continuation of existing research and methods devoted to dynamic risk models developing. It combines parametric and nonparametric statistical approaches to time series modeling. Practical application of the method shows its effectiveness in the case of risk modeling for highly volatile financial time series. The simplicity of the method and its background make it possible to recommend it for using in various fields.

However, the determination of the restrictions on applicability of this method, as well as the automation of the procedure for adjusting its parameters, requires further mathematical research, in particular, the construction of asymptotic estimates of the convergence of the model.

4 Conclusions

The paper considers the problem of dynamic VaR and CVaR risk measures modeling and forecasting for financial time series. Since the VaR measure is a conditional quantile of the distribution function of a given level, and CVaR for continuous distributions can be specified as the average of the quantile function it is proposed to use QLGARCH as a model for risk measures forecasting. The advantage of using of this model is the ability to estimate and predict not the full distribution, but the values of the quantiles of the required levels. Since the risk measures are determined for the tail part of the distribution, an non uniform grid is used in the work, which makes it possible to detail the quantiles with a low level. To smooth point values, it is proposed to fit a set of quantiles with metalog distribution. This approach is also convenient due to the presence of explicit analytical expressions VaR and CVaR for the metalog distribution. The proposed method for dynamic VaR and CVaR risk measures forecasting is formulated in the form of a step-by-step algorithm.

The proposed methodology was tested on the time series of daily log return of the Dow Jones Industrial Average (DJI) index for the period from 2007-02-16 to 2021-01-11. Using the formulated algorithm, a set of one-step forecasts of risk measures was obtained. An analysis of the quality of the forecasts was carried out using various standard backtesting techniques on real data. The results were compared with the forecasts obtained by standard methods that are based on the LGARCH model and various assumptions about distribution of the residuals. The carried out qualitative analysis of the obtained predicted values showed the effectiveness of using the method proposed in this work and its advantage in comparison with standard methods.

The results of the work can be directly applied in dynamic risk modeling for highly volatile time series, in particular, financial time series, and also can serve as the basis for the development of new methods and algorithms for random processes prediction.

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