

INDUCED VACUUM MAGNETIC FLUX IN THE BACKGROUND OF A LINEAR TOPOLOGICAL DEFECT. THE CASE OF FERMION MATTER

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Abstract

A topological defect in the form of a magnetic vortex is modeled as a magnetic flux-carrying tube that is impenetrable to quantum matter. The spinor matter field is quantized in the background of the vortex, with the most general set of boundary conditions ensuring both the impenetrability of the vortex core and the self-adjointness of the Dirac Hamiltonian operator. We compute the induced vacuum magnetic flux along the vortex axis in 3+1-dimensional space-time. Requiring the total induced vacuum flux to be finite restricts the set of admissible boundary conditions to a single option: the MIT bag boundary condition. The dependence of the results on the transverse size of the vortex was investigated.

Keywords:

Introduction

The description of the evolution of Universe necessarily leads to the existence of stable topological defects [1]. First of all, their formation is required by the theories of the Grand Unification, in which phase transitions in early Universe, during which the splitting of fundamental interactions occurred, occurred with a delay in different parts of it, while at the boundaries of the regions where they occurred, folds of space-time were formed, which formed the so-called linear topological defects — cosmic strings ([2, 3, 4]). Inside the strings, the matter remained in the state in which it was at the time of their formation, that is, a vacuum with an ultrahigh density. The existence of topological defects is also assumed in condensed matter physics, and here the possibility of their experimental detection is much greater today.

An important effect in the theory of cosmic strings is the induction of magnetic flux inside the string by fermionic matter outside. Theoretical study of this process allows one to limit the values of free parameters — boundary conditions parameters when solving string problems. Since analytical solutions for this process are extremely complex, numerical modeling is necessary for their analysis, which is the subject of this work.

1. Cosmic Strings

The energy of which is higher than the phase transition separating the strong interaction from the elec-

troweak one, the phase $\alpha(\mathbf{r})$ of the Higgs scalar field ϕ is uniformly distributed, vacuum medium $\langle \phi \rangle = 0$. After the phase transition, at which the vacuum medium becomes $\langle \phi \rangle \neq 0$, its value is fixed and at the same time it turns out to be different at different points. However, for the uniqueness of the field, it is necessary that the phase when traveling along any closed circuit can change only by a multiple of 2π :

$$\Delta\alpha = 2\pi N,$$

where N — number of windings (topological invariant). Under $N \neq 0$, we have topological defect — a string, and for the continuity of the field, it is necessary for the strings to be either open, approximately straight and fixed to the horizon of the Universe, or closed, and within the string, the vacuum remains in the state before the phase transition $\langle \phi \rangle = 0$. It is also necessary to note about additional global characteristic of string — flux

$$\Phi = \int_{\text{core}} d\sigma \partial \times \mathbf{A} = \oint d\mathbf{r} \cdot \mathbf{A},$$

where $\mathbf{A}$ is the vector potential of the gauge field corresponding to the spontaneously broken gauge symmetry, and the line integral is over a closed contour encircling the string core once. At the same time, there is a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ inside the string, the magnetic flux flowing along the string is quantized and also depends on the topological invariant

$$\int \mathbf{B} d\mathbf{S} = \frac{2\pi}{e} N.$$

Analysis of the equations of the gravitational field for infinite strings shows that there is no gravitational

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field outside the core of the string, i.e., space-time remains locally flat, but the geometry becomes conical due to the deficit of the polar angle $\varphi : 0 \leq \varphi < 2\pi(1 - 4G\mu)$, where μ is string tension (energy per unit of its length). An upper limit $\sim 10^{-6}$ rad for the value of the angular deficit can be estimated from above from observations of the anisotropy of the relic radiation [5]. This circumstance makes it possible to build the main method for the detection of cosmic strings today — the search for gravitational lensing by the string, which occurs as a result of the formation of a double image of an object located behind the string

Cosmic strings almost inevitably emerge in the framework of superstring and supergravity models [6]. String models are used not only to describe the dynamics and structure of the Universe, but also in the physics of hadrons and continuous media. In particular, structures analogous to cosmic strings arise in the theory of superconductivity. Here they are called Abrikosov vortices — tubes of magnetic flux in superconductors of the second kind at a temperature below the critical.

2. Induced vacuum magnetic field in the cosmic string background

In the paper [7] shown that in 2+1 space-time, a straight cosmic string of nonvanishing transverse size induces a magnetic field in the vacuum of the quantum relativistic charged spinor matter field. In the case of the magnetic string background, the Dirac hamiltonian takes form

$$H = -i\alpha \cdot (\partial - ieV) + \beta m,$$

and the only component of the vector potential is non-vanishing $V_\varphi = \Phi/(2\pi)$.

A certain boundary condition must be imposed on the surface of the magnetic tube, which is impermeable to the fields, to model the cosmic string. The peculiarity of fermionic matter is that this boundary condition cannot be the usual Neumann, Dirichlet, or Robin condition. In this case, the self-adjointness condition of the Dirac Hamiltonian should be applied. For the two-component spinor function

$$\psi_E(\mathbf{x}) = \sum_{n \in \mathbb{Z}} \begin{pmatrix} f_n(r, E)e^{in\varphi} \\ g_n(r, E)e^{i(n+1)\varphi} \end{pmatrix}$$

this boundary condition is written as

$$\cos\left(\frac{\theta}{2} + \frac{\pi}{4}\right) f_n(r_0, E) = -\sin\left(\frac{\theta}{2} + \frac{\pi}{4}\right) g_n(r_0, E),$$

where θ is a self-adjoint extension parameter. It was shown that only for values $\theta = 0$ and $\theta = \pi$ the induced vacuum magnetic flux is finite. For other values of boundary condition induced magnetic flux is infinite.

In paper [8], a similar analysis was carried out within the framework of 3+1 space-time. In this case, one can decompose the four-component function into the two-component ones. In general, the condition of the self-adjoint extension of the Dirac operator in 3+1 space-time is characterized by 4 parameters of the boundary

condition. By requiring invariance under the CPT transformation, we obtain the dependence of the boundary condition on only one boundary condition parameter, just as in the case of a (2+1)-dimensional space.

The corresponding analytical dependencies for the induced magnetic flux were obtained for 3+1 space-time. However, the analytical dependencies, which are quite complex for analysis even in 2+1 space-time, in the case of generalization to 3 spatial dimensions, become prohibitively complex for direct calculation and analysis. Therefore, the analysis of the obtained dependencies and the exclusion of non-physical boundary conditions on their basis is an independent problem for numerical calculations, which is the subject of this work.

The obtained in [8] induced vacuum magnetic flux Φ_I in (3 + 1)-dimensional space-time is expressed via the induced vacuum magnetic flux $\Phi_I^{(d=2)}$ in (2 + 1)-dimensional space-time:

$$\Phi_I = \int_0^\infty \frac{dk_3}{\pi} \Phi_I^{(d=2)}(m_3), \quad m_3 = \sqrt{m^2 + k_3^2},$$

where $\Phi_I^{(d=2)}$ is given by (1).

$$F = \left\{ \frac{e\Phi}{2\pi} \right\}$$

is the fractional part of the magnetic flux inside the impenetrable magnetic tube, and parameter ν for the cosmic string can be taken as $\nu = 1$.

3. Numerical computation

To determine how the magnetic flux of the string behaves depending on it's thickness in 3+1 space, we had to analyse the function for 2+1 space (1). Directly integrating the expression has shown to be highly challenging due to it's complexity, so it was decided to find a similar function that would be easier to integrate. The numerical computations were performed using **Wolfram Mathematica 14.2.1**.

To get the target function over the interval [0,1], we evaluated a set of points within the range and interpolated them using splines. The number of terms and the upper bound of integration were individually selected for each point to optimise computation time and precision.

Since one of the function's terms is a sum of an infinite series, that converges, a finite number of terms is sufficient for evaluation. On intervals where artefacts of interpolation appeared, more points were added. In points where flux approached zero, we raised the upper bound of integration and a number of terms.

An increase in accuracy for values of flux close to zero required more computational effort. The function beyond [0,1] was extrapolated using interpolating function. From a logarithmic plot it's easy to see that for larger values of argument, from a certain point the function behaves similar to an exponent.

For the extrapolation, a part of the data was approximated in different functions using **FindFit**. The best function that described the data was $e^{az}(bz^2 + cz + d)$.

$$\begin{aligned}
 \Phi_1^{(d=2)}(m) = & \frac{e}{2m} \left[\frac{1}{2\pi} \int_0^\infty \frac{du}{ch^3(u/2)} \exp(-2mr_0 \operatorname{ch}(u/2)) \Omega_{\operatorname{sgn}(F-1/2)}(u) + \frac{1}{v} \sum_{p=1}^{\lfloor v/2 \rfloor} \exp(-2mr_0 \sin(p\pi/v)) \frac{\sin[(2F-1)p\pi]}{\sin^3(p\pi/v)} - \right. \\
 & \left. - \frac{1}{4N} e^{-2mr_0 \sin[(2F-1)N\pi]} \delta_{v,2N} \right] + \frac{e}{\pi} r_0 \int_{mr_0}^\infty \frac{dv v}{\sqrt{v^2 - m^2 r_0^2}} \times \left[\operatorname{sgn}\left(F - \frac{1}{2}\right) C_{\lambda(0, |F-1/2|)}^{(\pm)}(v, mr_0) D_{\lambda(0, |F-1/2|)}(v) + \right. \\
 & \left. + \sum_{l=1}^\infty \left(C_{\lambda(l, F-1/2)}^{(\pm)}(v, mr_0) D_{\lambda(l, F-1/2)}(v) - C_{\lambda(l, 1/2-F)}^{(\pm)}(v, mr_0) \times D_{\lambda(l, 1/2-F)}(v) \right) \right], \\
 C_\rho^{(\pm)}(v, w) = & \frac{v I_\rho(v) K_\rho(v) \pm w [I_\rho(v) K_{\rho-1}(v) - I_{\rho-1}(v) K_\rho(v)] - v I_{\rho-1}(v) K_{\rho-1}(v)}{v K_\rho^2(v) \pm 2w K_\rho(v) K_{\rho-1}(v) + v K_{\rho-1}^2(v)} \quad (1)
 \end{aligned}$$

Finally we unified the two functions using Heaviside step function. The resulting function closely matches the original flux function, it was integrated as shown in the images below.

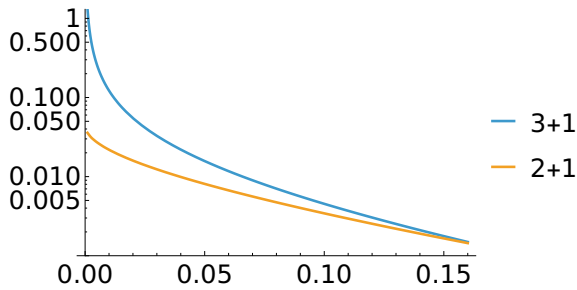


Fig. 1. Induced vacuum magnetic flux in dimensionless units in 2+1 space-time ($em^{-1}\Phi_1^{(d=2)}$), and in 3+1 space-time $e\pi\Phi_1^{(d=3)}$ as function of the tube thickness mr_0 .

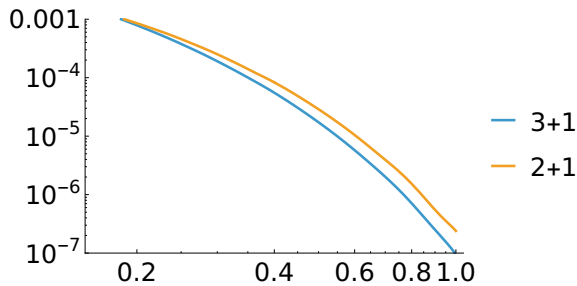


Fig. 2. Induced vacuum magnetic flux in dimensionless units in 2+1 and 3+1 space-times in the case of big tube radius ($mr_0 > 0.16$).

Conclusions

We investigate the induced magnetic flux in the vacuum of a fermionic field in the presence of a magnetic cosmic string background and compute it numerically. The results are presented in Fig.1 and Fig.2. As shown, the induced magnetic flux in (3 + 1)-dimensional space-time (in dimensionless units) is significantly larger than in (2 + 1) dimensions for small values of the tube radius, specifically when $mr_0 < 0.16$.

The requirement that the total induced vacuum flux be finite restricts the set of admissible boundary conditions to a single choice: the MIT bag boundary condition, $\theta = 0$. In contrast, the case $\theta = \pi$, which yields a finite induced magnetic flux in (2 + 1)-dimensional space-time, leads to an infinite induced flux in (3 + 1) dimensions.

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