ISSN (print) 2312–4547, ISSN (on-line) 2415–7325.
ВІСНИК ДНУ. Серія "Моделювання". 2016. Вип. 8, № 8. С. 20–25
DOI 10.15421/141602

УДК 517.9

ON FINITE DIMENSIONAL DYNAMICS UP TO A SMALL PARAMETER OF REACTION-DIFFUSION INCLUSION IN UNBOUNDED DOMAIN

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Communicated by Prof. P. I. Kogut

We consider the reaction-diffusion equation with multi-valued interaction function. We investigate the qualitative behavior of all weak solutions under the standard growth and sign conditions. We prove that dynamics of all weak solutions for the investigated problem is finite dimensional up to a small parameter.

Keywords: reaction-diffusion equation, weak solution, multi-valued semi-flow.

1. Introduction and Setting of the Problem

In this note we examine the long-time behavior of all weak solutions for reaction-diffusion inclusion. In particular, we study the questions related to the finite dimensionality of dynamics for considered problem. There are a lot of papers on qualitative behavior of solutions for evolution systems of reaction-diffusion type. It caused by the theoretical and applied significance of such objects [5,8,13,16,20]. The particular cases of reaction-diffusion problems are Kolmogorov – Petrovskiy – Piskunov equations (the problem on the gene diffusion) [9], models of Belousov – Zhabotinsky reaction [3, 15], Gause – Vitta models [17, 18], Selkov model for glycolysis [10, 14] etc. Reaction-diffusion equations are actively used for modelling of various biological and chemical processes.

We study the case when the conditions on the parameters of the problem do not guarantee uniqueness of solution for corresponding Cauchy problem. So, we need to use the methods of nonlinear analysis [11, 20], multi-valued analysis and theory of multi-valued semi-flows [1, 6, 20].

Consider the reaction-diffusion inclusion in unbounded domain $\mathbb{R}^N$, $N \geq 1$:

$$u_t(x,t) - \triangle u(x,t) \in f(x, u(x,t)), \quad x \in \mathbb{R}^N,\, t > 0, \tag{1.1}$$

where $u(\cdot,\cdot)$ is unknown function, $f : \mathbb{R}^{N+1} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ is possibly discontinuous or multi-valued function.

Assume that the following conditions hold:

(A) there exist functions $\underline{f}, \overline{f} : \mathbb{R}^{N+1} \to \mathbb{R}$ such that for a.a. $x \in \mathbb{R}^N$:

- the function $\overline{f}(x, \cdot)$ is upper semicontinuous;
- the function $\underline{f}(x, \cdot)$ is lower semicontinuous;
- $\underline{f}(x, y) \leq \overline{f}(x, y)$ for each $y \in \mathbb{R}$;
- $f(x, y) = [\underline{f}(x, y), \overline{f}(x, y)]$ for each $y \in \mathbb{R}$;

(B) there exist a function $C_1 \in L^1(\mathbb{R}^N)$ and a constant $\alpha > 0$ such that for a.e. $x \in \mathbb{R}^N$ and for all $y \in \mathbb{R}$ the following inequalities hold:

$$\underline{f}(x, y)y \geq \alpha|y|^2 - C_1(x), \ y \geq 0,$$

$$\overline{f}(x, y)y \geq \alpha|y|^2 - C_1(x), \ y \leq 0;$$

(C) there exist a non-negative function $C_2 \in L^1(\mathbb{R}^N)$ and constants $\beta > 0, \gamma > 0$ such that for a.e. $x \in \mathbb{R}^N$ and for all $y \in \mathbb{R}$ the following relations hold:

$$|\underline{f}(x, y)|^2 \leq C_2(x) + \beta|y|^2,$$

$$|\overline{f}(x, y)|^2 \leq C_2(x) + \gamma|y|^2.$$

Consider the standard spaces $H := L^2(\mathbb{R}^N)$, $V := H^1(\mathbb{R}^N)$ and $V^* := H^{-1}(\mathbb{R}^N)$ with respective norms and inner products [4]. Let $(V; H; V^*)$ be a Gelfand triple. Note that none of the embeddings $V \subset H \subset V^*$ is compact.

Definition 1.1. Let $\tau < T$. A function $\varphi : \mathbb{R}^N \times [\tau, T] \to \mathbb{R}$ is called a *weak solution* of inclusion (1.1) on $[\tau, T]$ if the following two properties hold together:

(i) $\varphi(\cdot) \in L^2(\tau, T; V)$;

(ii) there exists a measurable function $d : \mathbb{R}^N \times (\tau, T) \to \mathbb{R}$ satisfying the following two properties:

(a) $d(x, t) \in f(x, \varphi(x, t))$ for a.e. $(x, t) \in \mathbb{R}^N \times (\tau, T)$;

(b) the following equality holds:

$$-\int_\tau^T \int_{\mathbb{R}^N} \varphi(x, t) \cdot v_t(x, t) dx dt - \int_\tau^T \int_{\mathbb{R}^N} \varphi(x, t) \cdot \triangle v(x, t) dx dt + \int_\tau^T \int_{\mathbb{R}^N} d(x, t) \cdot v(x, t) dx dt = 0, \quad (1.2)$$

for each $v \in C_0^\infty(\mathbb{R}^N \times (\tau, T))$.

The main purpose of this note is to examine the finite dimensionality of the solution dynamics for Problem (1.1) up to a small parameter (Theorem 2.1).

2. Preliminaries and main results

Note that [16, Lemmas 3.1 and 3.2] yield that under assumptions 1–3 the following a priory estimates holds for each weak solution $u(\cdot)$ of Problem (1.1) on $[\tau, T]$:

$$\|u(\cdot)\|^2_{X_{\tau,T}} \le \frac{3}{2}\left(\|u(\tau)\|^2_H + 2\bar{C}_1(T-\tau)\right), \tag{2.1}$$

$$\|u_t(\cdot)\|_{Y_{\tau,T}} \le \sqrt{\frac{3}{2}\left(\|u(\tau)\|^2_H + 2\bar{C}_1(T-\tau)\right)} + c_1\sqrt{K_1(T-\tau)\left(1+\frac{3}{2}\left(\|u(\tau)\|^2_H + 2\bar{C}_1(T-\tau)\right)\right)}, \tag{2.2}$$

$$\|u(t)\|^2_H + 2\int_\tau^t \int_{\mathbb{R}^N} e^{-2\alpha(t-s)}\|\nabla u(x,t)\|^2_{\mathbb{R}^N} dxds \le \|u(\tau)\|^2_H e^{-2\alpha(t-\tau)} + D, \quad \forall t \in [\tau, T], \tag{2.3}$$

where $\bar{C}_1 := \int\limits_{\mathbb{R}^N} C_1(x)dx < +\infty$, $C_1(\cdot) \in L^1(\mathbb{R}^N)$, $D = \|C_1\|_{L^1(\mathbb{R}^N)}/\alpha$ (see assumption 2).

For each weak solution of inclusion (1.1) on $[\tau, T]$, $\tau < T$, consider the following initial data:

$$u(x,\tau) = u_\tau(x), \ \ x \in \mathbb{R}^N, \tag{2.4}$$

where $u_\tau(\cdot) \in H$.

Estimates (2.1)–(2.3) imply the existence of at least one weak solution of Cauchy Problem (1.1), (2.4) on $[\tau, T]$ for each $\tau < T$ and $u_\tau \in H$; [16, Theorem 3.1]. We note also that each weak solution of Problem (1.1), (2.4) on $[\tau, T]$ is regular in the following sense :

$$u(\cdot) \in C([\tau+\varepsilon, T]; V) \cap L^2(\tau+\varepsilon, T; H^2(\mathbb{R}^N) \cap V) \text{ and } u_t(\cdot) \in L^2(\tau+\varepsilon, T; H)$$

for each $\varepsilon \in (0, T-\tau)$; [5, Theorem 15.3].

Since $\tau < T$ are arbitrary, then translation and concatenation of weak solutions are also weak solutions on respective time intervals. Therefore, according to autonomy of Problem (1.1), we obtain that each weak solution of Problem (1.1) defined on $[0, T]$, $T > 0$, can be extended to the global one, defined on $[0, +\infty)$.

For each $u_0(\cdot) \in H$ we denote by $\mathcal{D}(u_0)$ the set of all global weak solutions of Problem (1.1) on $[0, +\infty)$ with initial data

$$u(x,0) = u_0(x) \text{ for a.e. } x \in \mathbb{R}^N. \tag{2.5}$$

Then $\mathcal{D}(u_0) \subset L^2_{loc}(0,+\infty;V)\cap C([0,+\infty),H)$. Moreover, $\mathcal{D}(u_0) \subset L^\infty(0,+\infty;H)$ for each $u_0 \in H$.

Denote by $\mathcal{K}^+$ the family of all weak solutions of Problem (1.1) defined on $[0,+\infty)$, that is, $\mathcal{K}^+ = \bigcup_{u_0\in H} \mathcal{D}(u_0)$; [19,20]. For arbitrary $u(\cdot) \in \mathcal{K}^+$ and $s \geq 0$ we remark that $u(\cdot + s) \in \mathcal{K}^+$.

Let us define multi-valued (in the general case) map $G : \mathbb{R}_+ \times H \to 2^H \setminus \{\emptyset\}$ as follows:

$$G(t,u_0) := \{u(t) \in H \,|\, u(\cdot) \in \mathcal{K}^+ : \ u(0) = u_0\}. \tag{2.6}$$

Note that (2.6) is equivalent to

$$G(t,u_0) := \{z \in H \,|\, \exists u(\cdot) \in \mathcal{D}(u_0) : u(t) = z\}.$$

Definition 2.1. ([20, Definition 1.1], [7,12]). The multi-valued map $G : \mathbb{R}_+ \times H \to 2^H \setminus \emptyset$ is called a multi-valued *semi-flow* if the following two properties hold together:

(i) $G(0,\cdot) = I$ is the identity map;

(ii) for all $t,s \in \mathbb{R}_+$ and for all $x \in H$

$$G(t+s,x) \subset G(t,G(s,x)),$$

where $G(t,B) = \cup_{x\in B} G(t,x)$, $B \subset H$.

The multi-valued semi-flow G is called *strict* if, moreover, for all $t,s \in \mathbb{R}_+$ and for all $x \in H$

$$G(t+s,x) = G(t,G(s,x)).$$

From [6, Lemmas 2.4 and 2.6] it follows that solution dynamics of Problem (1.1) is finite dimensional up to a small parameter if the multi-valued semi-flow generated by all weak solutions of Problem (1.1) is asymptotically compact.

Definition 2.2. ([20, Definition 1.4], [7,12]) The multi-valued semi-flow $G : \mathbb{R}_+ \times H \to 2^H \setminus \emptyset$ is called an *asymptotically compact* if for arbitrary nonempty bounded set $B \subset H$ there exists nonempty compact set $A(B) \subset H$ such that

$$dist_H(G(t,B),A(B)) \to 0 \text{ as } t \to +\infty.$$

Remark 2.1. ([2, p.35], [20]) If for arbitrary nonempty bounded set $B \subset H$ the set $\bigcup_{t\geq T} G(t,B)$ is bounded for some $T = T(B)$, then the multi-valued semi-flow G is asymptotically compact if and only if the arbitrary sequence $\{\xi_n\}_{n\geq 1}$ such that for all $n \geq 1$

$$\xi_n \in G(t_n,B), \quad t_n \to +\infty,$$

is precompact in H.

Let $B_r(x)$ is closed ball centered in $x \in H$ with radius $r > 0$. The following theorem is the main result of this note.

Theorem 2.1. *Let assumptions (A), (B) and (C) hold. Then the multi-valued semi-flow G defined in (2.6) satisfies the following condition: for each bounded set $B \subset H$ and $\varepsilon > 0$ there exist a moment of time $t_0(B, \varepsilon)$ and a finite dimensional subspace E of H such that for a bounded projector $P : H \to E$, $P(\cup_{t \geq t_0} G(t, B))$ is a bounded set in H and $(I - P)(\cup_{t \geq t_0} G(t, B)) \subset B_\varepsilon(\bar{0})$, where I is the identity map in H.*

The following three lemmas from [5] will be used in the proof of Theorem 2.1.

Lemma 2.1. *The map $G : \mathbb{R}_+ \times H \to 2^H \setminus \{\emptyset\}$ defined by (2.6) is strict multi-valued semi-flow.*

Lemma 2.2. *For arbitrary fixed nonempty bounded set $B \subset H$ and arbitrary fixed $\varepsilon > 0$ there exist the constants $T = T(\varepsilon, B) > 0$ and $K = K(\varepsilon, B) > 0$ such that for arbitrary $u_0 \in B$, for any global weak solution $u(\cdot) \in \mathcal{D}(u_0)$ on $[0, +\infty)$ and for all $t \geq T(\varepsilon, B)$ and $k \geq K(\varepsilon, B)$ the next inequality holds:*

$$\int_{\left\{x \,:\, \|x\|^2_{\mathbb{R}^N} \geq 2\pi k^2\right\}} |u(x,t)|^2 dx \leq \varepsilon.$$

Lemma 2.3. *For arbitrary fixed $t \geq 0$ the graph of $G(t, \cdot) : H \to 2^H \setminus \{\emptyset\}$ is weakly closed, that is, for each sequences $\{y_n\}_{n\geq 1} \subset H$ and $\{z_n\}_{n \geq 1} \subset H$ satisfying*

(a) *$y_n \in G(t, z_n)$ for each $n \geq 1$;*

(b) *$y_n \to y$ weakly in H and $z_n \to z$ weakly in H as $n \to +\infty$,*

the inclusion $y \in G(t, z)$ holds.

Proof of Theorem 2.1. Let us provide the sketch of the proof for Theorem 2.1. According to [6, Lemmas 2.4 and 2.6], the main purpose is to provide the asymptotically compactness of multi-valued semi-flow G defined in (2.6). This property follows from Lemmas 2.1–2.3 (see, also, [5]). Therefore, [6, Lemmas 2.4, 2.6] yields the necessary statement.

Remark 2.2. If we consider Problem (1.1) with Dirichlet boundary conditions in a bounded domain $\Omega \in \mathbb{R}^N$, then the statement of Theorem 2.1 holds. Indeed, the asymptotically compactness of multi-valued semi-flow follows from [8, Theorems 4–6]. So, from [6, Lemmas 2.4 and 2.6] we obtain the statement of Theorem 2.1.

Acknowledgments.
This work was partially supported by the National Academy of Sciences of Ukraine under Grant 2284.

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Надійшла до редколегії 10.05.2016