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“Igor Sikorsky Kyiv Polytechnic Institute”

HIGHER MATHEMATICS

Differential Calculus of a Function of One Variable

Elements of Theory

Kyiv
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HIGHER MATHEMATICS

Differential Calculus

of a Function of One Variable.

Elements of Theory.

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This textbook is designed for students of the first year of technical university. It covers two content areas to be studied in the first semester: Theory of Limits and Differential Calculus of a Function of One Variable.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of Calculus are explained and illustrated by figures and examples.

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Introduction

This textbook is designed for students of the first year of technical university. It covers two content areas to be studied in the first semester: Theory of Limits and Differential Calculus of a Function of One Variable.

The manual can be helpful for students who want to understand and be able to use standard differentiation techniques, analyze the behavior of a function and so on.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of Calculus are explained and illustrated by figures and examples.

The first part deal with the main definitions and concepts of the theory of limits: limit of a numerical sequence, limit of a function, concept of infinitesimals, concept of continuous function, the points of discontinuity of function.

The second part is concerned with the bases of differential calculus of function of one variable: the derivative and the differential of a function, the tangent and the normal line to the curve, applications of derivatives: monotonicity, extrema and concavity, L'Hospital's rule.

There are also four appendices. In Appendix 1 it is presented the simplest bases of set theory. The Appendix 2 is a rapid presentation of the concepts of functions, including the properties and graphs of elementary functions. The remaining appendices can be considered as giving some information about the polar coordinates and parametric representation of a function.

1. Theory of Limits

1.1 The Limit of a Numerical Sequence

I. *Numerical Sequences*

The numerical sequence is a set of numbers enumerated by natural index in ascending order of values of the index. The number of elements (possibly infinite) is called *the length* of the sequence. Further we will consider infinite sequences.

Members of the sequences are called *elements* or *terms*. The sequence is *ordered* in the sense that there is a *first term* (a_1), a *second term* (a_2), a *tenth term* (a_{10}), and, if n denotes an arbitrary positive integer, the *nth term* (a_n). The sequence usually has *the rule* (*formula*) for the nth term which is a way to evaluate each term.

$$\{a_1, a_2, a_3, ..., a_n, ...\} = \{a_n, n \in \mathbb{N}\}.$$

Also the sequence can be represented by points on the numerical axis (Fig. 1).

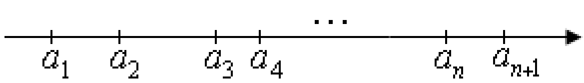
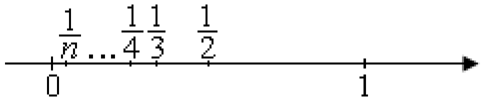


Figure 1.

Example.

$$\begin{array}{ccccccc} 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \dots, & \frac{1}{n}, \dots & \\ \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \Rightarrow a_n = \frac{1}{n} \quad \Rightarrow \left\{ a_n = \frac{1}{n}, n \in \mathbb{N} \right\} \Rightarrow \\ a_1 & a_2 & a_3 & a_4 & & a_n & \text{(rule)} \end{array}$$

$\Rightarrow$ 

The sequence $\{a_n\}$ is *an upper-bounded sequence,* if there exists such real number M that for any natural n: $a_n \le M$. ($\exists M \in \mathbb{R}\ \forall n \in \mathrm{N} : a_n \le M$.)

The sequence $\{a_n\}$ is *a lower-bounded sequence,* if $\exists K \in \mathbb{R}\ \forall n \in \mathbb{N} : a_n \ge K$.

The sequence $\{a_n\}$ is a *bounded sequence*, if $\exists M, K \in \mathbb{R}\ \forall n \in \mathbb{N} : K \le a_n \le M$.

The sequence $\{a_n\}$ is a *monotone increasing sequence*, if $\forall n \in \mathbb{N} : a_n \le a_{n+1}$.

The sequence $\{a_n\}$ is a *monotone decreasing sequence*, if $\forall n \in \mathbb{N} : a_n \ge a_{n+1}$.

II. *Limit of the Numerical Sequence*

Consider the numerical sequence $\{a_n,\ n \in \mathbb{N}\}$. What happens if n becomes larger and larger ($n \to \infty$)? For example, terms of sequence $\left\{a_n = \dfrac{1}{n}, n \in \mathbb{N}\right\}$ become smaller $\left\{1,\ \dfrac{1}{2},\ \dfrac{1}{3},\ \ldots,\ \dfrac{1}{10},\ \ldots,\ \dfrac{1}{1000},\ \ldots,\ \dfrac{1}{10^{100}},\ \ldots\right\}$ (closer and closer to zero), but terms of sequence $\{a_n = n, n \in \mathbb{N}\}$ become larger $\{1,\ 2,\ \ldots, 10^{100},\ \ldots\}$ (increase infinitely). In the first case, the sequence is convergent and in the second it is divergent.

Definition. The sequence $\{a_n,\ n \in \mathbb{N}\}$ is convergent to the number a ($\lim\limits_{n\to\infty} a_n = a$) if for every arbitrary small $\varepsilon > 0$ there exists a natural number N such that for all $n > N$: $|a_n - a| < \varepsilon$ (Fig. 2).

$$(\lim_{n\to\infty} a_n = a, \text{ if } \ \forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}\ \ \forall\, n > N : |a_n - a| < \varepsilon.)$$

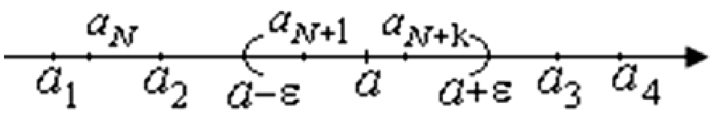


Figure 2.

If the sequence has a *finite* limit (a is a real number), then the sequence is *convergent* (the sequence converges to a), otherwise the sequence is *divergent* (the sequence diverges).

The sequence $\{a_n,\ n \in \mathbb{N}\}$ is *infinitely large* if $\lim\limits_{n\to\infty} a_n = \infty$.

The sequence $\{a_n,\ n \in \mathbb{N}\}$ is *infinitely small* (*infinitesimal*) if $\lim\limits_{n\to\infty} a_n = 0$.

For example $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0 \Rightarrow \left\{\dfrac{1}{n}, n \in \mathbb{N}\right\}$ is infinitesimal.

Lemma 1.1.

Every convergent sequence $\lim\limits_{n\to\infty} a_n = a$ can be expressed in the form $b_n = a + \alpha_n,\ \ n \geq 1,$ where $\{\alpha_n\}$ is infinitesimal $(\lim\limits_{n\to\infty} \alpha_n = 0)$.

Example. Find $\lim\limits_{n\to\infty}\dfrac{4n+1}{2n}$.

Let rewrite the general term of sequence as $\dfrac{4n+1}{2n}=\dfrac{4n}{2n}+\dfrac{1}{2n}=2+\dfrac{1}{2n}$.

Since $\dfrac{1}{2n}$ is an infinitesimal and $\lim\limits_{n\to\infty}\dfrac{1}{2n}=0$,

$$\lim_{n\to\infty}\frac{4n+1}{2n}=\lim_{n\to\infty}\left(2+\frac{1}{2n}\right)=2+\lim_{n\to\infty}\frac{1}{2n}=2.$$

Properties of infinitesimals.

Let $\{a_n,\ n\in\mathbb{N}\}$ and $\{b_n,\ n\in\mathbb{N}\}$ are infinitesimals, then

a) $\lim\limits_{n\to\infty}(a_n\pm b_n)=\lim\limits_{n\to\infty}a_n\pm\lim\limits_{n\to\infty}b_n=0\Rightarrow\{a_n\pm b_n\}$ is infinitesimal;

b) if $\{c_n,\ n\in\mathbb{N}\}$ is bounded then $\lim\limits_{n\to\infty}(a_n\cdot c_n)=0\Rightarrow\{a_n\cdot c_n\}$ is infinitesimal.

c) $\lim\limits_{n\to\infty}(a_n\cdot b_n)=\lim\limits_{n\to\infty}a_n\cdot\lim\limits_{n\to\infty}b_n=0\Rightarrow\{a_n\cdot b_n\}$ is infinitesimal;

d) $\forall\, c\in\mathbb{R}, \lim\limits_{n\to\infty}ca_n=c\lim\limits_{n\to\infty}a_n=0\Rightarrow\{ca_n\}$ is infinitesimal;

e) $\lim\limits_{n\to\infty}\dfrac{1}{a_n}=\infty\Rightarrow\left\{\dfrac{1}{a_n}\right\}$ is infinitely large sequence.

III. ***Basic Theorems About Limits***

In general, verifying the convergence directly from the definition is a difficult task. Thus, there are some methods to find limits of certain sequences and some sufficient conditions for the convergence of a sequence.

Theorem 1.1.

If $\lim\limits_{n\to\infty}a_n=a$ then a is unique.

Proof. Let $\lim\limits_{n\to\infty}a_n=a$ and $\lim\limits_{n\to\infty}a_n=b$ ($a<b$). Then a_n must satisfy, at one and the same time, two inequalities $|a_n-a|<\varepsilon$ and $|a_n-b|<\varepsilon$ for an arbitrary small $\varepsilon>0$. But it is impossible if $\varepsilon<\dfrac{a-b}{2}$.

Theorem 1.2.

Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then

a) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = ca, \quad c \in \mathbb{R}$;

b) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = a + b$;

c) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = ab$;

d) $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{a}{b}$, if $b \neq 0$.

Proof.

According to Lemma 1.1 $a_n = a + \alpha_n$ and $b_n = b + \beta_n$ where $\{\alpha_n,\ n \in \mathbb{N}\}$ and $\{\beta_n,\ n \in \mathbb{N}\}$ are infinitesimals.

a) Since $ca_n = c(a + \alpha_n) = ca + c\alpha_n$ and $\{c\alpha_n\}$ is infinitesimal we conclude that

$$\lim_{n\to\infty} ca_n = \lim_{n\to\infty} (ca + c\alpha_n) = ca.$$

b) Whereas $a_n + b_n = (a + \alpha_n) + (b + \beta_n) = (a + b) + (\alpha_n + \beta_n)$, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} ((a + b) + (\alpha_n + \beta_n)) = a + b + \lim_{n\to\infty} (\alpha_n + \beta_n) = a + b = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

c) Rewrite $a_n \cdot b_n = (a + \alpha_n)(b + \beta_n) = ab + b\alpha_n + a\beta_n + \alpha_n\beta_n$ where $\{b\alpha_n + a\beta_n + \alpha_n\beta_n\}$ is an infinitesimal.

Hence $\lim\limits_{n\to\infty} (a_n \cdot b_n) = ab = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$.

d) Let $\dfrac{a_n}{b_n} = \dfrac{a + \alpha_n}{b + \beta_n} = \dfrac{a}{b} + \left(\dfrac{a + \alpha_n}{b + \beta_n} - \dfrac{a}{b}\right) = \dfrac{a}{b} + \dfrac{b\alpha_n - a\beta_n}{b(b + \beta_n)}$. Here the fraction $\dfrac{a}{b}$ is a real number, while the fraction $\dfrac{b\alpha_n - a\beta_n}{b(b + \beta_n)}$ is an infinitesimal ($\{b\alpha_n - a\beta_n\}$ is an infinitesimal and $b(b + \beta_n) \to b^2 \neq 0, n \to \infty$).

Thus, $\lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{a}{b} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, if $b \neq 0$.

Theorem 1.3.

If $\lim\limits_{n\to\infty} a_n = a$, then $\exists\, C \in \mathbb{R}\ \ \forall\, n \geq 1\colon\ |a_n| \leq C.$

Proof.

According to definition of limit: for $\varepsilon = 1\ \exists\, N \in \mathbb{N}\ \ \forall\, \bar{n} > N : |a_{\bar{n}} - a| < 1$.

Let $C := \max\{\,|a_1|, |a_2|, \ldots |a_{\bar{n}-1}|, |a| + 1\}$. Then for $n,\ 1 \leq n \leq \bar{n} - 1,\ \ |a_n| \leq C$, and for $n \geq \bar{n}$, $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \leq C$. Hence $\forall\, n \geq 1\colon\ |a_n| \leq C.$

Theorem 1.4.

If $\lim\limits_{n\to\infty} a_n = a$ and $b > a,$ then $\exists\, N \in \mathbb{N}\ \ \forall\, n > N:\ a_n < b.$

Proof.

According to definition of limit: for $\varepsilon = b - a > 0\ \exists\, N \in \mathbb{N}\ \ \forall\, n > N : |a_n - a| < \varepsilon$.

Then $a - \varepsilon < a_n < a + \varepsilon = b \Rightarrow a_n < b$.

Theorem 1.5.

Let $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, and $\forall\, n > N\ \ a_n \leq b_n$, then $a \leq b$.

Proof.

It is given that $b_n - a_n \geq 0$. Evidently $\lim\limits_{n\to\infty}(b_n - a_n) \geq 0$ and

$\lim\limits_{n\to\infty}(b_n - a_n) = \lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n = b - a \geq 0 \Rightarrow a \leq b$.

Theorem 1.6.

Let $\forall\, n > N:\ a_n \leq c_n \leq b_n$ and $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = a$, then $\lim\limits_{n\to\infty} c_n = a$.

Proof.

Let an arbitrary $\varepsilon > 0$ be given.

According to definition of limit

$$\exists\, N_1 \in \mathbb{N}\ \ \forall\, n > N_1 : |a_n - a| < \varepsilon;$$

$$\exists\, N_2 \in \mathbb{N}\ \ \forall\, n > N_2 : |b_n - a| < \varepsilon.$$

Then for $\forall\, n > \max(N_1, N_2)$:

$$a-\varepsilon < a_n \le c_n \le b_n < a+\varepsilon \Rightarrow a-\varepsilon < c_n < a+\varepsilon \Rightarrow |c_n - a| < \varepsilon,$$

This means $\lim\limits_{n\to\infty} c_n = a$.

Theorem 1.7.

If the sequence $\{a_n,\ n\in\mathbb{N}\}$ is bounded and monotone, then $\exists\, a\in\mathbb{R}:\ \lim\limits_{n\to\infty} a_n = a.$

Proof.

Let $\{a_n,\ n\in\mathbb{N}\}$ be monotone increasing upper-bounded sequence and $a\in\mathbb{R}$ be the least upper bound of the sequence. It means that

- all terms of satisfy the condition $a_n \le a$;

- $\forall\varepsilon > 0$, $a-\varepsilon$ is not an upper bound of sequence $\{a_n,\ n\in\mathbb{N}\}$.

Therefore, $\exists\, N\in\mathbb{N}:\ a-\varepsilon < a_N$.

Since $\{a_n,\ n\in\mathbb{N}\}$ is monotone increasing, $a_N \le a_{N+1} \le a_{N+2} \le \ldots$ and $\forall\, n > N:\ a-\varepsilon < a_n \le a$.

Hence, $\lim\limits_{n\to\infty} a_n = a.$

IV. ***Indeterminate Forms***

The limits can be calculated using theorem 1.2. But sometimes usage of this method is impossible.

For example, $\lim\limits_{n\to\infty}\dfrac{2n-1}{3n+1} \Rightarrow \dfrac{(2n-1)\to\infty}{(3n+1)\to\infty} \Rightarrow \dfrac{\infty}{\infty}$.

An expression $\dfrac{\infty}{\infty}$ is an indeterminate form, and evaluating the limit requires a special method:

$$\lim_{n\to\infty}\frac{2n-1}{3n+1} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty}\frac{\cancel{n}\left(2-\overset{\nearrow 0}{\frac{1}{n}}\right)}{\cancel{n}\left(3+\underset{\searrow 0}{\frac{1}{n}}\right)} = \frac{2}{3}.$$

There are several types of indeterminate forms for sequences:

$$\frac{\infty}{\infty},\ \ \infty-\infty,\ \ 0\cdot\infty,\ \ 1^{\infty}.$$

Every indeterminate form has its own special method for computing. Most of these methods are based on transformation of mathematical expressions and the most important limits such as

$$\lim_{n\to\infty} n^m = \begin{cases} 0, & m<0, \\ \infty, & m>0; \end{cases} \qquad \lim_{n\to\infty} a^n = \begin{cases} 0, & 0<a<1, \\ \infty, & a>1; \end{cases} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \qquad (1.1)$$

Examples.

1. $\lim\limits_{n\to\infty}\frac{n^2-1}{3n+1}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{n\to\infty}\frac{n^2\left(1-\frac{1}{n^2}\right)}{n\left(3+\frac{1}{n}\right)}=\lim\limits_{n\to\infty}\frac{n}{3}=\infty;$

2. $\lim\limits_{n\to\infty}\frac{n-1}{3n^2+1}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{n\to\infty}\frac{n\left(1-\frac{1}{n}\right)}{n^2\left(3+\frac{1}{n^2}\right)}=\lim\limits_{n\to\infty}\frac{1}{3n}=0;$

3. $\lim\limits_{n\to\infty}\frac{2^n-1}{2^{n+1}+1}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{n\to\infty}\frac{2^n\left(1-\frac{1}{2^n}\right)}{2^n\left(2+\frac{1}{2^n}\right)}=\frac{1}{2};$

4. $\lim\limits_{n\to\infty}(n-\sqrt{n^2+1})=[\infty-\infty]=\lim\limits_{n\to\infty}\frac{(n-\sqrt{n^2+1})(n+\sqrt{n^2+1})}{(n+\sqrt{n^2+1})}=\lim\limits_{n\to\infty}\frac{n^2-n^2-1}{(n+\sqrt{n^2+1})}=$

$=\lim\limits_{n\to\infty}\frac{-1}{(n+\sqrt{n^2+1})}=0.$

1.2 The Limit of a Function

I. ***The Limit of a Function***

Consider the function $y = f(x)$, $x \in D(f) \subset \mathbb{R}$, and the point $x_0 \in D(f) \subset \mathbb{R}$.

Definition.

The limit of the function $f(x)$ as x tends to x_0, is $a \in \mathbb{R}$ $(\lim\limits_{x\to x_0} f(x) = a)$ if for every $\varepsilon > 0$, no matter how small, it exists $\delta > 0$ such that $\forall\, x \neq x_0$, $|x - x_0| < \delta$ we have inequality $|f(x) - a| < \varepsilon$ (Fig. 3).

$(\ \forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \neq x_0,\ |x - x_0| < \delta : |f(x) - a| < \varepsilon.)$

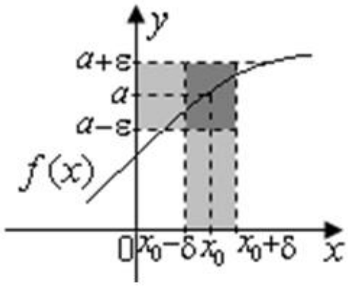


Figure 3.

For example, let consider the function $f(x) = x^2 - 9$ at the point $x_0 = 3$. It is easy to see that y approaches 0 as x becomes closer to 3 (from both the left and the right sides) (Fig. 4):

$$\lim_{x\to 3}(x^2 - 9) = 0.$$

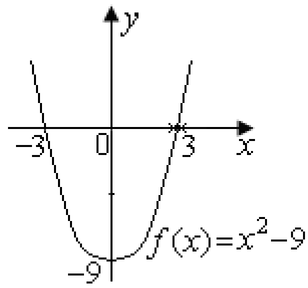


Figure 4.

Saying informally, the limit of function $f(x)$ at the point x_0 $(\lim\limits_{x\to x_0} f(x))$ is the value of y that the function approaches as its argument approaches x_0.

For the function to have a limit as x tends to x_0, it is not necessary that the function be defined at the point $x = x_0$. When finding the limit, we consider the values of the function at the points near the point x_0 that are different from x_0.

For example, the function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not defined for $x_0 = 3$ (Fig. 5) but

$$\lim_{x\to 3}\frac{x^2 - 9}{x - 3} = \lim_{x\to 3}\frac{(x+3)(x-3)}{x - 3} = \lim_{x\to 3}(x+3) = 6.$$

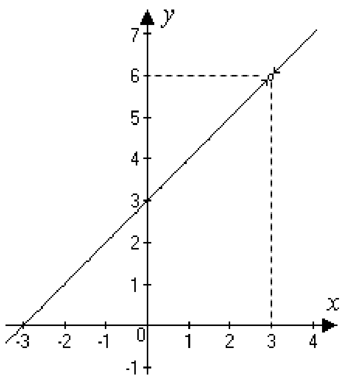


Figure 5.

Definition.

The limit of the function $f(x)$ as x tends to ∞, is $a \in \mathbb{R}$ $(\lim\limits_{x\to\infty} f(x) = a)$ if

$$\forall\, \varepsilon > 0 \;\; \exists C \in \mathbb{R} \;\; \forall\, x > C: \; \left| f(x) - a \right| < \varepsilon.$$

If $\lim\limits_{x\to x_0} f(x) = 0$, then the function $f(x)$ is *infinitely small* (*infinitesimal*) as x approaches to x_0.

Examples.

$\lim\limits_{x\to 0} x = 0;$ $\quad \lim\limits_{x\to 2}(x^2 - 4) = 0;$ $\quad \lim\limits_{x\to\infty} \dfrac{1}{x} = 0;$

$\lim\limits_{x\to 0} \sin x = 0;$ $\quad \lim\limits_{x\to 0}(1 - \cos x) = 0;$ $\quad \lim\limits_{x\to \frac{\pi}{2}} \cos x = 0;$

$\lim\limits_{x\to 0}(e^x - 1) = 0;$ $\quad \lim\limits_{x\to 1} \ln x = 0;$ $\quad \lim\limits_{x\to 0} \ln(x+1) = 0.$

If $\lim\limits_{x\to x_0} f(x) = \infty$, then the function $f(x)$ is *infinitely large* as x tends to x_0.

Examples.

$\lim\limits_{x\to 0} \dfrac{1}{x} = \pm\infty;$ $\quad \lim\limits_{x\to\infty} x^2 = +\infty;$ $\quad \lim\limits_{x\to+\infty} e^x = +\infty;$

$\lim\limits_{x\to 0} \cot x = \pm\infty;$ $\quad \lim\limits_{x\to \frac{\pi}{2}} \tan x = \pm\infty;$ $\quad \lim\limits_{x\to+\infty} \ln x = +\infty;$

1.3 *Infinitesimals and Their Properties*

If $\lim\limits_{x\to x_0} \alpha(x) = 0$, then the function $\alpha(x)$ is *infinitely small* (*infinitesimal*) as x approaches to x_0.

If $\lim\limits_{x\to x_0} f(x) = a$, then $\lim\limits_{x\to x_0} (f(x) - a) = 0$ and $f(x) - a$ is infinitesimal as $x \to x_0$.

Lemma 1.2.

The function $f(x)$ could be expressed as $f(x) = a + \alpha(x)$, where $\alpha(x)$ is infinitesimal as $x \to x_0$.

Properties of infinitesimals

Let $\alpha(x)$ and $\beta(x)$ are infinitesimals as $x \to x_0$, then

a) $\alpha(x)+\beta(x)$ is infinitesimal as $x \to x_0$;

b) if $f(x)$ is bounded as $x \to x_0$ then $\alpha(x)\cdot f(x)$ is infinitesimal as $x \to x_0$;

c) $\alpha(x)\cdot\beta(x)$ is infinitesimal as $x \to x_0$;

d) $\forall\, c \in \mathbb{R}$, $c\alpha(x)$ is infinitesimal as $x \to x_0$;

e) $\dfrac{1}{\alpha(x)}$ is infinitely large function.

Classification of infinitesimals

Let $\alpha(x)$ and $\beta(x)$ be two infinitesimal functions as $x \to x_0$.

1. If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = 0$, then $\alpha(x)$ has *a higher order of smallness* with respect to $\beta(x)$ as $x \to x_0$, $\alpha = o(\beta)$.

2. If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = 1$, then $\alpha(x)$ and $\beta(x)$ are equivalent as $x \to x_0$, $\alpha \sim \beta$.

3. If $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{\beta(x)} = c$, $c \neq 0$, $c \neq 1$, then $\alpha(x)$ and $\beta(x)$ have *the same order of smallness* as $x \to x_0$.

1.4 *Basic Theorems About Limits of Functions*

Theorem 1.8.

If $\lim\limits_{x\to x_0} f(x) = a$ and $a < C$, then $\exists\, \delta > 0\ \forall\, x \neq x_0,\ |x - x_0| < \delta:\ f(x) < C.$

Proof.

According to definition of limit of function: for given $\varepsilon = C - a > 0$,

$$\exists\, \delta > 0\ \forall\, x \neq x_0,\ |x - x_0| < \delta:\ |\, f(x) - a \,| < \varepsilon \Rightarrow f(x) < \varepsilon + a = C.$$

Theorem 1.9.

Let $f, g : D \to \mathbb{R}$ and $\forall\, x \in D:\ f(x) \le g(x)$.

If $\lim\limits_{x\to x_0} f(x) = a$ and $\lim\limits_{x\to x_0} g(x) = b$, then $a \le b$.

Proof.

It is given that $f(x) - g(x) \ge 0$. Evidently

$$\lim_{x\to x_0} (f(x) - g(x)) \ge 0$$

and

$$\lim_{x\to x_0} (f(x) - g(x)) = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = b - a \ge 0 \Rightarrow a \le b .$$

Theorem 1.10.

Let $f, g, h : D \to \mathbb{R}$ and $\forall\, x \in D:\ f(x) \le h(x) \le g(x)$.

If $\lim\limits_{x\to x_0} f(x) = a$ and $\lim\limits_{x\to x_0} g(x) = a$, then $\lim\limits_{x\to x_0} h(x) = a$.

Proof.

From the inequality $f(x) \le h(x) \le g(x)$ follow inequalities

$$f(x) - a \le h(x) - a \le g(x) - a .$$

Let an arbitrary $\varepsilon > 0$ be given. According to definition of limit

$$\exists\, \delta_1 > 0\ \ \forall\, x \ne x_0,\ |x - x_0| < \delta_1 :\ |f(x) - a| < \varepsilon;$$

$$\exists\, \delta_2 > 0\ \ \forall\, x \ne x_0,\ |x - x_0| < \delta_2 :\ |g(x) - a| < \varepsilon .$$

Then for $\delta = \max(\delta_1, \delta_2)$ $\forall\, x \ne x_0,\ |x - x_0| < \delta :$

$$-\varepsilon < f(x) - a < \varepsilon$$

and

$$-\varepsilon < g(x) - a < \varepsilon$$

and thus the inequalities

$$-\varepsilon < h(x) - a < \varepsilon$$

will be fulfilled.

This means $\lim\limits_{n\to\infty} c_n = a$.

Theorem 1.11.

Let $\lim\limits_{x\to x_0} f(x)=a$ and $\lim\limits_{x\to x_0} g(x)=b$, then

a) $\lim\limits_{x\to x_0} cf(x)=ca,\quad c\in\mathbb{R}$;

b) $\lim\limits_{x\to x_0} (f(x)\pm g(x))=\lim\limits_{x\to x_0} f(x)\pm \lim\limits_{x\to x_0} g(x)=a\pm b$;

c) $\lim\limits_{x\to x_0} (f(x)\cdot g(x))=\lim\limits_{x\to x_0} f(x)\cdot \lim\limits_{x\to x_0} g(x)=a\cdot b$;

d) $\lim\limits_{x\to x_0}\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}=\dfrac{a}{b}$, if $b\neq 0$.

Proof.

Proving this theorem is similar to that for theorem 1.2. We suggest to prove it on their own.

In the following two sections we shall consider the limits of two functions that find wide practical application.

1.5 *The Limit of the Function* $\dfrac{\sin x}{x}$ *as* $x\to 0$.

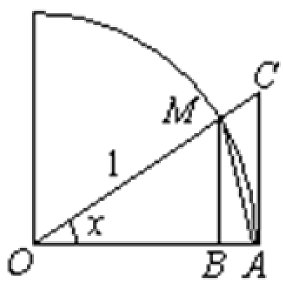


Figure 6.

This function is not defined at point $x=0$. Let us find the limit as $x\to 0$.

Let x be the central angle MOB $\left(0<x<\dfrac{\pi}{2}\right)$ of the unit circle. Note that function $\dfrac{\sin x}{x}$ is even, that's why we consider only the case of positive values of x.

Compare areas of triangle *OBM*, of sector *OAM* and triangle *OAC* (Fig. 6):

area of triangle *OBM* < area of sector *OAM* < area of triangle *OAC*

$$\frac{1}{2}OA\cdot MB<\frac{1}{2}OA\cdot \overset{\frown}{AM}<\frac{1}{2}OA\cdot AC\Rightarrow \frac{1}{2}\cdot 1\cdot \sin x<\frac{1}{2}\cdot 1\cdot x<\frac{1}{2}\cdot 1\cdot \tan x\Rightarrow$$

$$\sin x<x<\tan x.$$

Divide all terms by $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or

$$\cos x < \frac{\sin x}{x} < 1.$$

As $x \to 0$ we obtain $\lim\limits_{x\to 0} \cos x = 1$ and $\lim\limits_{x\to 0} 1 = 1$. Hence, according to Theorem 1.10

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{1.2}$$

The graph of the function $y = \frac{\sin x}{x}$ is shown in figure 7.

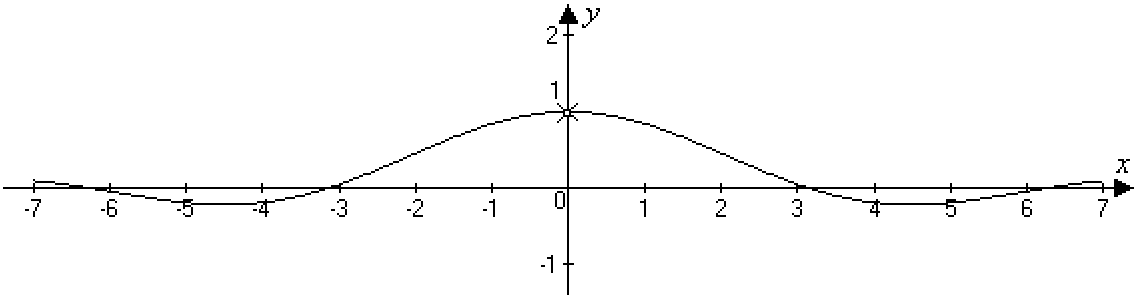


Figure 7.

Corollary:

$$\lim_{x\to 0}\frac{\tan x}{x} = 1; \quad \lim_{x\to 0}\frac{\arcsin x}{x} = 1; \quad \lim_{x\to 0}\frac{\arctan x}{x} = 1;$$

$$\lim_{x\to 0}\frac{x}{\sin x} = 1; \quad \lim_{x\to 0}\frac{x}{\tan x} = 1; \quad \lim_{x\to 0}\frac{x}{\arcsin x} = 1; \quad \lim_{x\to 0}\frac{x}{\arctan x} = 1.$$

1.6 *Number e*

Theorem 1.12.

The sequence $\left\{\left(1+\frac{1}{n}\right)^n, n \in \mathbb{N}\right\}$ is monotone increasing bounded sequence and has a finite limit between the numbers 2 and 3 as $n \to \infty$.

Proof.

1. Let prove the sequence is monotone increasing.

By Newton's binomial formula we have

$$\left(1+\frac{1}{n}\right)^n=1+\frac{n}{1}\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\left(\frac{1}{n}\right)^3+\ldots$$
$$\ldots+\frac{n(n-1)(n-2)\ldots(n-(n-1))}{1\cdot 2\cdot 3\cdot\ldots\cdot n}\cdot\left(\frac{1}{n}\right)^n.$$

Making simple transformation we obtain

$$a_n=\left(1+\frac{1}{n}\right)^n=1+1+\frac{n(n-1)}{n^2}\cdot\frac{1}{1\cdot 2}+\frac{n(n-1)(n-2)}{n^3}\cdot\frac{1}{1\cdot 2\cdot 3}+\ldots$$
$$\ldots+\frac{n(n-1)(n-2)\ldots(n-(n-1))}{n^n}\cdot\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot n}=$$
$$=1+1+\frac{1}{2!}\cdot\left(1-\frac{1}{n}\right)+\frac{1}{3!}\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)+\ldots+\frac{1}{n!}\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right).$$

Substituting $(n+1)$ for n we get the expression for the next term a_{n+1}:

$$a_{n+1}=\left(1+\frac{1}{n+1}\right)^{n+!}=1+1+\frac{1}{2!}\cdot\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)+\ldots$$
$$\ldots+\frac{1}{n!}\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n+1}\right)+$$
$$+\frac{1}{(n+1)!}\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n+1}\right)\cdot\left(1-\frac{n}{n+1}\right)$$

Comparing expressions for a_n and a_{n+1} we can conclude that

- each element of the sequence is positive;

- each term of a_{n+1} is greater than the corresponding term of the sum for a_n:

$$\left(1-\frac{1}{n}\right)<\left(1-\frac{1}{n+1}\right),\ \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)<\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right),$$ and so on;

- another term for a_{n+1} is added.

Thus, the sequence $\left\{\left(1+\frac{1}{n}\right)^n, n\in\mathbb{N}\right\}$ is monotone increasing.

2. Let prove the sequence is bounded.

From the expression for a_n it follows that $\left(1+\frac{1}{n}\right)^n\geq 2$.

Noting that $\left(1-\frac{1}{n}\right)<1$, $\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)<1$, etc., we obtain

$$a_n = \left(1+\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}.$$

Further noting that $\frac{1}{1\cdot 2\cdot 3} < \frac{1}{2^2}$; $\frac{1}{1\cdot 2\cdot 3\cdot 4} < \frac{1}{2^3}$; $\ldots$ $\frac{1}{1\cdot 2\cdot 3\cdot \ldots \cdot n} < \frac{1}{2^{n-1}}$,

we can write the inequality

$$a_n = \left(1+\frac{1}{n}\right)^n < 1+\underbrace{\left(1+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}\right)}_{\text{nth sum of geometric progression}} = 1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1+\left(2-\frac{1}{2^{n-1}}\right) < 3.$$

Consequently, $\forall n \in \mathbb{N}$

$$2 < \left(1+\frac{1}{n}\right)^n < 3.$$

This proves that the sequence is bounded.

The sequence $\left\{\left(1+\frac{1}{n}\right)^n, n \in \mathbb{N}\right\}$ is increasing and bounded, and according to theorem 1.7, it has a limit. This limit is denoted by the letter $\boldsymbol{e}$.

Thus

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e, \tag{1.3}$$
$$e = 2.7182818284\ldots$$

Theorem 1.13.

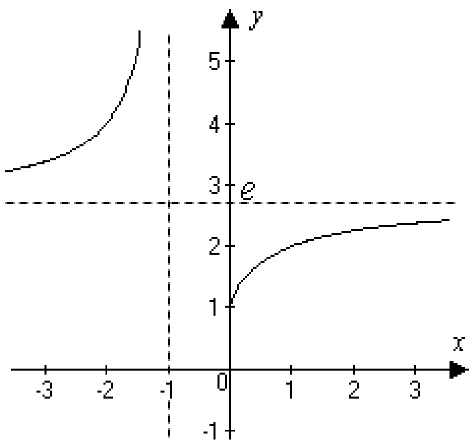


Figure 8.

The function $y = \left(1+\frac{1}{x}\right)^x$ approaches the limit $\boldsymbol{e}$ as x tends to ∞.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e. \tag{1.4}$$

The graph of the function is shown in Figure 8.

Proof. (See detailed proof in [1])

Theorem 1.14.

The function $y=(1+x)^{\frac{1}{x}}$ approaches the limit e as x tends to 0.

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e. \tag{1.5}$$

Proof.

If in the limit $\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t=e$ we make a substitution $x=\frac{1}{t}$, then as $t\to\infty$ we have $x\to 0$ and we get $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$

1.7 *Calculating the Limit at the Point*

1. If $f(x)$ is defined at the point x_0, then $\lim_{x\to x_0} f(x)=f(x_0).$

The limit of an expression involving addition, multiplication or division of functions can often be calculated by taking the limits of these functions separately (using theorem 1.11). However sometimes, usage of this method is impossible because common limit can not be determined from the limits of these functions. Such case is called *an indeterminate form*.

For example

$$\lim_{x\to 3}\frac{x^2-9}{x-3}\Rightarrow \begin{array}{c} f(x)=x^2-9\to 0,\ x\to 3 \\ g(x)=x-3\to 0,\ x\to 3 \end{array}\Rightarrow \frac{f(x)}{g(x)}\to\frac{0}{0},\ x\to 3$$

expression $\frac{0}{0}$ is not meaningful (an indeterminate form) and evaluating the limit requires a definite special method

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\left[\frac{0}{0}\right]=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

There are several types of indeterminate forms:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty-\infty,\ 0\cdot\infty,\ 1^{\infty},\ 0^0,\ \infty^0.$$

Every indeterminate form has its own special method for computing. Most of these methods are based on transformation of mathematical expressions and the most important limits:

$$\lim_{x\to0}\frac{\sin x}{x}=1; \qquad \lim_{x\to0}\frac{\tan x}{x}=1; \qquad \lim_{x\to0}\frac{\arcsin x}{x}=1;$$

$$\lim_{x\to0}\frac{\arctan x}{x}=1; \qquad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e; \qquad \lim_{x\to0}(1+x)^{\frac{1}{x}}=e;$$

$$\lim_{x\to0}\frac{e^x-1}{x}=1; \qquad \lim_{x\to0}\frac{\ln(1+x)}{x}=1; \qquad \lim_{x\to0}\frac{(1+x)^m-1}{x}=m.$$

2. The main idea of application equivalent infinitesimals to finding limits is replacing an infinitesimal by an equivalent one. It could simplify the expression.

The table of equivalent functions

$$z(x)\to0, \quad x\to x_0$$

$\sin z(x)\sim z(x)$	$\arcsin z(x)\sim z(x)$	$e^{z(x)}-1\sim z(x)$
$\tan z(x)\sim z(x)$	$\arctan z(x)\sim z(x)$	$a^{z(x)}-1\sim z(x)\ln a$
$\ln(1+z(x))\sim z(x)$	$\log_a(1+z(x))\sim\dfrac{z(x)}{\ln a}$	$(1+z(x))^m-1\sim m\,z(x)$

Example.

Calculate the limit

$$\lim_{x\to0}\frac{\sin 2x-\arctan^2 3x+e^x-1}{5\tan x^2-3\arcsin^2\sqrt{x}+\ln(1-5x)}=\left[\frac{0}{0}\right]=$$

$$\left|\begin{array}{lll}\sin 2x\sim 2x, & \arctan 3x\sim 3x, & e^x-1\sim x\\ \tan x^2\sim x^2, & \arcsin\sqrt{x}\sim\sqrt{x}, & \ln(1-5x)\sim -5x\end{array}\right|$$

$$=\lim_{x\to0}\frac{2x-(3x)^2+x}{5x^2-3(\sqrt{x})^2-5x}=\left[\frac{0}{0}\right]=\lim_{x\to0}\frac{-9x^2+3x}{5x^2-8x}=\lim_{x\to0}\frac{(-9x+3)x}{(5x-8)x}=-\frac{3}{8}.$$

1.8 Continuity of Functions

I. *One-Sided Limits*

The one-sided limit considers the values of function $f(x)$ when x approaches to x_0 from either the right or the left side.

The right side limit of the function $f(x)$ as x tends to x_0 from the right side (Fig. 9) is $a \in \mathbb{R}$

$$\lim_{x \to x_0+} f(x) = a$$

if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \neq x_0,\ x \in (x_0, x_0 + \delta):\ |f(x) - a| < \varepsilon.$$

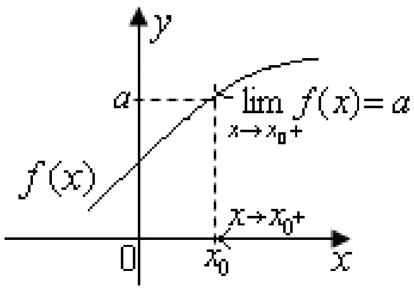


Figure 9.

The left side limit of the function $f(x)$ as x tends to x_0 from the left side (Fig. 10), is $a \in \mathbb{R}$

$$\lim_{x \to x_0-} f(x) = a$$

if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, x \neq x_0,\ x \in (x_0 - \delta, x_0):\ |f(x) - a| < \varepsilon.$$

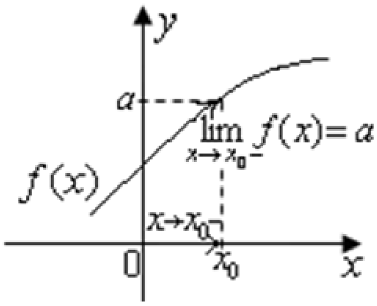


Figure 10.

II. *Continuity of Functions*

Let $y = f(x)$ be some function and $x_0 \in D(f)$.

Definition. The function $f(x)$ is called *continuous at the point* x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In other words the function $f(x)$ is called *continuous at the point* x_0 (Fig. 11) if and only if :

$$\lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x) = f(x_0) = \lim_{x \to x_0} f(x).$$

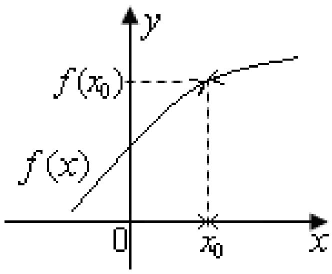


Figure 11.

Properties of Continuous at the Point Functions

Theorem 1.15.

If $f(x)$ and $g(x)$ are continuous at the point x_0 functions, then $f(x)+g(x)$, $f(x)\cdot g(x)$, $\dfrac{f(x)}{g(x)}$ (if $g(x_0)\neq 0$) are continuous at the point x_0 functions.

Proof.

The proof follows from the definition of the continuity of the function at the point and the theorem 1.11 on the properties of the limit of the function.

Theorem 1.16.

If $g(x)$ is continuous at x_0 and if $f(x)$ at $b=g(x_0)$, then

1) $\lim\limits_{x\to x_0} f(g(x)) = f(\lim\limits_{x\to x_0} g(x)) = f(b) = f(g(x_0))$;

2) the composite function $f(g(x))$ is continuous at the point x_0.

Theorem 1.17.

All elementary functions x^n, a^x, $\log_a x$, $\sin x$, $\cos x$, $\tan x$, $\arcsin x$, $\arccos x$, $\arctan x$ are continuous at each point at which it is defined.

In order to calculate the limit of a continuous function as $x\to x_0$ it is sufficient to substitute into expression of the function the value of the argument (x_0) and evaluate the value of y.

Definitions. The function $f(x)$ is called *continuous on the right at the point* x_0 if

$$\lim_{x\to x_0+} f(x) = f(x_0).$$

The function $f(x)$ is called *continuous on the left at the point* x_0 if

$$\lim_{x\to x_0-} f(x) = f(x_0).$$

The function $f(x)$ is called *continuous over the interval* $[a,b]$ if it is continuous at every point of this interval and if, in addition, $\lim\limits_{x\to a+} f(x) = f(a)$ and $\lim\limits_{x\to b-} f(x) = f(b)$.

Definition. If at some point $x = x_0$, at least one of the conditions of continuity is not fulfilled for the function $y = f(x)$ ($\lim\limits_{x \to x_0} f(x) = \pm\infty$ or it does not exist) then the function $f(x)$ is called *discontinuous at the point* x_0. Such points are called *the points of discontinuity*.

II. ***Classification of the Points of Discontinuity***

1. If $f(x_0)$ does not exist and $\lim\limits_{x \to x_0+} f(x) = a \in \mathbb{R}$, $\lim\limits_{x \to x_0-} f(x) = b \in \mathbb{R}$ and $a \neq b$ then point x_0 is *the point of ordinary discontinuity or the points of discontinuity of the first kind* (*jump*) (Fig. 12).

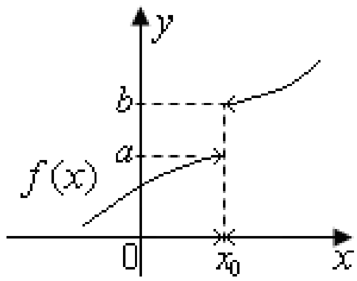


Figure 12.

2. If $f(x_0)$ does not exist and $\lim\limits_{x \to x_0+} f(x) = \lim\limits_{x \to x_0-} f(x) = a \in \mathbb{R}$, then point x_0 is *the point of discontinuity of the first kind* (*removable)* (Fig. 13).

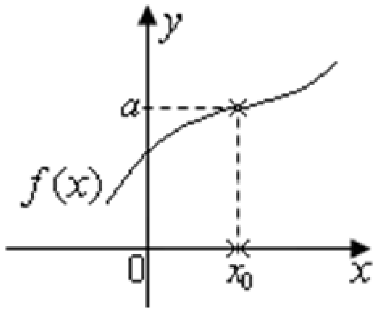


Figure 13.

3. If at least one of the limits $\lim\limits_{x \to x_0+} f(x)$ and $\lim\limits_{x \to x_0-} f(x)$ is $\pm\infty$ or does not exist, then point x_0 is *the point of nonremovable discontinuity or the points of discontinuity of the second kind* (Fig. 14).

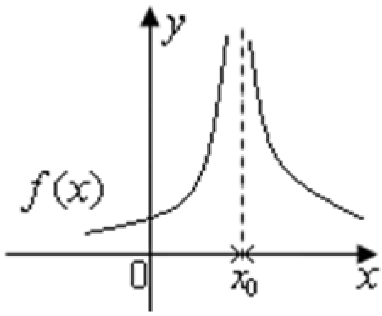


Figure 14.

1.9 *Certain Properties of Continuous Functions*

Theorem 1.18.

If $y = f(x)$ is continuous over the interval $[a,b]$, $a \le x \le b$, there exists at least one point x^* such that $\forall x \in [a,b]$: $f(x^*) \ge f(x)$ and there exists at least one point x_* such that $\forall x \in [a,b]$: $f(x_*) \le f(x)$.

We call the value $f(x^*)$ *the greatest value* of the function $y = f(x)$ on the interval $[a,b]$ $\left(\max\limits_{[a,b]} f(x) = f(x^*)\right)$, and the value $f(x_*)$ *the smallest value* of the function $y = f(x)$ on the interval $[a,b]$ $\left(\min\limits_{[a,b]} f(x) = f(x_*)\right)$. The meaning of this theorem is illustrated in Fig. 15.

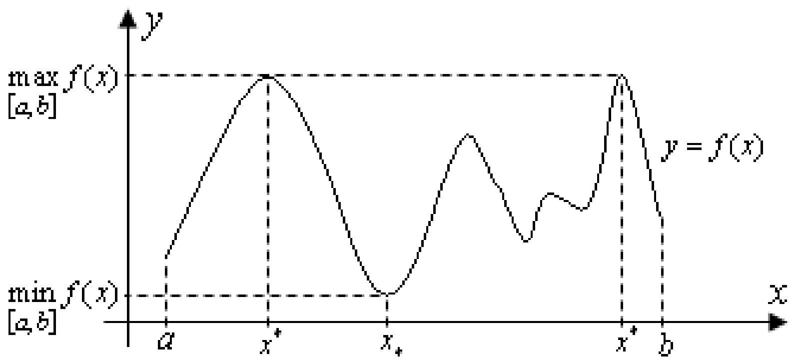


Figure 15.

Theorem 1.19.

If $y = f(x)$ is continuous over the interval $[a,b]$, $a \le x \le b$ and one of the following inequalities is fulfilled $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there exists at least one point x_0 such that $f(x_0) = 0$.

The geometrical meaning of this theorem is shown in Fig. 16. The graph of a continuous function $y = f(x)$ joining the points $(a, f(a))$ and $(b, f(b))$, where $f(a) < 0$ and $f(b) > 0$, cuts the x-axis at one point.

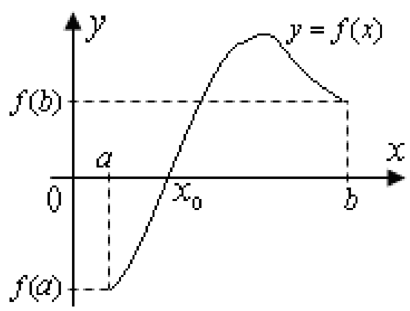


Figure 16.

Theorem 1.20.

If $y = f(x)$ is continuous over the interval $[a,b]$, $a \le x \le b$ and $f(a) \ne f(b)$, then no matter what the number C between numbers $f(a)$ and $f(b)$, there exists a point $x = c$, $a < c < b$ such that $f(c) = C$.

The meaning of this theorem is illustrated in Fig. 17.

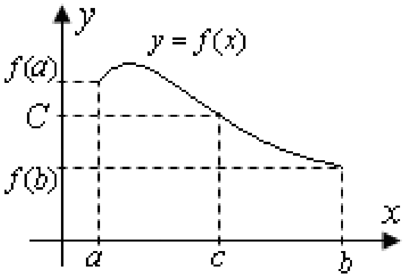


Figure 17.

Corollary of Theorem 1.20.

If $y = f(x)$ is continuous over the interval $[a,b]$, $a \le x \le b$ and takes on a greatest value and a smallest value, then in this interval it takes on, at least once, any value lying between the greatest and smallest values.

Theorem 1.21.

If a monotone function $y = f(x)$ is continuous on the interval $[a,b]$, where $f(a) = c$, $f(b) = d$, then the inverse function $x = g(y)$ is defined and is continuous on the interval $[c,d]$.

If the functions $y = f(x)$ and $x = g(y)$ are reciprocal, their graphs are represented by the single curve. But if we denote the argument of the inverse function by x and the function by y, then, constructing them in the single coordinate system, we get two different graphs. The graphs are symmetric about the bisector of the first quadrant's angle.

Example.

Given the function $y = e^x$. This function is increasing on $(-\infty, +\infty)$. It has an inverse function $x = \ln y$. The domain of inverse function is $(0, +\infty)$. (Fig.18)

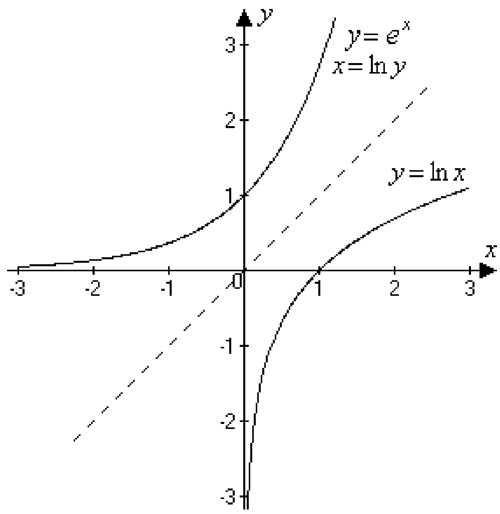


Figure 18.

2. Differential Calculus of a Function of one variable.

2.1 *Definition of Derivative.*

Consider the function $y = f(x)$ determined in some interval. If argument x increases by Δx, then the function $f(x)$ has the growth $\Delta y = f(x+\Delta x) - f(x)$ on the interval $[x, x+\Delta x]$ (Fig. 19).

The average rate of change of the function $y = f(x)$ over the interval $[x, x+\Delta x]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

It is also known as the *difference quotient*.

The average rate of change does not tell us about the function's behavior between points x and $x+\Delta x$,

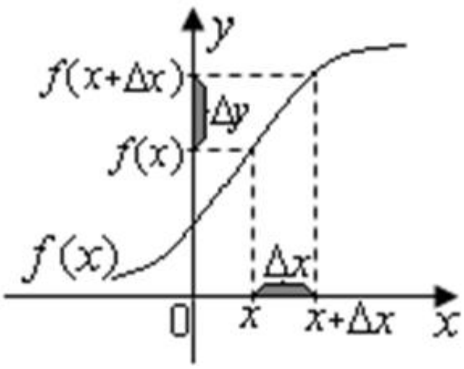


Figure 19.

however, if we make Δx small enough, the average rate will be more precise.

If $\Delta x \to 0$, then $x + \Delta x \to x$ and the rate of change becomes instantaneous.

The instantaneous rate of change of the function $y = f(x)$ is called derivative of $f(x)$ with respect to x and denoted by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}. \qquad (2.1)$$

Another notation: $\dfrac{df}{dx}, \ \dfrac{d}{dx} f(x), \ Df(x)$.

The derivative at $x = x_0$ can be expressed in such way:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \qquad (2.2)$$

In fact, the derivative is a number (the value of limit) for the given point. For some points, the limit (2.1) does not exist. In this case, the derivative does not exist too.

If the limit (2.1) exists, then function $f(x)$ is *differentiable* at the point *x*. If the function $f(x)$ is *differentiable* at every point of interval [*a*, *b*], then $f'(x)$ exists in every point of interval $[a, b]$ and $f'(x)$ is defined as the function on the interval [*a*, *b*].

The operation of finding the derivative of a function $f(x)$ is called *differentiation* of function.

The designation $f'(x)$ (f prime of x) is not the only one used for a derivative. Alternative symbols are $\dfrac{dy}{dx}$, y', y'_x, Dy.

Theorem 2.1

If a function $f(x)$ is differentiable at point $x = x_0$, then it is continuous at this point.

Proof.

If $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = f'(x)$ then $\dfrac{\Delta y}{\Delta x} = f'(x) + \alpha(x)$ where $\alpha(x)$ is an infinitesimal as $\Delta x \to 0$.

Hence $\Delta y = f'(x)\Delta x + \alpha(x)\Delta x$ and it follows that $\Delta y \to 0$ as $\Delta x \to 0$. This means that $f(x)$ is continuous function at the point $x = x_0$.

In other words, a function cannot have a derivative at points of discontinuity. The converse is not true.

II. *Geometric Interpretation of Derivative*

Consider the graph of the function $y = f(x)$.

The straight line that goes thought two points M and M_0 on the graph is called *the secant line* (Fig. 20). The position of the secant line is determined by $\tan\varphi$ – slope of the line:

$$\tan\varphi = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \ \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

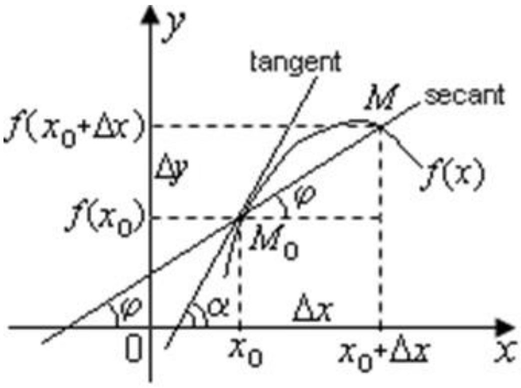


Figure 20.

Assume that the point M_0 moves along the curve to the point M. The limit position of the secant line MM_0 as the distance between the two points goes to zero and is called *the tangent line* and the slope of the tangent line is

$$\tan\alpha = \lim_{M \to M_0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x).$$

The equation of the tangent line for the function $y = f(x)$ at the point x_0 is:

$$y = f'(x_0)(x - x_0) + f(x_0). \tag{2.3}$$

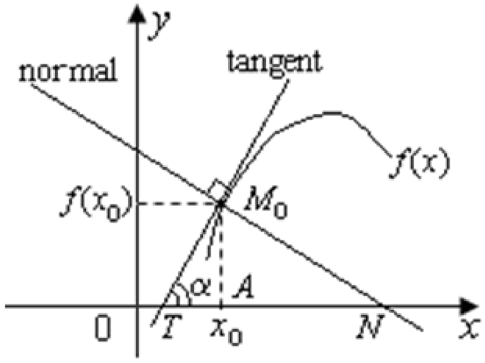


Figure 21.

A straight line passing through the point M_0 perpendicularly to the tangent line is called *the normal line to the curve* (Fig. 21). The equation of the normal line is

$$y = -\frac{1}{f'(x_0)}(x - x_0) + f(x_0), \tag{2.4}$$

$$f'(x_0) \neq 0.$$

The segments *AT*, *AN* are called the subtangent and subnormal, respectively. The lengths of the indicated segments can be calculated by formulas

$$AT = \left|\frac{f(x_0)}{f'(x_0)}\right|; \quad AN = \left|f(x_0)\, f'(x_0)\right|.$$

III. ***Physical Interpretation of Derivative***

According to definition of derivative, if the function $f(x)$ describes a definite physical process, then the derivative shows the rate of changes in this process. This makes it very useful for solving physics problems.

For example, if $s(t)$ describes the position of a moving particle at the time t (*motion*), then $\frac{ds}{dt}$ is the velocity of the particle at the time t.

If q is the amount of electric charge, the derivative $\frac{dq}{dt}$ is the change in that charge over time, or the electric current.

2.2 *Derivatives of Basic Elementary Functions*

$C' = 0 \ \ \forall C \in \mathbb{R};$	$(x)' = 1;$
$(x^n)' = nx^{n-1};$ $\left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2};$	$\left(\sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}};$
$(e^x)' = e^x;$	$(a^x)' = a^x \ln a;$
$(\ln x)' = \dfrac{1}{x};$	$(\log_a x)' = \dfrac{1}{x \ln a};$
$(\sin x)' = \cos x;$	$(\cos x)' = -\sin x;$
$(\tan x)' = \dfrac{1}{\cos^2 x};$	$(\cot x)' = -\dfrac{1}{\sin^2 x};$
$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}};$	$(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}};$
$(\arctan x)' = \dfrac{1}{1+x^2};$	$(\text{arccot} x)' = -\dfrac{1}{1+x^2};$
$(\sinh x)' = \cosh x;$	$(\cosh x)' = \sinh x;$
$(\tanh x)' = \dfrac{1}{\cosh^2 x};$	$(\coth x)' = -\dfrac{1}{\sinh^2 x};$

Let prove some formulas:

1. $$(x^n)' = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - (x)^n}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^n\left(\left(1+\frac{\Delta x}{x}\right)^n - 1\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^{n-1}\left(n\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = nx^{n-1}.$$

2. $$(e^x)' = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x \Delta x}{\Delta x} = e^x.$$

3. $$(\sin x)' = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x}{2}\cos x}{\frac{\Delta x}{2}} = \cos x.$$

2.3 ***Basic Rules of Differentiation.***

To compute derivatives using the definition means to compute the limits. For elementary functions it is not complicated but for more complex functions using the definition of the derivative would be an almost impossible task.

However we have a lot of formulas and properties that we can use to simplify the operation of differentiation.

Theorem 2.2.

Let the functions $f(x)$ and $g(x)$ be differentiable $\left(\exists\, f',\ g'\right)$ at the point x_0.

1. $\forall C \in \mathbb{R}\ \ (C\cdot f)' = C\cdot f'$;

2. $(f+g)' = f'+g'$;

3. $(f\cdot g)' = f'\cdot g + f\cdot g'$;

4. if $g(x)\neq 0$, $\left(\dfrac{f}{\mathrm{g}}\right)' = \dfrac{f'\cdot g - f\cdot g'}{g^2}$.

Proof:

If the functions $f(x)$ and $g(x)$ are differentiable at the point x_0, then

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \quad\text{and}\quad g'(x_0)=\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}.$$

1. $$(Cf)'(x_0)=\lim_{x\to x_0}\frac{Cf(x)-Cf(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{C(f(x)-f(x_0))}{x-x_0}=Cf'(x_0).$$

2. $$(f+g)'(x_0)=\lim_{x\to x_0}\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{(f(x)+g(x))-(f(x_0)+g(x_0))}{x-x_0}=$$

$$=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=f'(x_0)+g'(x_0).$$

3. $$(f\cdot g)'(x_0)=\lim_{x\to x_0}\frac{(f\cdot g)(x)-(f\cdot g)(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)\cdot g(x)-f(x_0)\cdot g(x_0)}{x-x_0}=$$

$$=\lim_{x\to x_0}\frac{f(x)\cdot g(x)-f(x_0)\cdot g(x)+f(x_0)\cdot g(x)-f(x_0)\cdot g(x_0)}{x-x_0}=$$

$$=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\cdot g(x)+\lim_{x\to x_0}f(x)\cdot\frac{g(x)-g(x_0)}{x-x_0}=f'(x_0)\cdot g(x_0)+f(x_0)\cdot g'(x_0).$$

4. Let $g(x_0) \neq 0$.

$$\left(\frac{f}{\mathrm{g}}\right)'(x_0) = \lim_{x \to x_0} \frac{1}{x - x_0}\left(\frac{f(x)}{\mathrm{g}(x)} - \frac{f(x_0)}{\mathrm{g}(x_0)}\right) = \lim_{x \to x_0} \frac{1}{x - x_0} \cdot \frac{f(x)\cdot g(x_0) - f(x_0)\cdot g(x)}{\mathrm{g}(x)\cdot \mathrm{g}(x_0)} =$$

$$= \lim_{x \to x_0} \frac{1}{x - x_0} \cdot \frac{f(x)\cdot g(x_0) - f(x_0)\cdot g(x_0) - f(x_0)\cdot g(x) + f(x_0)\cdot g(x_0)}{\mathrm{g}(x)\cdot \mathrm{g}(x_0)} =$$

$$= \lim_{x \to x_0} \frac{1}{\mathrm{g}(x)\cdot \mathrm{g}(x_0)}\left(\frac{f(x) - f(x_0)}{x - x_0}\cdot g(x_0) - f(x_0)\cdot \frac{g(x) - g(x_0)}{x - x_0}\right) =$$

$$= \frac{f'(x_0)\cdot g(x_0) - f(x_0)\cdot g'(x_0)}{g^2(x_0)}.$$

Examples.

1. $$\left(x^5 + \sqrt[4]{x^7} - \frac{2}{x^5}\right)' = \left(x^5\right)' + \left(x^{\frac{7}{4}}\right)' - 2\left(x^{-5}\right)' = 5x^{5-1} + \frac{7}{4}x^{\frac{7}{4}-1} - 2\cdot(-5)x^{-5-1} =$$

$$= 5x^4 + \frac{7}{4}x^{\frac{3}{4}} + 10x^{-6} = 5x^4 + \frac{7}{4}\sqrt[4]{x^3} + \frac{10}{x^6};$$

2. $\left(e^x \cos x\right)' = (e^x)'\cos x + e^x(\cos x)' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x);$

3. $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

4. $$\left(\frac{\ln x}{\sin x}\right)' = \frac{(\ln x)'\sin x - \ln x(\sin x)'}{\sin^2 x} = \frac{\frac{1}{x}\sin x - \ln x\cos x}{\sin^2 x} = \frac{\sin x - x\ln x\cos x}{x\sin^2 x}.$$

Theorem 2.3.

Let the functions $y = f(x)$ be differentiable at the point x_0 $\left(\exists\ f'(x_0)\right)$ and there exists inverse function $x = g(y)$ such that it is continuous at the point y_0, $y_0 = f(x_0)$. Then there exists

$$g'(y_0) = \frac{1}{f'(x_0)}. \tag{2.5}$$

Example.

Given the function $y = \arcsin x$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $x \in [-1,1]$. Let prove the table formula $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ using theorem 2.3.

The inverse function is $x = \sin y$. Then $x'_y = (\sin y)'_y = \cos y$. By the rule for differentiating an inverse function,

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y}.$$

Since $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$, we have

$$y'_x = \frac{1}{\sqrt{1-x^2}}.$$

The sign in front of the radical is plus because the function $y = \arcsin x$ takes on values in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and, consequently, $\cos y > 0$.

VI. *Derivative of a Composite Function*

Given a composite function $y = y(x)$, that is, such that it may be represented in the following form:

$$y = f(u),\ u = g(x) \text{ or } y = f(g(x)).$$

In the expression $y = f(u)$, u is called the *intermediate argument*.

Theorem 2.4 (***Chain Rule).***

Let the functions $f(x)$ and $g(x)$ be differentiable at the point *x*.

$$y'_x = \left(f(g(x))\right)'_x = f'_g(g(x)) \cdot g'_x(x). \tag{2.6}$$

Proof.

For the increased value of argument $x + \Delta x$,

$$u + \Delta u = g(x + \Delta x), \qquad y + \Delta y = y(u + \Delta u).$$

Thus, to the increment Δx there corresponds an increment Δu, to which corresponds an increment Δy, whereby $\Delta u \to 0$ and $\Delta y \to 0$ as $\Delta x \to 0$. It is given that

$$y'_u(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}.$$

From this relation we get $\dfrac{\Delta y}{\Delta u} = y'_u + \alpha$ (for $\Delta u \neq 0$), where $\alpha \to 0$ as $\Delta u \to 0$. Then

$$\Delta y = y'_u \Delta u + \alpha \Delta u .$$

This equality also holds true when $\Delta u = 0$ for any arbitrary α, since it turns into identity, $0 \equiv 0$. For $\Delta u = 0$ we shall assume $\alpha = 0$. Divide all terms by Δx:

$$\frac{\Delta y}{\Delta x} = y'_u \frac{\Delta u}{\Delta x} + \alpha \frac{\Delta u}{\Delta x}.$$

It is given that

$$u'_x(x) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}, \qquad \lim_{\Delta x \to 0} \alpha = 0.$$

Passing to the limit as $\Delta x \to 0$, we get $y'_x = y'_u u'_x = f'_g(g(x)) \cdot g'_x(x)$.

Examples:

1. $\left(\sin(x^3 + x)\right)' = \cos(x^3 + x) \cdot (x^3 + x)' = \cos(x^3 + x) \cdot (3x^2 + 1);$
2. $\left(\sqrt{x + e^x}\right)' = \dfrac{1}{2\sqrt{x + e^x}} \cdot (x + e^x)' = \dfrac{1 + e^x}{2\sqrt{x + e^x}};$
3. $\left(\ln^4 \cos x\right)' = 4\ln^3 \cos x \cdot (\ln \cos x)' = 4\ln^3 \cos x \cdot \dfrac{1}{\cos x} \cdot (\cos x)' = 4\ln^3 \cos x \cdot \dfrac{\sin x}{\cos x} =$

   $= 4 \tan x \ln^3 \cos x$.

2.4 *The Derivative of an Implicit Function*

Let the values of two variables x and y be related by equation

$$F(x, y) = 0.$$

Then the function $y(x)$ is called an *implicit* function defined by this equation.

Sometimes this equation can be solved for y. It means that y can be explicitly expressed in terms of x: $y = f(x)$. For example, for $y + 2x - 4 = 0$ we can rewrite the equation in explicit form as $y = 4 - 2x$.

But some equations do not explicitly define y as a function of x and are not solvable for y. For example, for $e^{xy}+2xy^3-4x-1=0$ it is impossible to isolate y or x on one side of the equation.

The solutions of the equation $F(x,y)=0$ form a set of points (x,y). So it is possible to plot the graph of an implicit function.

For example, the equation

$$x^2+y^2-a^2=0$$

defines implicitly the following elementary functions

$$y=\sqrt{a^2-x^2},$$

$$y=-\sqrt{a^2-x^2}$$

and the graph is a circle of radius a (Fig. 22).

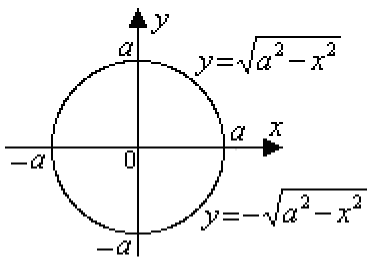


Figure 22.

The technique of implicit differentiation allows us to find the derivative of y with respect to x without transforming $F(x,y)=0$ it into an explicit one. The chain rule must be used whenever the function y is being differentiated because of our assumption that y is a function of x. Then we solve the obtained equation $\frac{d}{dx}F(x,y)=0$ with respect to y'_x.

Example.

Consider the function

$$x^2+y^2-a^2=0.$$

Differentiate both sides of this identity with respect to x:

$$\left(x^2\right)'_x+\left(y^2\right)'_x-\left(a^2\right)'_x=0.$$

Regarding y as a function of x and using the rule of differentiating a composite function, we get

$$2x+2yy'_x=0,$$

whence

$$y'_x=-\frac{x}{y}.$$

Notice that the derivative is a function of both y and x. But if we need to find derivative at some point (x_0, y_0): $y'_x = -\dfrac{x_0}{y_0}$ $(y_0 \neq 0)$.

2.5 ***The Logarithmic Differentiation***

Consider the function

$$y = \left(f(x)\right)^{g(x)}.$$

Such function in which both the base and the exponent are functions of x is called *a composite exponential function*. For example, $y = x^x$, $y = x^{\tan x}$, $y = (\sin x)^{x^2+2}$.

The process of finding the derivative of a composite exponential function is quite complicated because we can not use the ordinary rules of differentiation. Taking the derivatives of such functions is called logarithmic differentiation. It is based on the properties of logarithms.

Begin with

$$y = \left(f(x)\right)^{g(x)}.$$

First we apply natural logarithms of the left and right side of the equation

$$\ln y = \ln\left(f(x)\right)^{g(x)} = g(x)\ln f(x).$$

Next we differentiate both sides of resultant equation with respect to x. The left side requires the chain rule since y is a function of x.

$$\left(\ln y\right)'_x = \left(g(x)\ln f(x)\right)'_x$$

$$\frac{y'}{y} = \left(g(x)\right)'_x \ln f(x) + g(x)\left(\ln f(x)\right)'_x = g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}$$

whence

$$y' = y\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right).$$

Substitute the original function instead of y in the right side of equation and obtain:

$$y' = \left(f(x)\right)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right).$$

Example.

Let we differentiate the function $y = x^x$.

Taking logarithms of the both sides of equation

$$\ln y = \ln x^x = x\ln x.$$

Differentiate both sides of the equation

$$\left(\ln y\right)' = \left(x\ln x\right)'$$

and obtain

$$\frac{y'}{y} = (x)'\ln x + x(\ln x)' = \ln x + x\frac{1}{x} = \ln x + 1.$$

Finally we get

$$y' = y(\ln x + 1)$$

and

$$y' = x^x(\ln x + 1).$$

Logarithmic differentiation is used not only for differentiating composite exponential functions but also for simplifying calculations during finding derivatives.

Example.

Find the derivative of the function

$$y = \frac{(x-1)^2 x^3}{\sqrt{2+x^2}}.$$

Taking logarithms and using the properties of logarithms we obtain

$$\ln y = \ln\frac{(x-1)^2 x^3}{\sqrt{2+x^2}} = \ln(x-1)^2 + \ln x^3 - \ln\sqrt{2+x^2} = 2\ln(x-1) + 3\ln x - \frac{1}{2}\ln(2+x^2).$$

Differentiate the equality

$$\left(\ln y\right)' = \left(2\ln(x-1) + 3\ln x - \frac{1}{2}\ln(2+x^2)\right)'$$

$$\frac{y'}{y} = \frac{2}{x-1} + \frac{3}{x} - \frac{x}{(2+x^2)}.$$

Hence

$$y' = \frac{(x-1)^2 x^3}{\sqrt{2+x^2}} \left(\frac{2}{x-1} + \frac{3}{x} - \frac{x}{(2+x^2)} \right).$$

2.6 *The Derivative of a Function Represented Parametrically*

Consider a function y of x represented by the parametric equation

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases} \quad t \in [T_1,\ T_2]. \tag{2.7}$$

Let us suppose that functions $\varphi(t)$ and $\psi(t)$ have derivatives with respect to t and that the function $x = \varphi(t)$ has an inverse $t = \Phi(x)$, which has derivative with respect to x. Then the function $t = \Phi(x)$ defined by the parametric equation may be considered as a composite function

$$y = \psi(t) = \psi(\Phi(x)).$$

Using the chain rule we get

$$y'_x = \psi'_t(t) \cdot \Phi'_x(x). \tag{2.8}$$

By the theorem 2.3 about the differentiating the inverse function we obtain

$$\Phi'_x(x) = \frac{1}{\varphi'_t(t)}.$$

Finally, putting this expression into (2.8), we get

$$y'_x = \frac{\psi'_t(t)}{\varphi'_t(t)}$$

or

$$y'_x = \frac{y'_t}{x'_t}. \tag{2.9}$$

Formula (2.9) permits finding the derivative of y with respect to x without having to find the expression of y as a function of x.

Example.

Find the derivative of the function

$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} \quad t \in [0,\ \pi].$$

Since $y'_t = b\cos t$ and $x'_t = -a\sin t$, according to (2.9) we have

$$y'_x = \frac{y'_t}{x'_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

Notice that the derivative is a function of t and not of x. But it is not a problem if we need to find derivative at some point. For example,

$$y'_x\Big|_{t=\frac{\pi}{4}} = -\frac{b}{a}\cot t\Big|_{t=\frac{\pi}{4}} = -\frac{b}{a}\cot\frac{\pi}{4} = -\frac{b}{a}.$$

2.7 *The Differential*

Let the function $y = f(x)$ be differentiable on the interval $[a,b]$. The derivative of this function at the point $x \in [a,b]$ is determined by following limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x),$$

where Δx is increment of the independent variable and Δy is the increment of the function corresponding to the change of the independent variable.

As $\Delta x \to 0$, the ratio $\dfrac{\Delta y}{\Delta x}$ tends to a definite number $f'(x)$ and, according to properties of limits and infinitesimals, this ratio could be represented as

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha(x),$$

where $\alpha(x) \to 0,\ \ \Delta x \to 0$.

Hence, the increment Δy can be represented as a sum:

$$\Delta y = f'(x)\Delta x + \alpha(x)\Delta x$$

where the first term is called the *principal part* of the increment and it is linear relative to Δx, and the second term has a higher order of smallness with respect to Δx.

The expression $f'(x)\Delta x$ is called the *differential of function* and is denoted by dy or $df(x)$.

Since $x'=1$, the differential of the independent variable dx coincides with its increment Δx.

Then we can write the formula for differential of the function $f(x)$:

$$dy = f'(x)dx. \tag{2.10}$$

Properties of the Differentials

Let $f(x)$ and $g(x)$ be functions of the variable x. Then

1. $\forall\, C \in \mathbb{R}$, $d(Cf) = Cdf$.

2. $d(f \pm g) = df \pm dg$.

3. $d(f \cdot g) = gdf + fdg$.

4. $d\left(\dfrac{f}{g}\right) = \dfrac{gdf - fdg}{g^2}$.

5. Let $y = f(u)$ and $u = g(x)$, then the differential of a composite function is

$$dy = f'(u)du = f'_u(u)g'_x(x)dx.$$

The fifth property of differential is called invariance of the form of the differential.

6. Since $\Delta y \approx dy$, that is $f(x+\Delta x) - f(x) \approx f'(x)\Delta x$, we get a formula for approximate calculations

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x. \tag{2.11}$$

2.8 *Derivatives of Higher Orders*

I. *Higher Order Derivatives*

Let the function $y = f(x)$ has a derivative on some interval. Since the values of derivative $y' = f'(x)$ depend on x, the first derivative is also the function of x. If this function $f'(x)$ is differentiable, then we can take the derivative of $f'(x)$. This new function is called *the second derivative (the derivative of second order)* of $f(x)$ and is denoted as $f''(x)$ $\left(\dfrac{d^2 f}{dx^2},\ D^2 f(x)\right)$. This process of differentiation can be continued to find the third, fourth and successive derivatives of $f(x)$ called *higher order derivatives* of the function $f(x)$.

$y' = f'(x)$ – *the first derivative*;

$y'' = f''(x)$ – *the second derivative*;

$y''' = f'''(x)$ – *the third derivative*;

$y^{IV} = y^{(4)} = f^{(4)}(x)$ – *the fourth derivative*;

$y^{V} = y^{(5)} = f^{(5)}(x)$ – *the fifth derivative*;

. . .

$y^{(n)} = f^{(n)}(x)$ – *the higher order derivative*;

. . . .

Example.

Provide the successive differentiation of function $y = x^6 + 2x^3 - x^2 + 5x + 3$ till $y^{(5)}$.

$y' = (x^6 + 2x^3 - x^2 + 5x + 3)' = 6x^5 + 6x^2 - 2x + 5$;

$y'' = (6x^5 + 6x^2 - 2x + 5)' = 30x^4 + 12x - 2$;

$y''' = (30x^4 + 12x - 2)' = 120x^3 + 12$;

$y^{(4)} = (120x^3 + 12)' = 360x^2$;

$y^{(5)} = (360x^2)' = 720x$.

II. *Rules of Finding Higher Order Derivatives*

The rules given for finding first order derivatives are generalized to the case of any order. In this case we get

$$(Cf(x))^{(n)} = Cf^{(n)}(x); \qquad (2.12)$$

$$(f(x) \pm g(x))^{(n)} = f^{(n)}(x) \pm g^{(n)}(x). \qquad (2.13)$$

To obtain a formula for nth derivative of the product of functions let us find several derivatives and deduce the general rule.

$$y = f \cdot g\,;$$

$$y' = f' \cdot g + f \cdot g'\,;$$

$$y'' = (f' \cdot g + f \cdot g')' = f'' \cdot g + 2f' \cdot g' + f \cdot g''\,;$$

$$y''' = f''' \cdot g + 3f'' \cdot g' + 3f' \cdot g'' + f \cdot g'''\,;$$

$$y^{IV} = f^{IV} \cdot g + 4f''' \cdot g' + 6f'' \cdot g'' + 4f' \cdot g''' + f \cdot g^{IV}\,.$$

The formulas obtained are similar to the formulas for the expansion $(f+g)^n$ by the binomial theorem. If in the expansion the exponents of the powers of f and g are replaced by the orders of derivatives we obtain the rule for nth derivative of the product of two functions.

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g + nf^{(n-1)} \cdot g' + \frac{n(n-1)}{1 \cdot 2} f^{(n-2)} \cdot g'' + \ldots + f \cdot g^{(n)}\,. \qquad (2.14)$$

This is the *Leibniz rule*.

The rigorous proof of this rule is performed by the method of mathematical induction.

Example. Find the nth derivative of function $y = x^2 e^{2x}$.

$$\begin{array}{ll}
g = e^{2x}, & f = x^2, \\
g' = 2e^{2x}, & f' = 2x, \\
g'' = 2^2 e^{2x}, & f'' = 2, \\
\ldots, & \\
g^{(n)} = 2^n e^{2x}, & f''' = \ldots = f^{(n)} = 0,
\end{array}$$

$$y^{(n)} = \left(x^2 e^{2x}\right)^{(n)} = 2^n e^{2x} x^2 + n2^{n-1} e^{2x} 2x + \frac{n(n-1)}{1 \cdot 2} 2^{n-2} e^{2x} 2 = 2^n e^{2x}(x^2 + nx + 2^{-2})\,.$$

III. ***Higher Order Derivatives of Implicit Function***

Let us illustrate the process of finding second order derivative of implicit function by example.

Consider an implicit function y of x defined by formula

$$y^3 - 2x^2 + 2 = 0\,.$$

Differentiate the equality with respect to x

$$3y^2y' - 4x = 0$$

and find first derivative

$$y' = \frac{4x}{3y^2}. \tag{2.15}$$

Again differentiate this equality with respect to x

$$y'' = \frac{(4x)'(3y^2) - (3y^2)'(4x)}{(3y^2)^2} = \frac{12y^2 - 24xyy'}{9y^4} = \frac{4y - 8xy'}{3y^3}.$$

Substituting, in the place of the derivative y', its expression from (2.15), we obtain

$$y'' = \frac{4y - 8x \cdot \dfrac{4x}{3y^2}}{3y^3}.$$

Then, after simplifying,

$$y'' = \frac{12y^3 - 32x^2}{9y^5}.$$

It is possible to continue the differentiating in the same manner if we need the derivatives of higher orders.

IV. ***Higher Order Derivatives of Functions Represented Parametrically***

Let the function y of x be represented by the parametric equation

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases} \quad t \in [T_1,\ T_2].$$

Let us suppose that functions $\varphi(t)$ and $\psi(t)$ have derivatives with respect to t and that the function $x = \varphi(t)$ has an inverse $t = \Phi(x)$, which has derivative with respect to x.

It was proved (see formula 2.9 from part 2.6.) that

$$y'_x = \frac{y'_t}{x'_t}.$$

To find the second derivative, differentiate this expression with respect to x, bearing in the mind that t is a function of x:

$$y''_{xx} = (y'_x)'_x = \frac{d}{dx}\left(\frac{y'_t}{x'_t}\right) = \frac{d}{dt}\left(\frac{y'_t}{x'_t}\right) \cdot \frac{1}{x'_t}, \tag{2.16}$$

or

$$y''_{xx} = \frac{y''_{tt}x'_t - x''_{tt}y'_t}{(x'_t)^3}. \tag{2.17}$$

In similar fashion we can find the derivatives

$$\frac{d^3y}{dx^3} = \frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{1}{x'_t},$$

$$\frac{d^4y}{dx^4} = \frac{d}{dt}\left(\frac{d^3y}{dx^3}\right)\cdot\frac{1}{x'_t}$$

and so forth.

Example.

Find the derivative of the function

$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} \quad t \in [0,\ \pi].$$

Since $y'_t = b\cos t$ and $x'_t = -a\sin t$, according to (2.9) we have

$$y'_x = \frac{y'_t}{x'_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t.$$

Then

$$y''_{xx} = \frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\cdot\frac{1}{-a\sin t} = \frac{b}{a\sin^2 t}\cdot\frac{1}{-a\sin t} = -\frac{b}{a^2\sin^3 t}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dt}\left(-\frac{b}{a^2\sin^3 t}\right)\cdot\frac{1}{-a\sin t} = \frac{3b\cos t}{a^2\sin^4 t}\cdot\frac{1}{-a\sin t} = -\frac{3b\cos t}{a^3\sin^5 t}.$$

2.9 *Basic Theorems of the Differential Calculus*

Theorem 2.5.

Let $f(x)$ be defined on (a,b) and

$$f(x_0) = \max_{x\in(a,b)} f(x) \text{ or } f(x_0) = \min_{x\in(a,b)} f(x).$$

If the function is differentiable at the point $x_0 \in (a,b)$, then $f'(x_0) = 0$.

Proof.

Let $f(x_{0_1}) = \max\limits_{x\in(a,b)} f(x)$. Then for $x \neq x_{0_1}$ we get $f(x) - f(x_{0_1}) \le 0$ (Fig. 3).

Since, according to definition of derivative,

$$f'(x_{0_1}) = f'_-(x_{0_1}) = \lim_{x \to x_{0_1}-} \frac{f(x_{0_1}) - f(x)}{x - x_{0_1}} \ge 0$$

and

$$f'(x_{0_1}) = f'_+(x_{0_1}) = \lim_{x \to x_{0_1}+} \frac{f(x_{0_1}) - f(x)}{x - x_{0_1}} \le 0,$$

we obtain $f'(x_{0_1}) = 0$.

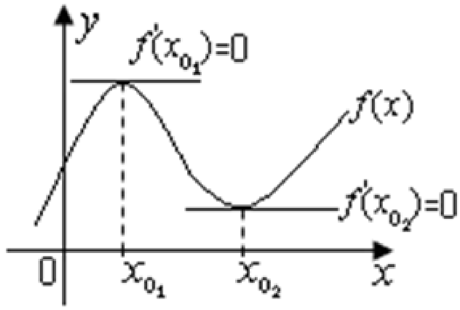


Figure 23.

Note: The tangent line of the function $f(x)$ at this point x_0 is a horizontal line. Hence, its slope is equal to zero, that is, the derivative $f'(x_0) = 0$ (Fig. 23).

Theorem 2.6 **(Rolle's Theorem).**

Let $f(x)$ be continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one point c in (a,b) where $f'(c) = 0$.

Proof.

Since function $f(x)$ is continuous over the interval $[a,b]$, there exists at least one point x^* such that $f(x^*) = \max\limits_{x \in (a,b)} f(x)$ and there exists at least one point x_* such that $f(x_*) = \min\limits_{x \in (a,b)} f(x)$ (Theorem 1.18). Let $c = x^*$ (Fig. 24).

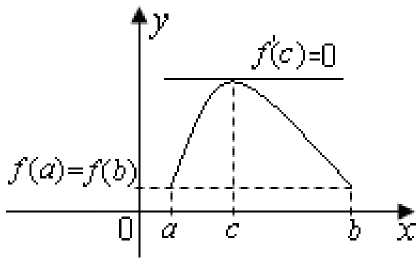


Figure 24.

According to theorem 2.5 $f'(c) = 0$.

Theorem 2.7 **(The Mean Value Theorem, Lagrange's Theorem).**

Let $f(x)$ be continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b). Then, there is at least one point c in (a,b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \qquad (2.18)$$

Proof.

Let us consider the auxiliary function

$$F(x) = f(x) - f(a) - (x-a)\frac{f(b) - f(a)}{b-a}.$$

This function satisfies the conditions of the Rolle's Theorem. By this theorem, there exists within the interval a point $x = c$ such that $F'(c) = 0$.

But

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}.$$

And

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0.$$

Whence

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Note: the Mean Value Theorem tells us: if it all points of arc MM_0 there is a tangent line, then there will be, on this arc, a point between M and M_0 at which the tangent is parallel to the secant MM_0. The slope of secant and the slope of the tangent line must be equal (Fig. 25).

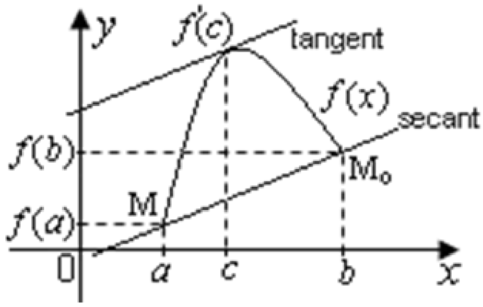


Figure 25.

The geometric significance of the auxiliary function: it is an equation of secant MM_0.

***Theorem 2.8* (Cauchy's Theorem).**

Let functions $f(x)$ and $g(x)$ be continuous on the interval $[a,b]$ and differentiable within it. If $g'(x) \neq 0$, $x \in (a,b)$, then, there exists a point c in (a,b) at which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \qquad (2.19)$$

Proof.

Note that $g(b) - g(a) \neq 0$, since otherwise $g(b) = g(a)$, and then, by the Rolle's Theorem, $g'(c) = 0$ for some $c \in (a,b)$), that contradicts to the statement of the theorem.

Let us consider the auxiliary function

$$F(x)=f(x)-f(a)-(g(x)-g(a))\frac{f(b)-f(a)}{g(b)-g(a)}.$$

This function satisfies the conditions of the Rolle's Theorem. We conclude that there exists a point $x=c$, $c\in(a,b)$ such that $F'(c)=0$.

But

$$F'(x)=f'(x)-g'(x)\frac{f(b)-f(a)}{g(b)-g(a)}.$$

And

$$F'(c)=f'(c)-g'(c)\frac{f(b)-f(a)}{g(b)-g(a)}=0.$$

Whence

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

2.10 *The L'Hospital's Rule*

Previously, we have learned the concept of the limit of an indeterminate form $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$. This is an expression involving two functions that have their limits impossible to find from the limits of the particular functions. Now we proceed to the one of the most powerful methods for finding limits concerned with application of the derivatives.

The L'Hopital Rule for an indeterminate form $\frac{0}{0}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on the interval (a,b), so that:

1. $g'(x)\neq 0$ on the interval (a,b);
2. $\lim\limits_{x\to x_0} f(x)=0$ and $\lim\limits_{x\to x_0} g(x)=0$;
3. $\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists.

Then $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\left[\frac{0}{0}\right]=\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$.

Proof.

Applying the formula (2.19) (Cauchy's theorem) we have

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(c)}{g'(c)}$$

for some $x\in(a,b)$, $x\neq x_0$, and c between x and x_0.

But in view of assumption 2) we put $f(x_0)=0$ and $g(x_0)=0$. Then

$$\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}.$$

Taking the limit of this equality as $x\to x_0$. If $x\to x_0$ then $c\to x_0$ also, since c lies between x and x_0.

Whence

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{c\to x_0}\frac{f'(c)}{g'(c)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

Note that if an expression $\dfrac{f'(x)}{g'(x)}$ is again an indeterminate form $\dfrac{0}{0}$, then we can apply the L'Hopital Rule repeatedly.

The L'Hopital Rule is also applicable if $\lim\limits_{x\to\infty} f(x)=0$ and $\lim\limits_{x\to\infty} g(x)=0$:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\left[\frac{0}{0}\right]=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Examples.

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{(\sin x)'}{(x)'}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1;$

2. $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{(x-\sin x)'}{(x^3)'}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{(1-\cos x)'}{(3x^2)'}=$

$=\lim\limits_{x\to 0}\dfrac{\sin x}{6x}=\dfrac{1}{6};$

3. $\lim\limits_{x\to\infty}\dfrac{\sin\frac{2}{x}}{\frac{3}{x}}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{\left(\sin\frac{2}{x}\right)'}{\left(\frac{3}{x}\right)'}=\lim\limits_{x\to 0}\dfrac{\cos\frac{2}{x}\cdot\left(-\frac{2}{x^2}\right)}{-\frac{3}{x^2}}=\dfrac{2}{3}.$

The L'Hopital Rule for an indeterminate form $\frac{\infty}{\infty}$.

Let functions $f(x)$ and $g(x)$ be defined and differentiable on the interval (a,b), so that:

1. $\lim\limits_{x\to x_0} f(x)=\infty$ and $\lim\limits_{x\to x_0} g(x)=\infty$;

2. $\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists.

Then $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to x_0}\frac{f'(x)}{g'(x)}$.

Proof.

Let we reduce an indeterminate form $\frac{\infty}{\infty}$ to the form $\frac{0}{0}$ as follows: $\frac{f(x)}{g(x)}=\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$.

Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to x_0}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}=\left[\frac{0}{0}\right]=\lim_{x\to x_0}\frac{\left(\frac{1}{g(x)}\right)'}{\left(\frac{1}{f(x)}\right)'}=\left|\begin{array}{l}\left(\frac{1}{g(x)}\right)'=-\frac{g'(x)}{g^2(x)}\\ \left(\frac{1}{f(x)}\right)'=-\frac{f'(x)}{f^2(x)}\end{array}\right|=$$

$$=\lim_{x\to x_0}\frac{-\frac{g'(x)}{g^2(x)}}{-\frac{f'(x)}{f^2(x)}}=\lim_{x\to x_0}\frac{f^2(x)g'(x)}{g^2(x)f'(x)}=\left(\lim_{x\to x_0}\frac{f(x)}{g(x)}\right)^2\lim_{x\to x_0}\frac{g'(x)}{f'(x)}.$$

Let $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=A$.

Hence

$$A=\lim_{x\to x_0}\frac{f(x)}{g(x)}=\left(\lim_{x\to x_0}\frac{f(x)}{g(x)}\right)^2\lim_{x\to x_0}\frac{g'(x)}{f'(x)}=A^2\lim_{x\to x_0}\frac{g'(x)}{f'(x)};$$

$$\lim_{x\to x_0}\frac{g'(x)}{f'(x)}=\frac{1}{A}.$$

Finally

$$A=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

Example.

$$\lim_{x\to 0}\frac{\ln x}{\cot x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0}\frac{(\ln x)'}{(\cot x)'}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}}=\lim_{x\to 0}\frac{-\sin^2 x}{x}=-\lim_{x\to 0}\frac{\sin x}{x}\sin x=0.$$

This rule can also be applied for limits at infinity:

Example.

$$\lim_{x\to\infty}\frac{x^2-x}{e^{3x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(x^2-x)'}{(e^{3x})'}=\lim_{x\to\infty}\frac{2x-1}{3e^{3x}}=\left[\frac{\infty}{\infty}\right]=\lim_{x\to\infty}\frac{(2x-1)'}{(3e^{3x})'}=\lim_{x\to\infty}\frac{2}{9e^{3x}}=0.$$

The other indeterminate forms can be reduced to one of the forms: $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

1. $[0\cdot\infty]$.

Let $\lim\limits_{x\to x_0} f(x)=0$ and $\lim\limits_{x\to x_0} g(x)=\infty$. It is required to find

$$\lim_{x\to x_0}\left(f(x)g(x)\right)=[0\cdot\infty]$$

If the expression is rewritten as follows

$$\lim_{x\to x_0}\left(f(x)g(x)\right)=\lim_{x\to x_0}\left(\frac{f(x)}{\frac{1}{g(x)}}\right)$$

or

$$\lim_{x\to x_0}\left(f(x)g(x)\right)=\lim_{x\to x_0}\left(\frac{g(x)}{\frac{1}{f(x)}}\right),$$

then as x tends to x_0 we obtain the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ respectively.

Example.

$$\lim_{x\to 0}\left(x^2\ln x\right)=[0\cdot\infty]=\lim_{x\to 0}\left(\frac{\ln x}{\frac{1}{x^2}}\right)=\left[\frac{\infty}{\infty}\right]=\lim_{x\to 0}\frac{(\ln x)'}{\left(\frac{1}{x^2}\right)'}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=-\lim_{x\to 0}\frac{x^3}{2x}=0.$$

2. $[\infty-\infty]$.

The indeterminate form $[\infty-\infty]$ should be reduced to the form $\frac{0}{0}$ by algebraic transformation.

Example.

$$\lim_{x\to\frac{\pi}{2}}\left(tgx-\frac{1}{\cos x}\right)=[\infty-\infty]=\lim_{x\to\frac{\pi}{2}}\left(\frac{\sin x}{\cos x}-\frac{1}{\cos x}\right)=\lim_{x\to\frac{\pi}{2}}\left(\frac{\sin x-1}{\cos x}\right)=\left[\frac{0}{0}\right]=$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{(\sin x-1)'}{(\cos x)'}=\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=0.$$

3. $[1^\infty]$, $[0^0]$, $[\infty^0]$.

These indeterminate form can be reduced to the form $[0\cdot\infty]$ by transformation

$$f(x)^{g(x)}=\exp(g(x)\ln f(x)).$$

Examples.

1. $$\lim_{x\to0}x^x=[0^0]=\lim_{x\to0}\exp(x\ln x)=[0\cdot\infty]=\exp\lim_{x\to0}\frac{\ln x}{\frac{1}{x}}=\left[\frac{\infty}{\infty}\right]=\exp\lim_{x\to0}\frac{(\ln x)'}{\left(\frac{1}{x}\right)'}=$$

$$=\exp\lim_{x\to0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\exp\lim_{x\to0}\frac{-x^2}{x}=e^0=1.$$

2. $$\lim_{x\to0}\left(\frac{1}{x}\right)^{\sin x}=[\infty^0]=\lim_{x\to0}\exp(\sin x\ln\frac{1}{x})=[0\cdot\infty]=\exp\lim_{x\to0}\frac{-\ln x}{\frac{1}{\sin x}}=\left[\frac{\infty}{\infty}\right]=$$

$$=\exp\lim_{x\to0}\frac{(-\ln x)'}{\left(\frac{1}{\sin x}\right)'}=\exp\lim_{x\to0}\frac{-\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}}=\exp\lim_{x\to0}\frac{\sin^2 x}{x\cos x}=e^0=1.$$

2.11 *Taylor's and Maclaurin's Formulas*

Taylor's formula is one of the methods for approximating a given function by polynomials.

Let $f(x)$ be a function such that $f(x)$ and its first *n* derivatives are determined and are continuous on $[a,b]$. Furthermore, let $f^{(n+1)}(x)$ exist for all $x\in(a,b)$. Then $\forall\, x_0 \in (a,b)$

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{P_n(x,x_0)} + R_n(x,x_0), \quad (2.20)$$

where $P_n(x,x_0)$ – *the n-th degree Taylor's polynomial* of the function $f(x)$ in the region near the point x_0, $R_n(x,x_0)$ – *remainder term* associated with $P_n(x,x_0)$ using for evaluation of error of approximation (Fig. 26).

There are several forms for remainder $R_n(x,x_0)$, but the most common one is *the Lagrange form*:

$$R_n(x,x_0) = \frac{f^{(n+1)}(x_0 + \xi(x-x_0))}{(n+1)!}(x-x_0)^{n+1},$$

where $\xi \in (0,1)$.

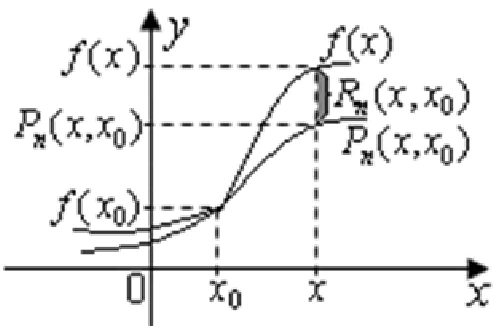


Figure 26.

If $x_0 = 0$, then Taylor's formula simplifies to

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi x)}{(n+1)!}x^{n+1}, \ \xi \in (0,1), \quad (2.21)$$

which is called *the Maclaurin's formula*.

The remainder in the Maclaurin's formula has the form

$$R_n(x) = \frac{f^{(n+1)}(\xi x)}{(n+1)!}x^{n+1}, \ \xi \in (0,1).$$

Examples.

1. Expansion of the function $f(x) = e^x$.

The function $f(x) = e^x$ has a derivative of any order

$$f^{(n)}(x) = e^x.$$

Therefore, the Maclaurin's formula is applicable to this function. Let us compute the values of the function and its first n derivatives at the point $x=0$, and the value of $n+1$th derivative at the point $\xi x,\ \xi\in(0,1)$:

$$f(0)=f'(0)=f''(0)=\ldots=f^{(n)}(0)=e^0=1;$$
$$f^{(n+1)}(\xi x)=e^{\xi x},\ \xi\in(0,1).$$

Whence

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+R_n(x),$$

where

$$R_n(x)=\frac{e^{\xi x}}{(n+1)!}x^{n+1},\ \xi\in(0,1).$$

Here $\lim\limits_{n\to\infty}R_n(x)=\lim\limits_{n\to\infty}\frac{e^{\xi x}}{(n+1)!}x^{n+1}=0$ for all values of x.

2. Expansion of the function $f(x)=\sin x$.

Let us find the successive derivatives of $f(x)=\sin x$.

$f(x)=\sin x,$ $\qquad f(0)=0;$

$f'(x)=\cos x=\sin\left(x+\frac{\pi}{2}\right),$ $\qquad f'(0)=1;$

$f''(x)=-\sin x=\sin\left(x+2\frac{\pi}{2}\right),$ $\qquad f''(0)=0;$

$f'''(x)=-\cos x=\sin\left(x+3\frac{\pi}{2}\right),$ $\qquad f'''(0)=-1;$

$f^{IV}(x)=\sin x=\sin\left(x+4\frac{\pi}{2}\right),$ $\qquad f^{IV}(0)=0;$

....

$f^{(n)}(x)=\sin\left(x+n\frac{\pi}{2}\right),$ $\qquad f^{(n)}(0)=\sin n\frac{\pi}{2};$

$f^{(n+1)}(x)=\sin\left(x+(n+1)\frac{\pi}{2}\right),$ $\qquad f^{(n+1)}(0)=\sin\left(\xi x+(n+1)\frac{\pi}{2}\right),\quad \xi\in(0,1);$

Substituting the values into (2.21) we get an expansion:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_n(x),$$

where

$$R_n(x) = \frac{\sin\left(\xi x + (n+1)\frac{\pi}{2}\right)}{(n+1)!} x^{n+1}, \ \xi \in (0,1).$$

Since $\left|\sin\left(\xi x + (n+1)\frac{\pi}{2}\right)\right| \le 1$, we have $\lim\limits_{n\to\infty} R_n(x) = 0$ for all values of x.

Figure 27 shows the graph of the function $f(x) = \sin x$ and the first three approximations: $S_1(x) = x,\ S_2(x) = x - \frac{x^3}{3!},\ S_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

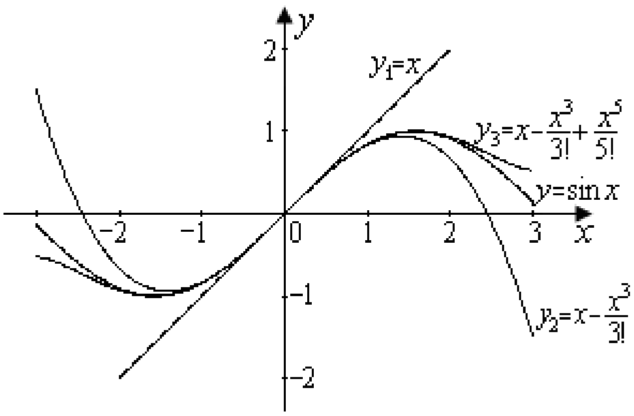


Figure 27.

3. Expansion of the function $f(x) = \cos x$**.**

Finding the values of the successive derivatives for $x = 0$ of the function $f(x) = \cos x$ and substituting them into (2.21), we obtain the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + R_n(x),$$

where

$$R_n(x) = \frac{\cos\left(\xi x + (n+1)\frac{\pi}{2}\right)}{(n+1)!} x^{n+1}, \ \xi \in (0,1).$$

Since $\left|\cos\left(\xi x + (n+1)\frac{\pi}{2}\right)\right| \le 1$, we have $\lim\limits_{n\to\infty} R_n(x) = 0,\ \forall x$.

4. Another examples of expansions:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + R_n(x);$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x);$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + R_n(x).$$

2.12 *The Monotonicity of a Function*

If the function $y = f(x)$ is such that

$$\forall\ \{x_1, x_2\} \in E(f),\ x_1 < x_2:$$
$$f(x_1) \le f(x_2),$$

then the function $y = f(x)$ is called *increasing* (Fig. 28).

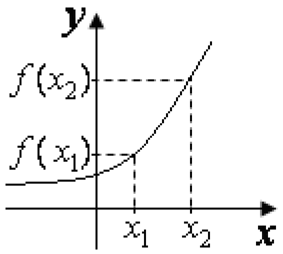


Figure 28.

If the function $y = f(x)$ is such that

$$\forall\ \{x_1, x_2\} \in E(f),\ x_1 < x_2:$$
$$f(x_1) \ge f(x_2),$$

then the function $y = f(x)$ is called *decreasing* (Fig. 29).

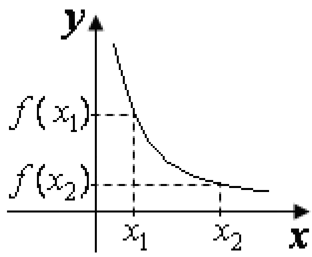


Figure 29.

It is possible to apply the concept of derivative to investigate the increase and decrease of a function.

Theorem 2.9.

Let the function $f(x)$ be defined and differentiable on the interval (a,b).

If $f'(x) \ge 0\ \ \forall\, x \in (a,b)$, then $f(x)$ monotone increasing function on (a,b).

If $f'(x) \le 0\ \ \forall\, x \in (a,b)$, then $f(x)$ monotone decreasing function on (a,b).

Proof.

Let $\{x_1, x_2\} \in (a,b),\ x_1 < x_2\ \ (x_2 - x_1 > 0)$. By the Mean Value Theorem for $c \in (x_1, x_2)$:

$$\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} = f'(c).$$

Then for increasing function ($f(x_2) - f(x_1) > 0$) we obtain

$$f'(c) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} > 0.$$

Similarly, for decreasing function ($f(x_2) - f(x_1) < 0$):

$$f'(c) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} < 0.$$

Note. This theorem has the following geometric meaning (Fig. 30 and 31).

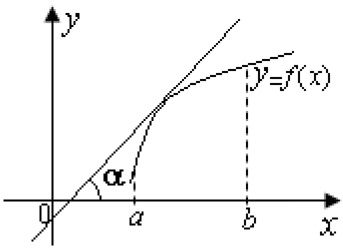


Figure 30.

For increasing function, the slope of tangent is not negative $(f'(x) \geq 0)$.

The angle α is acute or $0°$.

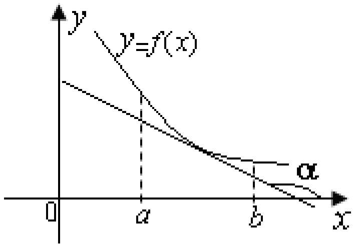


Figure 31.

For decreasing function, the slope of tangent is not positive $f'(x) \leq 0$.

The angle α is obtuse or $0°$.

This theorem permits judging the nature of the monotonicity of a function by the sign of its derivative.

Example.

Find the intervals on which function $y = (x-1)^2(x+2)$ is increasing or decreasing (intervals of monotonicity).

Find the derivative

$$y' = 2(x-1)(x+2) + (x-1)^2 = (x-1)(3x+3).$$

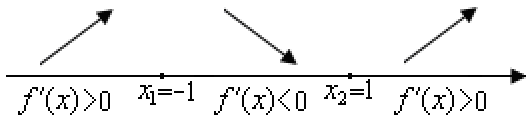
Find the critical points solving the equation

$$y' = (x-1)(3x+3) = 0;$$

$$(3x+3) = 0, \quad (x-1) = 0;$$

$$x_1 = -1, \qquad x_2 = 1. \text{ (critical points)}$$

Investigate the sign of derivative on the intervals between the critical points.



$f(x)$ monotone increases on $(-\infty,-1]$ and $[1,+\infty)$;

$f(x)$ monotone decreases on $[-1,1]$.

2.13 *Local Extrema of a Function*

Consider the function $y = f(x),\ x \in (a,b)$.

The function $f(x)$ has *a local maximum* (Fig. 32) at the point x_0 if

$$\exists\,\delta > 0\ \ \forall\, x \in (x_0 - \delta, x_0 + \delta):\ f(x) < f(x_0),$$

$$f(x_0) = \underset{x\in(a,b)}{\mathrm{locmax}}\ f(x).$$

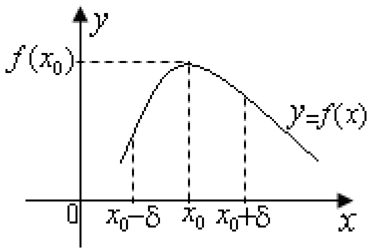


Figure 32.

The function $f(x)$ has *a local minimum* (Fig. 33) at the point x_0 if

$$\exists\,\delta > 0\ \ \forall\, x \in (x_0 - \delta, x_0 + \delta):\ f(x) > f(x_0),$$

$$f(x_0) = \underset{x\in(a,b)}{\mathrm{locmin}}\ f(x).$$

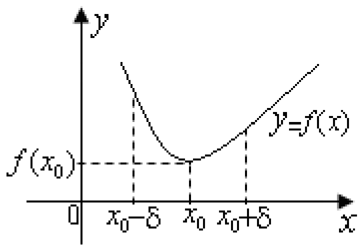


Figure 33.

The extreme point (extremum) is the point where the function attains either its local maximum or local minimum.

Theorem 2.10 (A necessary condition for the existence of an extremum).

Let the function $y = f(x)$ be continuous and differentiable in a definite region $((x_0-\delta, x_0+\delta),\ \delta > 0)$ of the point x_0. If x_0 is the extreme point of the function $y = f(x)$ (Fig. 34), then

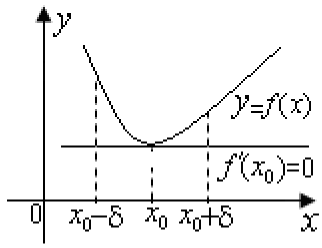


Figure 34.

or $f'(x_0) = 0$ (x_0 – *stationary point*);

or $f'(x_0)$ does not exist (x_0 – *singular point*).

Proof. Analogically to the proof of *Theorem 2.5*.

Points where $f'(x_0) = 0$ or $f'(x_0)$ do not exist are called *critical*.

Theorem 2.11 (Sufficient conditions for the existence of an extremum).

Let the function $y = f(x)$ be continuous on some interval $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$, and differentiable at all points of the region $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$, $\delta > 0$, where the point x_0 is critical point for the function $y = f(x)$. If in moving from left to right through this point the sign of $f'(x)$ changes from "–" to "+" at x_0, then $f(x)$ has a local minimum at x_0, if it changes from "+" to "–", then $f(x)$ has a local maximum at x_0.

Proof.

Let us consider the case when the derivative changes the sign from "–" to "+" at x_0, that is

$$f'(x) > 0, \text{ when } x < x_0,$$
$$f'(x) < 0, \text{ when } x > x_0.$$

Applying the Lagrange theorem to $f(x) - f(x_0)$ we have

$$f(x) - f(x_0) = f'(c)(x - x_0).$$

where c is a point between x and x_0.

Let $x < x_0$, then $c < x_0$, $f'(c) > 0$, $f'(c)(x - x_0) < 0 \Rightarrow f(x) - f(x_0) < 0$,

and

$$f(x) < f(x_0). \tag{2.22}$$

Let $x > x_0$, then $c > x_0$, $f'(c) < 0$, $f'(c)(x - x_0) < 0 \Rightarrow f(x) - f(x_0) < 0$,

and

$$f(x) < f(x_0). \tag{2.23}$$

The inequalities (2.22) and (2.23) show that for all values of x sufficiently close to the point x_0

$$f(x) < f(x_0).$$

Hence, the function $y = f(x)$ has a local maximum at point x_0 (Fig. 35).

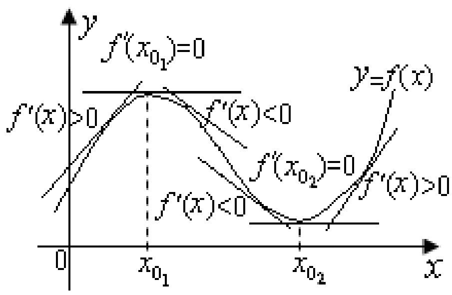


Figure 35.

The sufficient condition for a local minimum is proved in similar fashion.

There is another method to solve the problem of finding extrema, which is based on the investigation the sign of second derivative at critical points.

If $f''(x_0) < 0$ then the critical point x_0 is a point of a local maximum.

If $f''(x_0) > 0$ then the critical point x_0 is a point of a local minimum.

If $f''(x_0) = 0$ then the second derivative test does not give any answer.

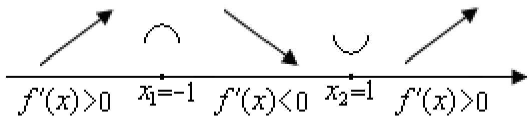
Example.

Find local extrema for the function $y = (x-1)^2(x+2)$.

Find the derivative and the critical points solving the equation

$$y' = (x-1)(3x+3) = 0 \Rightarrow \ x_1 = -1, \ x_2 = 1.$$

Investigate the sign of derivative on the intervals between the critical points and state the type of critical points.



Local maximum at $x_1 = -1$:

$$f(x_1) = f(-1) = (-1-1)^2(-1+2) = 4.$$

Local minimum at $x_2 = 1$:

$$f(x_2) = f(1) = (1-1)^2(1+2) = 0.$$

The graph of function is sketched in Figure 36.

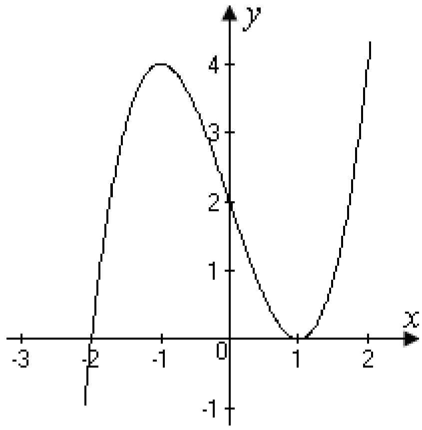


Figure 36.

2.14 *Concavity of a Curve. Points of Inflection*

Let $y = f(x)$ be differentiable on some interval $[a,b]$.

The function $f(x)$ *concaves downward* on the interval $[a,b]$, if all points of the curve lie above any tangent line to it on this interval.

The function $f(x)$ *concaves upward* on the interval $[a,b]$, if all points of the curve lie below any tangent line to it on this interval. (Fig. 37).

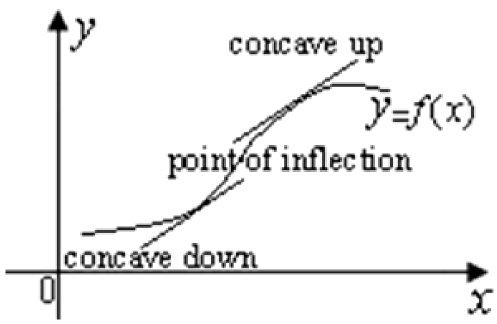


Figure 37.

The point of *inflection* of a function $f(x)$ is defined to be the point at which the concavity changes from upward to downward or vice-versa.

Theorem 2.12.

If at all points of an interval (a,b) the second derivative of the function $f(x)$ is negative $(f''(x)<0)$, the curve $y=f(x)$ on this interval concaves upward.

If at all points of an interval (a,b) the second derivative of the function $f(x)$ is positive $(f''(x)>0)$, the curve $y=f(x)$ on this interval concaves downward.

Proof.

Let us prove the first statement.

Let take a point $x_0\in(a,b)$ and draw the tangent line to the curve at the point $(x_0, f(x_0))$. The equation of the tangent line is

$$\bar{y}=f(x_0)+f'(x_0)(x-x_0).$$

Let us show that the ordinate of any point of the curve $y=f(x)$ is less than the ordinate $\bar{y}$ of the tangent line for one and the same value of x.

The difference of the ordinates of the curve and of the tangent for the same value of x is

$$y-\bar{y}=f(x)-f(x_0)-f'(x_0)(x-x_0).$$

Applying the Mean Value Theorem to the difference $f(x)-f(x_0)$ we get for c that lies between x and x_0

$$y-\bar{y}=f'(c)(x-x_0)-f'(x_0)(x-x_0)=(f'(c)-f'(x_0))(x-x_0).$$

Let us apply the Mean Value Theorem to the difference $f'(c)-f'(x_0)$ for c_1 that lies between c and x_0

$$y-\bar{y}=f''(c_1)(c-x_0)(x-x_0).$$

For the case $x_0<c<c_1<x$, since

$$f''(c_1)<0,\quad (c-x_0)>0,\quad (x-x_0)>0,$$

we have

$$y-\bar{y}<0.$$

For the case $x < c < c_1 < x_0$, since

$$f''(c_1) < 0, \quad (c - x_0) < 0, \quad (x - x_0) < 0,$$

we have

$$y - \bar{y} < 0.$$

We have thus proved that every point of the curve lies below the tangent line to the curve, no matter what values x and x_0 have on the interval (a,b). Hence, the curve is concave up.

The second statement is proved in similar fashion.

Theorem 2.13.

Let $y = f(x)$ be twice differentiable on some interval $[a,b]$. If $f''(x_0)$ or $f''(x_0)$ does not exist and if the second derivative $f''(x)$ changes sign when passing through $x = x_0$, then the point of the curve with abscissa $x = x_0$ is the point of inflection (Fig. 38).

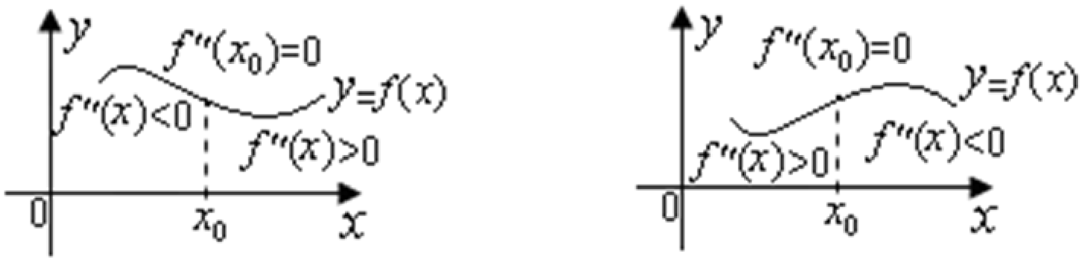


Figure 38.

Example.

Find the points of inflection and determine the intervals of concavity up and down of the curve $y = (x-1)^2(x+2)$.

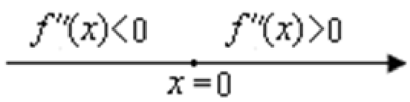
Find the second derivative

$$y'' = ((x-1)(3x+3))' = 6x$$

and solve the equation

$$y'' = 0 \Rightarrow 6x = 0 \Rightarrow x = 0.$$

Investigate the sign of second derivative to the right and left of the point $x = 0$.



The function *concaves downward* on the interval $(0,+\infty)$;

the function *concaves upward* on the interval $(-\infty,0)$;

the point of *inflection* $x=0 \Rightarrow f(0)=(0-1)^2(0+2)=2$.

The graph of function is sketched in Figure 36.

2.15 *Asymptotes*

The asymptote of a graph is the straight line such that the distance between the graph and the line approaches zero as they tend to infinity (a curve comes closer and closer to a line without actually crossing it).

Asymptotes can be *vertical*, *horizontal* and *inclined*.

I. Vertical asymptote

The straight line $x=a$ is the vertical asymptote of the function $y=f(x)$, if $x=a$ is a point of discontinuity of the second kind (at least one of the limits $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to a-} f(x)$ is $\pm\infty$) (Fig. 38).

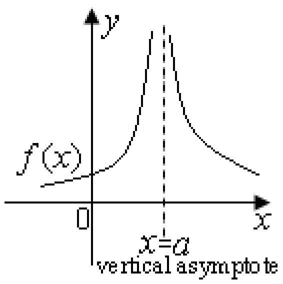


Figure 38.

II. Inclined asymptote

The straight line $y=kx+b$ (Fig. 7), where

$$k=\lim_{x\to\infty}\frac{f(x)}{x} \quad \text{and} \quad b=\lim_{x\to\infty}(f(x)-kx).$$

Here $k\neq 0$ and b are real numbers.

($x\to+\infty$ and $x\to-\infty$ should be considered separately)

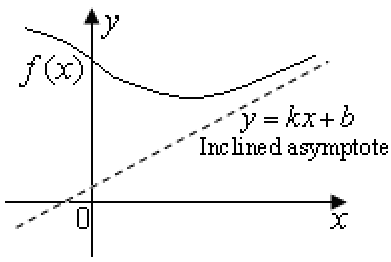


Figure 39.

III. Horizontal asymptote

The straight line $y=b$ is the horizontal asymptote (Fig. 40) of the function $y=f(x)$ if $b=\lim\limits_{x\to\pm\infty} f(x)$.

($x\to+\infty$ and $x\to-\infty$ should be considered separately)

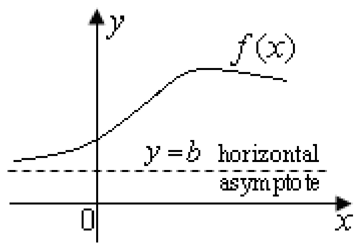


Figure 40.

2.16 *The General Plan for Investigating Functions*

1. Domain of the function.
2. Investigate whether the function is even or odd, periodic.
3. Intersections with the x-axis and y-axis.

   The intervals where the function holds the sign.
4. Points of discontinuities. Asymptotes.
5. Monotonicity.
6. Local maxima and local minima.
7. Concavity down and up. Inflection points.
8. Plot the graph.

Example.

Graph the function

$$y = \frac{x^3+4}{x^2}.$$

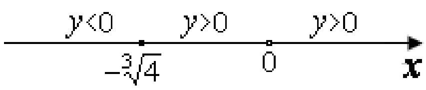
1. *Domain of the function* $\quad D(y) = (-\infty, 0) \cup (0, +\infty)$.
2. *Investigate whether the function is even or odd, periodic.*

$y(-x) = \dfrac{(-x)^3+4}{(-x)^2} = \dfrac{-x^3+4}{x^2} \Rightarrow y(-x) \neq y(x)$ and $y(-x) \neq -y(x) \Rightarrow$ function is neither odd nor even

3. *Intersections with the y-axis*

Since $x = 0$ is not the point of domain there is no points of intersection with y-axis.

Intersections with the x-axis $y = 0 \Rightarrow \dfrac{x^3+4}{x^2} = 0 \Rightarrow \begin{cases} x \neq 0, \\ x^3 = -4; \end{cases} \Rightarrow \begin{cases} x \neq 0, \\ x = -\sqrt[3]{4} \approx 1{,}6; \end{cases}$



$y > 0$ for $x \in (-\sqrt[3]{4}, 0) \cup (0, +\infty)$;

$y < 0$ for $x \in (-\infty, -\sqrt[3]{4})$.

4. *Asymptotes*

Since $x=0$ is a point of discontinuity and

$$\lim_{x\to 0+}\frac{x^3+4}{x^2}=\frac{(0+0)^3+4}{(0+0)^2}=\frac{4}{+0}=+\infty,$$

$$\lim_{x\to 0-}\frac{x^3+4}{x^2}=\frac{(0-0)^3+4}{(0-0)^2}=\frac{4}{+0}=+\infty,$$

then $x=0$ is vertical asymptote.

Let find inclined asymptote $y=kx+b$:

$$k=\lim_{x\to\pm\infty}\frac{\frac{x^3+4}{x^2}}{x}=\lim_{x\to\pm\infty}\frac{x^3+4}{x^3}=1,$$

$$b=\lim_{x\to\pm\infty}\left(\frac{x^3+4}{x^2}-x\right)=\lim_{x\to\pm\infty}\frac{x^3+4-x^3}{x^2}=\lim_{x\to\pm\infty}\frac{4}{x^2}=0.$$

The straight line $y=x$ the inclined asymptote for $x\to\pm\infty$.

5. *Monotonicity.*

Find the derivative

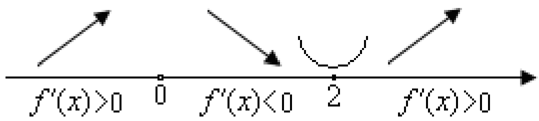
$$y'=\left(\frac{x^3+4}{x^2}\right)'=\left(x+\frac{4}{x^2}\right)'=1-\frac{8}{x^3}=\frac{x^3-8}{x^3}.$$

Find the critical points

$$y'=\frac{x^3-8}{x^3}=0;$$

$$x\neq 0,\qquad x^3=8;$$

$$x\neq 0,\qquad x=2.$$



$f(x)$ monotone increase on $(-\infty,0)$ and $(2,+\infty)$;

$f(x)$ monotone decrease on $(0,2)$.

6. *Maxima and minima*

Local minimum at $x=2$, $f(2)=3$.

7. *Concavity down and up. Inflection points*

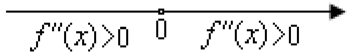
Find the second derivative

$$y''=\left(1-\frac{8}{x^3}\right)'=\frac{24}{x^4}.$$

Find the solution of equation

$$y'' = \frac{24}{x^4} = 0;$$

$$x \neq 0.$$



$f(x)$ concave down on $(-\infty,0)\cup(0,+\infty)$.

8. Graph

The graph of function is sketched in Figure 36.

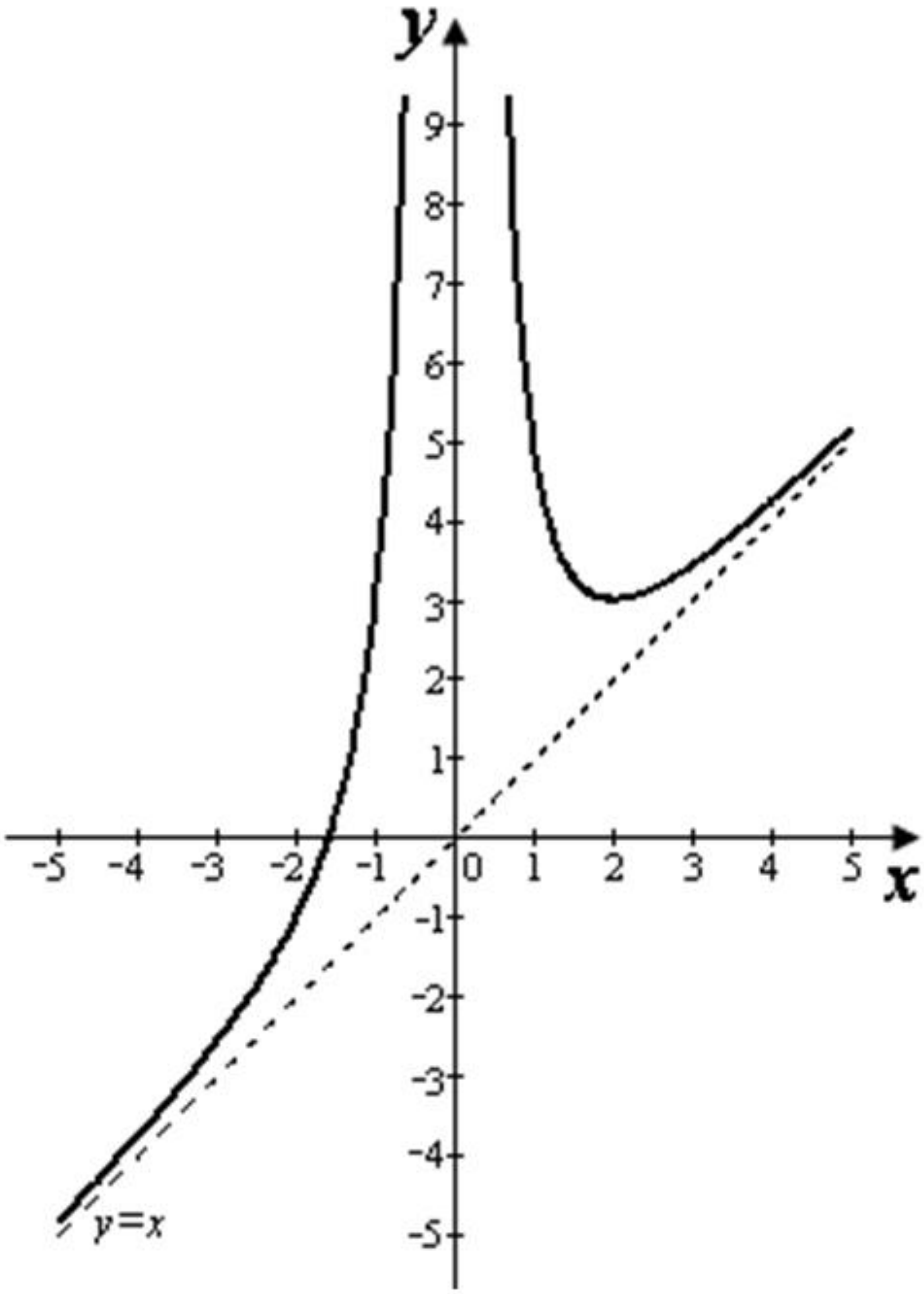


Figure 41.

Appendix 1. The Concept of Sets. Binary Operations with Sets

The set is a collection of distinct objects, considered as an object in its own right.

For example the set of digits {0,1,2,3,4,5,6,7,8,9}, set of colors of Ukrainian flag {blue, yellow}.

The empty set $\varnothing$ is a set that contains no elements.

Operations with Sets

Let **A** and **B** are two sets.

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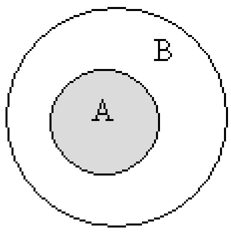

A is a *subset* of **B**

A⊂B

$A \subset A$
$\varnothing \subset A$
$A = B$ if and only if $A \subset B$ and $B \subset A$

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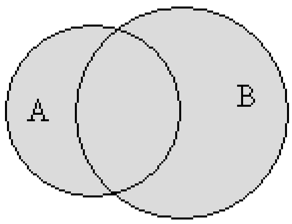

The *union* of **A** and **B**

A∪B

Properties of unions:

$A \cup B = B \cup A$
$A \cup (B \cup C) =$ $=(A \cup B) \cup C$
$A \subset (A \cup B)$
$A \cup A = A$
$A \cup \varnothing = A$
$A \subset B$ if and only if $A \cup B = B$

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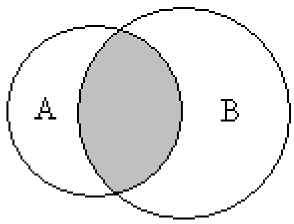

The *intersection* of **A** and **B**

A∩B

Properties of intersections:

$A \cap B = B \cap A$
$A \cap (B \cap C) = (A \cap B) \cap C$
$A \cap B \subset A$
$A \cap A = A$
$A \cap \varnothing = \varnothing$
$A \subset B$ if and only if $A \cap B = A$

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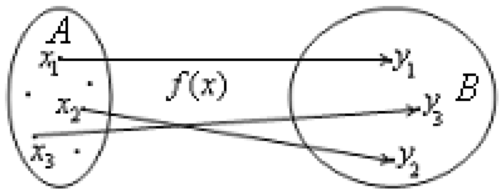
</table>

Appendix 2. The Concept of Function.
Basic Elementary Functions and Their Graphs

Let are sets $A \in \mathbb{R}$ and $B \in \mathbb{R}$. *Function* is a rule of relationship of each element x of the set A with exactly one element y of the set B. It is denoted by $y = f(x)$.



$$f \ : \ A \to B$$

$$\forall \ x \in A \ \exists! \ y \in B, \ y = f(x).$$

Domain, Range and Codomain

The set of x is called *domain* ($D(f)$). The number x belonging to the domain of the function is called the *independent variable or argument*.

The set containing $y = f(x)$ is called *codomain* ($E(f)$). The set of elements, that are the actual values produced by the function, is called *range* and the number y – *the dependent variable.*

The set of ordered pairs $(x, y) = (x, f(x))$ is called *graph*.

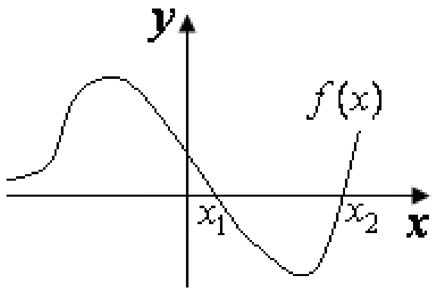
Input		Output
$x,\ t,\ \varphi,\ \theta\ldots$	Relationship	$f(x),\ g(x),\ h(x),\ y(x)\ \ldots$
domain elements	$f,\ g,\ h,\ y\ \ldots$	dependent variable
independent variable	$\longrightarrow$	value of function
argument		range elements
DOMAIN $D(f)$		RANGE $E(f)$

Zeros of a Function. Intervals of Positive and Negative Values of Function

Zeros of function ($f(x)=0$) is points where the graph crosses the *x*-axis.

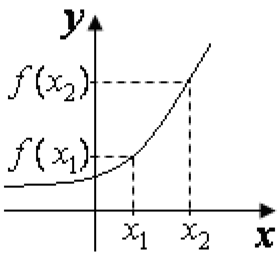
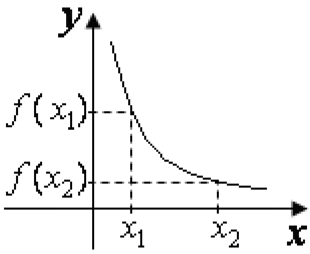
Function is positive $f(x)>0$ on intervals (on the *x*-axis), where the graph line lies *above the x-axis*.

Function is negative $f(x)<0$ on intervals (on the *x*-axis), where the graph line lies *below the x-axis*.

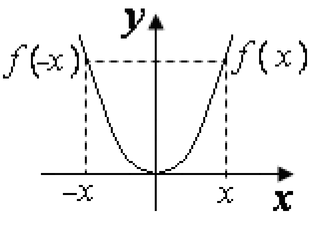
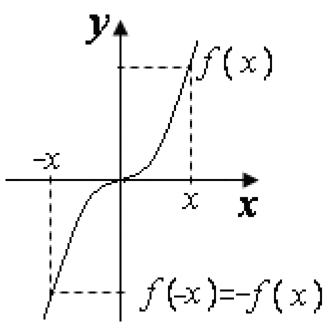


$f(x)>0$ for $x\in(-\infty,x_1)$ and $x\in(x_2,+\infty)$;
$f(x)<0$ for $x\in(x_1,x_2)$;
$f(x)=0$ for $x=x_1$ and $x=x_2$.

Increasing and Decreasing Functions.

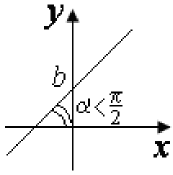
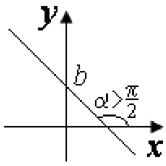
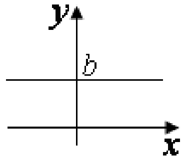
Increasing function	***Decreasing function***
$\forall\ \{x_1,x_2\}\in E(f)$, $x_1<x_2$: $f(x_1)\le f(x_2)$. 	$\forall\ \{x_1,x_2\}\in E(f)$, $x_1<x_2$: $f(x_1)\ge f(x_2)$. 

Symmetry of Function

Symmetry about y-axis	***Diagonal (origin) symmetry***
(Even Function). $f(-x)=f(x)$ 	***(Odd Function).*** $f(-x)=-f(x)$ 

In fact most functions are neither odd nor even.

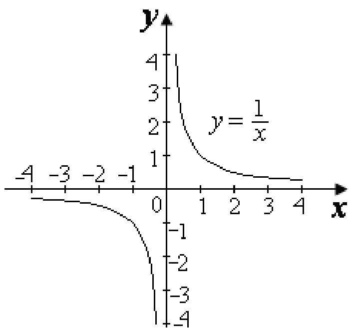
Linear Function

$y = kx + b$		
Slanted line for $k > 0$, for any b 	Slanted line for $k < 0$, for any b 	Horizontal line for $k = 0$, for any b 

Inverse Proportionality.

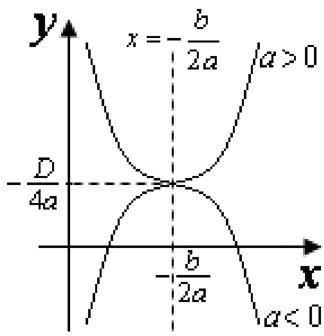
Direct proportionality is when one value increases and another value increases at the same rate ($y = kx$, k – constant of proportionality).

Inverse proportionality is when one value decreases at the same rate that the other increases. That relationship can be written as $y = \frac{k}{x}$ and called *reciprocal function*.



- Its graph is called *Hyperbola*;
- $D(f) = (-\infty, 0) \cup (0, +\infty)$;
- $E(f) = (-\infty, 0) \cup (0, +\infty)$;
- Line $y = 0$ is called *horizontal asymptote*;
- Line $x = 0$ is called v*ertical asymptote*.

Quadratic Function



$y = ax^2 + bx + c,\ a \neq 0$ (The graph is *Parabola*.)

- $D(f) = (-\infty, +\infty)$;
- $a > 0 \Rightarrow E(f) = \left(-\frac{D}{4a}, +\infty\right)$;
- $a < 0 \Rightarrow E(f) = \left(-\infty, -\frac{D}{4a}\right)$;
- It has vertex at point $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, where $D = b^2 - 4ac$;
- The curve has symmetry about the axis that passes through $x = -\frac{b}{2a}$.

Power Function

The function of form $y = x^n$, *n* – integer, is called *Power Function*.

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$y = x^n$ with even values of *n* behave the same:

- always above (or equal to) zero;
- always go through (0,0), (±1,1);
- it is even function;
- lager values of n flatten out near 0 and rise more sharply.

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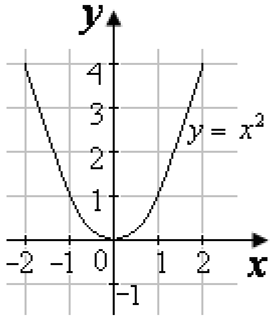

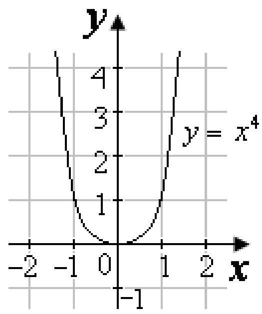

Parabola

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$y = x^n$ with odd values of *n* behave the same:

- always go from negative *x* and *y* to positive *x* and *y*;
- always go through (0,0), (1,1) and (–1,–1);
- it is odd function;
- lager values of n flatten out near 0 and rise more sharply.

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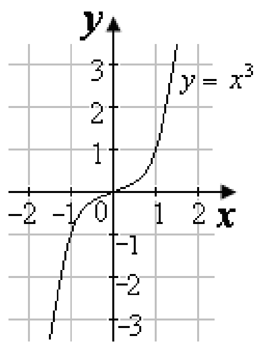

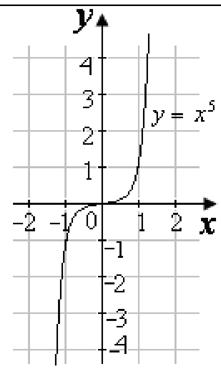

Cubic Parabola

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Inverse Power Functions

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$y = \sqrt[n]{x}$ with even values of *n* behave the same:

- domain $x \in [0,+\infty)$;
- always above (or equal to) zero;
- always go through (0,0), (1,1).

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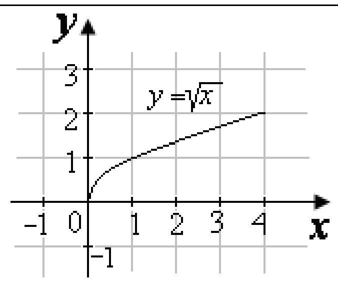

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<td>

$y = \sqrt[n]{x}$ with odd values of *n* behave the same:

- always go from negative *x* and *y* to positive *x* and *y*;
- always go through (0,0), (1,1) and (–1,–1);
- it is odd function.

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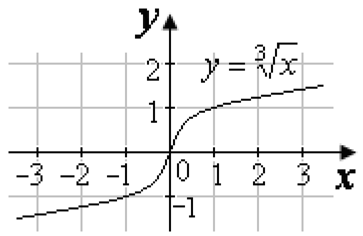

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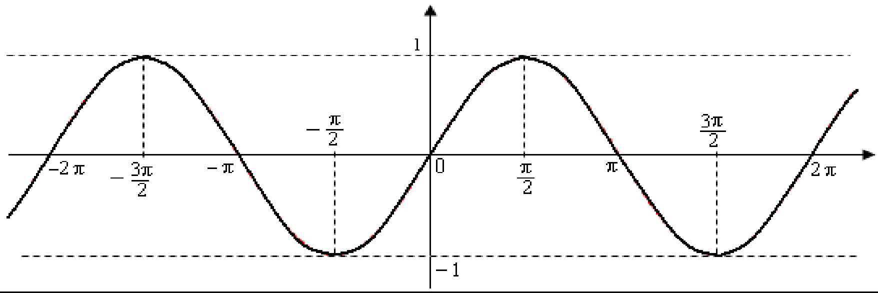
Trigonometric Functions

<table>
<tr>
<td>$y = \sin x$</td>
<td>

- $D(y) = (-\infty,\ +\infty)$; $E(y) = [-1, 1]$;
- $\sin(x + 2\pi n) = \sin x,\ n \in \mathbb{Z}$, $T = 2\pi$ – period;
- $\sin(-x) = -\sin x$, $y = \sin x$ – odd function;
- $\sin x = 0 \Rightarrow x = \pi n,\ n \in \mathbb{Z}$;
- $y_{\max} = 1,\ x_{\max} = \frac{\pi}{2} + 2\pi n$; $y_{\min} = -1,\ x_{\min} = -\frac{\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$;
- $\sin x > 0 \Rightarrow x \in (2\pi n, \pi + 2\pi n)$, $n \in \mathbb{Z}$;
- $\sin x < 0 \Rightarrow x \in (\pi + 2\pi n, 2\pi + 2\pi n)$, $n \in \mathbb{Z}$;
- $y = \sin x$ is increasing for $x \in \left(-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right)$, $n \in \mathbb{Z}$;
- $y = \sin x$ is decreasing for $x \in \left(\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right)$, $n \in \mathbb{Z}$;



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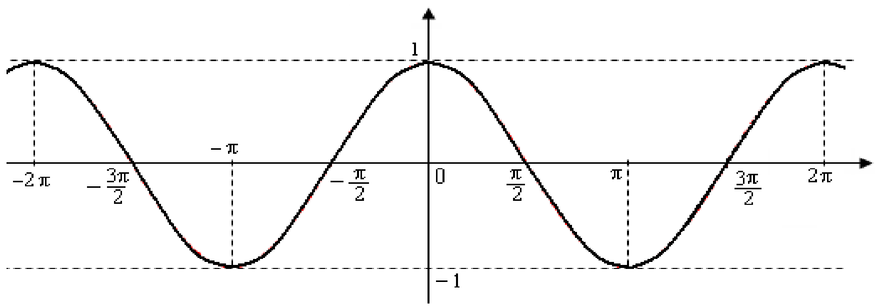
<tr>
<td>$y = \cos x$</td>
<td>

- $D(y) = (-\infty,\ +\infty)$; $E(y) = [-1, 1]$;
- $\cos(x + 2\pi n) = \cos x,\ n \in \mathbb{Z}$, $T = 2\pi n$– period;
- $\cos(-x) = \cos x$, $y = \cos x$ – even function;
- $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}$;
- $y_{\max} = 1,\ x_{\max} = 2\pi n$; $y_{\min} = -1,\ x_{\min} = \pi + 2\pi n$, $n \in \mathbb{Z}$;
- $\cos x > 0 \Rightarrow x \in \left(-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n\right)$, $n \in \mathbb{Z}$;
- $\cos x < 0 \Rightarrow x \in \left(\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right)$, $n \in \mathbb{Z}$;
- $y = \cos x$ is increasing for $x \in (\pi + 2\pi n, 2\pi + 2\pi n)$, $n \in \mathbb{Z}$;
- $y = \cos x$ is decreasing for $x \in (2\pi n, \pi + 2\pi n)$, $n \in \mathbb{Z}$;



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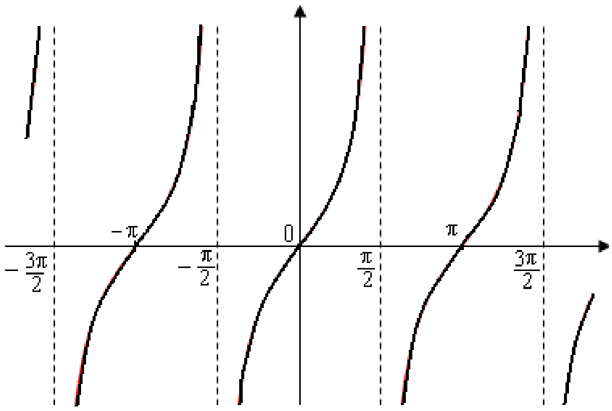
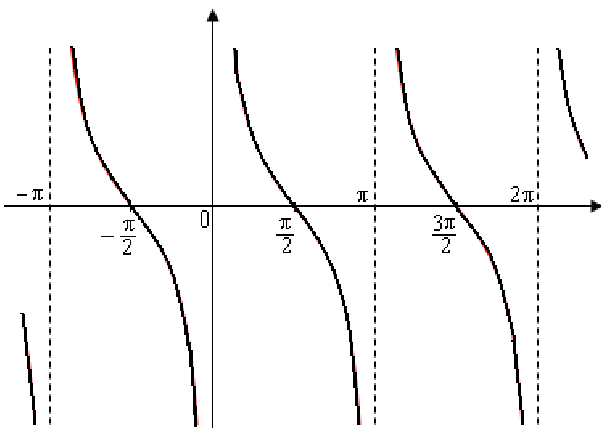
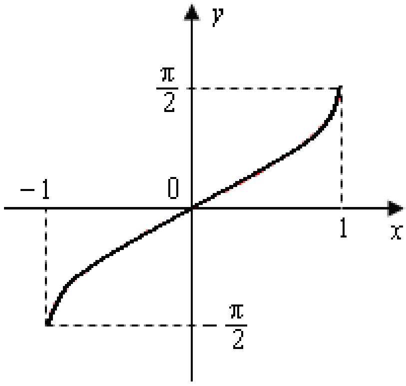
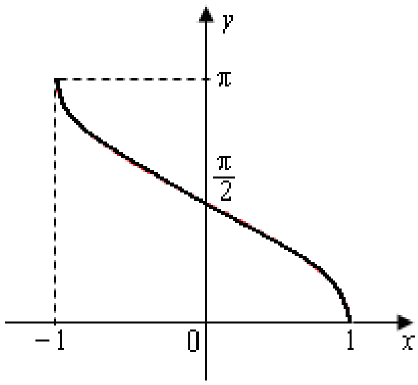
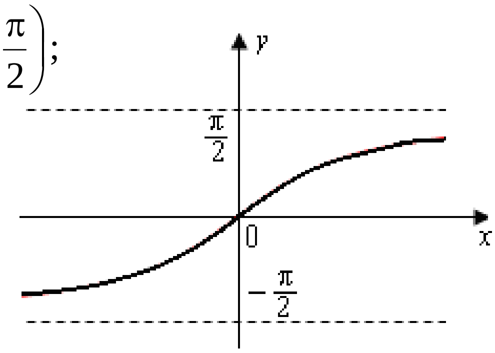
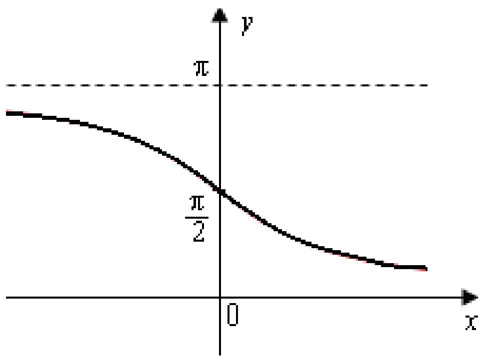
$y=\tan x$	• $D(y): x \neq \frac{\pi}{2}+\pi n,\ \ n\in\mathbb{Z};\ \ E(y)=(-\infty,\ +\infty);$ • $\tan(x+\pi n)=\tan x,\ n\in\mathbb{Z},\ T=\pi$ – period; • $\tan(-x)=-\tan x,\ \ y=\tan x$ – odd function; • $\tan x=0 \Rightarrow x=\pi n,\ n\in\mathbb{Z};$ • $x=\frac{\pi}{2}+\pi n,\ \ n\in\mathbb{Z}$, – vertical asymptotes; • $\tan x>0 \Rightarrow x\in\left(\pi n, \frac{\pi}{2}+\pi n\right),\ \tan x<0 \Rightarrow x\in\left(-\frac{\pi}{2}+\pi n, \pi n\right),\ n\in\mathbb{Z};$ • $y=\tan x$ is increasing for $x\in\left(-\frac{\pi}{2}+\pi n, \frac{\pi}{2}+\pi n\right),\ n\in\mathbb{Z};$ 
$y=\cot x$	• $D(y): x \neq \pi n,\ \ n\in\mathbb{Z};\ \ E(y)=(-\infty,\ +\infty);$ • $\cot(x+\pi n)=\cot x,\ n\in\mathbb{Z},\ T=\pi$ – period; • $\cot(-x)=-\cot x,\ \ y=\cot x$ – odd function; • $\cot x=0 \Rightarrow x=\frac{\pi}{2}+\pi n,\ n\in\mathbb{Z};$ • $x=\pi n,\ \ n\in\mathbb{Z}$, – vertical asymptotes; • $\cot x>0 \Rightarrow x\in\left(\pi n, \frac{\pi}{2}+\pi n\right);\ \cot x<0 \Rightarrow x\in\left(\frac{\pi}{2}+\pi n, \pi+\pi n\right),\ n\in\mathbb{Z};$ • $y=\cot x$ is decreasing for $x\in(\pi n, \pi+\pi n),\ n\in\mathbb{Z};$ 

Inverse Trigonometric Functions

Inverse Sine. $y = \arcsin x$ It answers the question "what **angle** has sine equal to *x*?"	• $D(y) = [-1, 1]$; $E(y) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; • $\arcsin(-x) = -\arcsin x$, $y = \arcsin x$ – odd function; • $\arcsin x = 0 \Rightarrow x = 0$; • $\arcsin x > 0 \Rightarrow x \in (0, 1]$; $\arcsin x < 0 \Rightarrow x \in [-1, 0)$; • $y = \sin x$ is increasing for $x \in [-1, 1]$; 
Inverse Cosine. $y = \arccos x$ It answers the question "what **angle** has cosine equal to *x*?"	• $D(y) = [-1, 1]$; $E(y) = [0, \pi]$; • $\arccos(-x) = \pi - \arccos x$; • $\arccos x = 0 \Rightarrow x = 1$; • $\arccos x > 0 \Rightarrow x \in [-1, 1)$; • $y = \arccos x$ is decreasing for $x \in [-1, 1]$; 
Inverse Tangent. $y = \arctan x$ It answers the question "what **angle** has tangent equal to *x*?"	• $D(y) = (-\infty, +\infty)$; $E(y) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; • $\arctan(-x) = -\arctan x$, $y = \arctan x$ – odd function; • $\tan x = 0 \Rightarrow x = 0$; • $y = \pm\frac{\pi}{2}$ – horizontal asymptotes;  • $\arctan x > 0 \Rightarrow x \in (0, +\infty)$; $\arctan x < 0 \Rightarrow x \in (-\infty, 0)$; • $y = \arctan x$ is increasing for $x \in (-\infty, +\infty)$;
Inverse Cotangent. $y = \operatorname{arccot} x$ It answers the question "what **angle** has cotangent equal to *x*?"	• $D(y) = (-\infty, +\infty)$; $E(y) = (0, \pi)$; • $\operatorname{arccot}(-x) = \pi - \operatorname{arccot} x$; • $\begin{matrix} x = 0, \\ x = \pi \end{matrix}$ – horizontal asymptotes; • $\operatorname{arccot} x > 0 \Rightarrow x \in (-\infty, +\infty)$; • $y = \operatorname{arccot} x$ is decreasing for $x \in (-\infty, +\infty)$; 

Exponential Functions

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<tr>
<th>$y = a^x$, $0 < a < 1$</th>
<th>$y = a^x$, $a > 1$</th>
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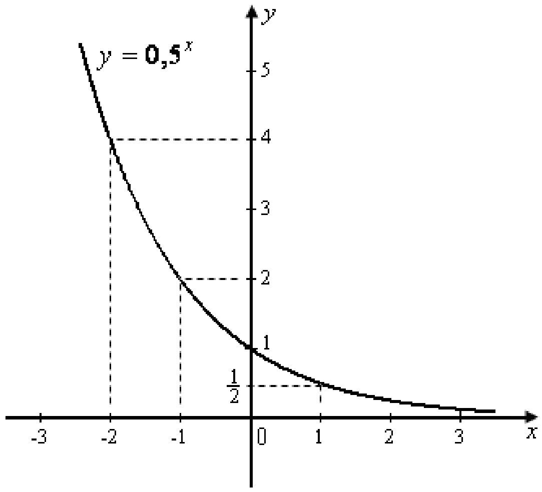
<td>

• $D(y) = (-\infty,\ +\infty)$;
• $E(y) = (0,\ +\infty)$;
• it is neither odd nor even function;
• it is always greater then zero and never intersects the x-axis;
• it intersects the y-axis at y=1;
• y=0 – horizontal asymptote;
• $y = a^x$ is **decreasing** for $x \in (-\infty,\ +\infty)$;



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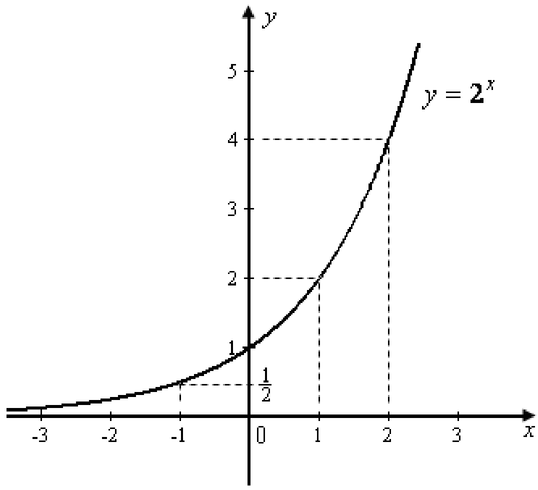
<td>

• $D(y) = (-\infty,\ +\infty)$;
• $E(y) = (0,\ +\infty)$;
• it is neither odd nor even function;
• it is always greater then zero and never intersects the x-axis;
• it intersects the y-axis at y=1;
• y=0 – horizontal asymptote;
• $y = a^x$ is **increasing** for $x \in (-\infty,\ +\infty)$;



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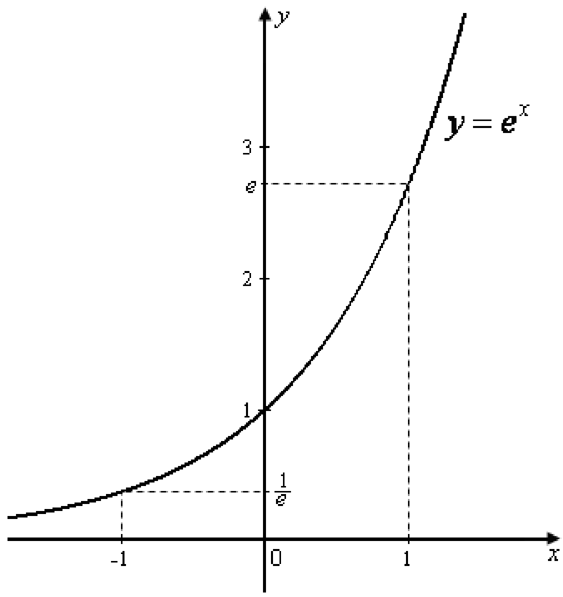
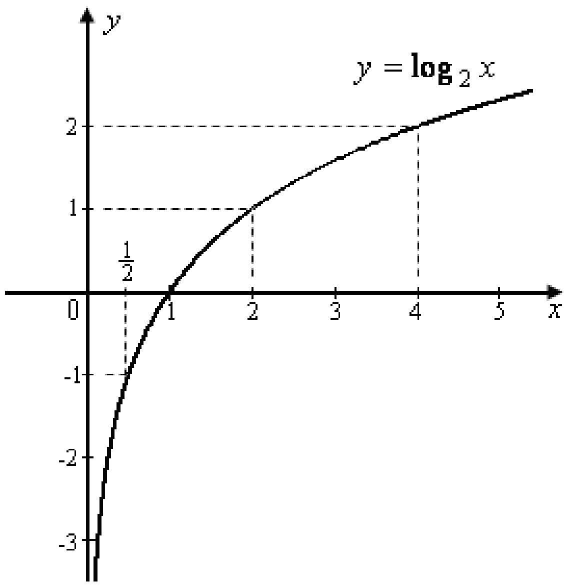
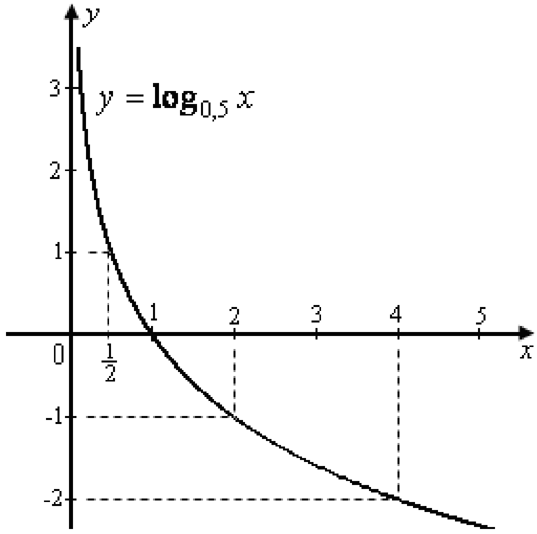
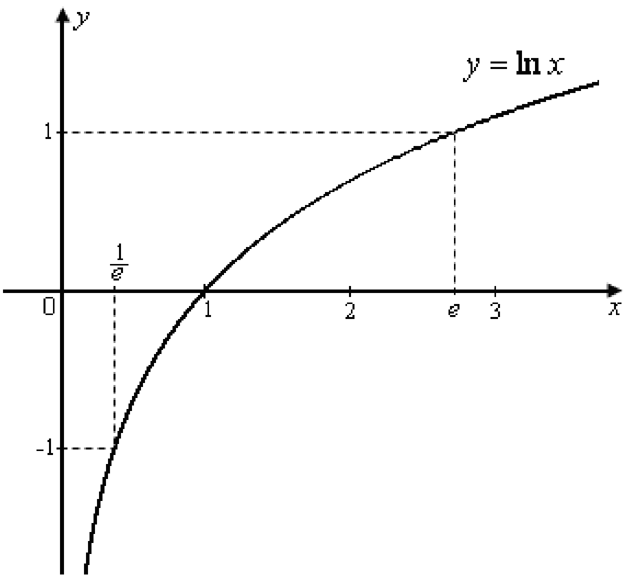
Natural Exponential Function $y = e^x$



Logarithmic Functions

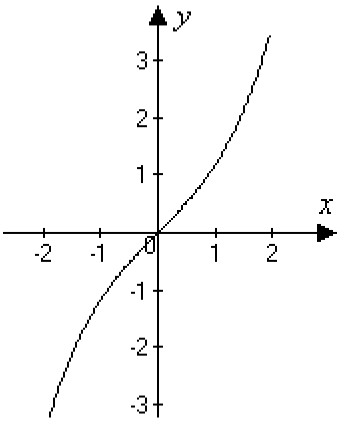
$y = \log_a x$, $a > 1$	$y = \log_a x$, $0 < a < 1$
• $D(y) = (0,\ +\infty)$; • $E(y) = (-\infty,\ +\infty)$; • it is neither odd nor even function; • it intersects the *x*-axis at *x*=1; • it is never intersects the *y*-axis; • *x*=0 – vertical asymptote; • $y = \log_a x$ is **increasing** for $x \in (0,\ +\infty)$;  	• $D(y) = (0,\ +\infty)$; • $E(y) = (-\infty,\ +\infty)$; • it is neither odd nor even function; • it intersects the *x*-axis at *x*=1; • it is never intersects the *y*-axis; • *x*=0 – vertical asymptote; • $y = \log_a x$ is **decreasing** for $x \in (0,\ +\infty)$; 

Natural Logarithm $y = \ln x$

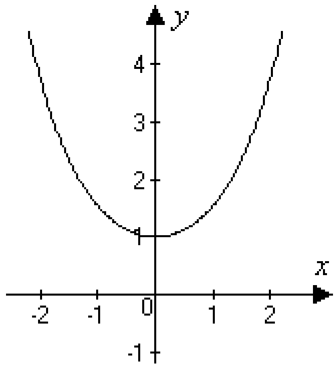


Hyperbolic Functions

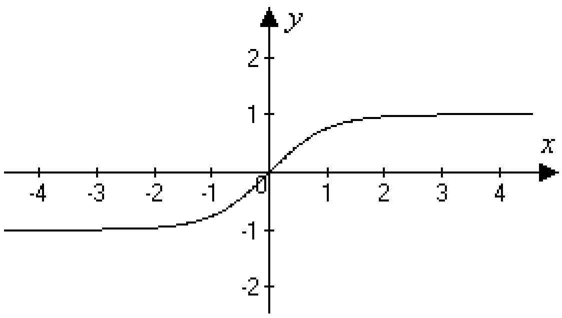
<table>
<tr><th>$y = \sinh x$</th><th>$y = \cosh x$</th></tr>
<tr><td>

• $D(y) = (-\infty,\ +\infty)$;
• $E(y) = (-\infty,\ +\infty)$;
• $\sinh x = \dfrac{e^x - e^{-x}}{2}$;


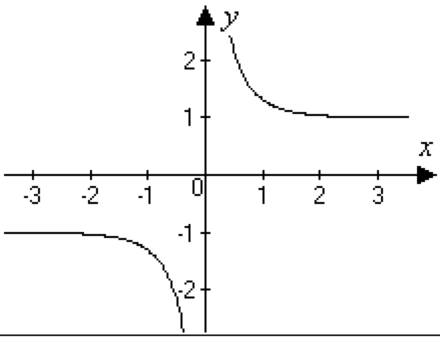
</td><td>

• $D(y) = (-\infty,\ +\infty)$;
• $E(y) = (1,\ +\infty)$;
• $\cosh x = \dfrac{e^x + e^{-x}}{2}$;


</td></tr>
<tr><th>$y = \tanh x$</th><th>$y = \coth x$</th></tr>
<tr><td>

• $D(y) = (-\infty,\ +\infty)$;
• $E(y) = (-1,\ 1)$;
• $\tanh x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$;
• $y = \pm 1$ – horizontal asymptotes;


</td><td>

• $D(y) = (-\infty, 0) \cup (0,\ +\infty)$;
• $E(y) = (-\infty, -1) \cup (1, +\infty)$;
• $\coth x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
• $y = \pm 1$ – horizontal asymptotes;
• x=0 – vertical asymptote;


</td></tr>
</table>

$$\cosh^2 x - \sinh^2 x = 1; \quad 1 - \tanh^2 x = \frac{1}{\cosh^2 x}; \quad 1 - \coth^2 x = \frac{1}{\sinh^2 x};$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x; \quad \sinh 2x = 2\cosh x \sinh x;$$

$$\cosh 2x - 1 = 2\sinh^2 x; \quad \cosh 2x + 1 = 2\cosh^2 x;$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y; \quad \sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y;$$

Appendix 3. Polar Coordinates

We determine the coordinate system as the way to define a point in the space. For example, in the Cartesian coordinate system we specify each point by a pair of numbers (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically.

But there is another way to define the position of a point on a plane. We can use the distance from the origin (*pole*) and the angle from the fixed direction (Fig.). The ray from the pole in reference direction is called the *polar axis*.

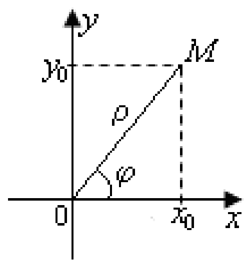


Figure .

The distance from the pole is called *the radius* ρ and the angle – the *polar angle* φ. Coordinates in this form are called the *polar coordinates.*

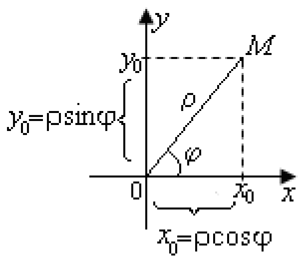
Converting Between Polar and Cartesian Coordinates

The relationship between Cartesian and polar coordinates is represented by the following formulas:

$$M(x, y) \qquad\qquad M(\rho, \varphi)$$

$$\begin{cases} x = \rho\cos\varphi, \\ y = \rho\sin\varphi; \end{cases} \quad \Rightarrow \quad \begin{cases} \rho = \sqrt{x^2 + y^2}, \\ \varphi = \arctan\dfrac{y}{x}. \end{cases}$$

$$\rho \geq 0,\ \ \varphi \in [-\pi, \pi)$$



Many curves can be described by a rather simple polar equation, whereas their Cartesian form is much more difficult. The best known of these curves are the circle, polar rose, Archimedean spiral, lemniscate and cardioid.

Examples.

1. Consider the unit circle $x^2 + y^2 = 1$. Apply the formulas of converting Cartesian and polar coordinates:

$$(\rho\cos\varphi)^2 + (\rho\sin\varphi)^2 = \rho^2(\cos^2\varphi + \sin^2\varphi) = \rho^2 = 1.$$

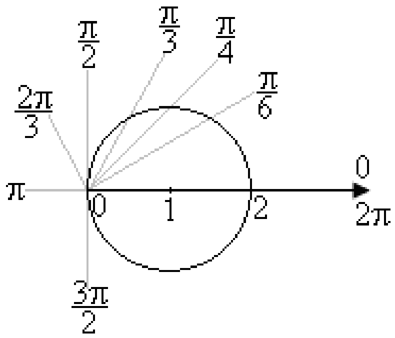
Hence $\rho = 1$ is an equation of unit circle in polar coordinates.

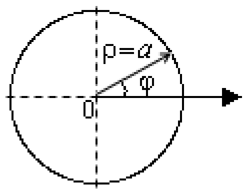
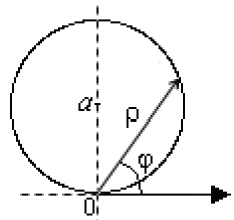
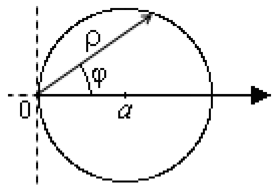
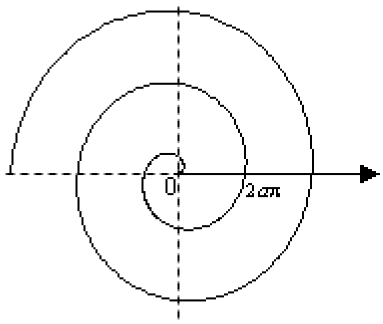
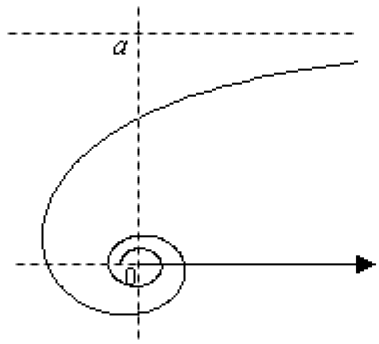
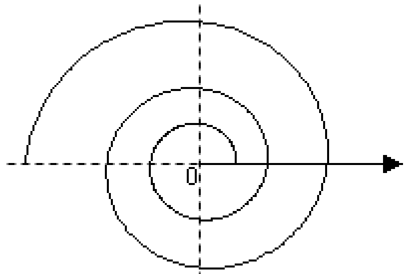
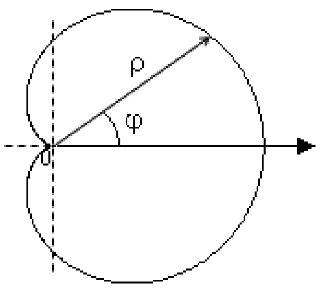
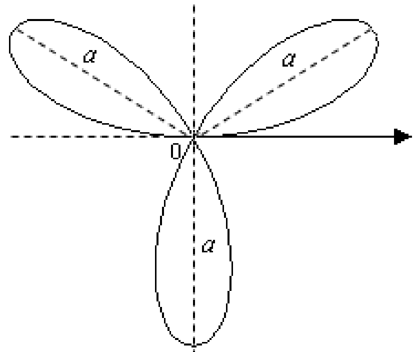
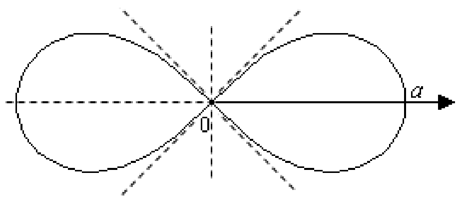
2. Convert to the polar coordinates and plot the curve $x^2 + y^2 = 2x$.

$x^2 + y^2 = 2x \Rightarrow (\rho\cos\varphi)^2 + (\rho\sin\varphi)^2 = 2\rho\cos\varphi \Rightarrow$

$\rho^2 = 2\rho\cos\varphi \Rightarrow \rho = 2\cos\varphi$. It is the circle $(x-1)^2 + y^2 = 1$.

φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
ρ	2	$\sqrt{3}$	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2	0	2



Graphs of certain functions in polar coordinates.		
 Circle $x^2 + y^2 = a^2$ $\rho = a$	 Circle $x^2 + y^2 = 2ay$ $\rho = 2a\sin\varphi$	 Circle $x^2 + y^2 = 2ax$ $\rho = 2a\cos\varphi$
 Archimedean spiral $\rho = a\varphi$	 Hyperbolic spiral $\rho = \frac{a}{\varphi}$	 Logarithmic spiral $\rho = e^{a\varphi}$
 Cardioid $\rho = a(1+\cos\varphi)$	 Triple-petaled rose $\rho = a\sin 3\varphi$	 Lemniscate of Bernoulli $\rho = a\sqrt{\cos 2\varphi}$

Appendix 4. Parametric Representation of a Function

There are some curves that we can't write down as a equation in terms of only x and y or it is too complicated. So, to deal with these curves we use parametric equations. Instead of defining y as a function of x ($y = y(x)$) or x as a function of y ($x = x(y)$) we define both x and y in terms of a third variable as follows

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases}$$

where t assumes values that lie in the interval $[T_1,\ T_2]$.

The variable t is called a *parameter* and equations $\begin{cases} x = \varphi(t), \\ y = \psi(t) \end{cases}$ are called *parametric equation* of some curve.

To each value of t there will be corresponded a definite point $(x, y) = (\varphi(t), \psi(t))$ in the plane that we can plot. When t varies from T_1 to T_2 this point will describe a certain curve and this curve is called the *parametric curve*.

If the function $x = \varphi(t)$ has an inverse, $t = \Phi(x)$, then y is a function in terms of x

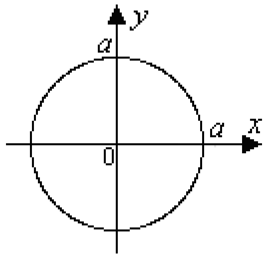
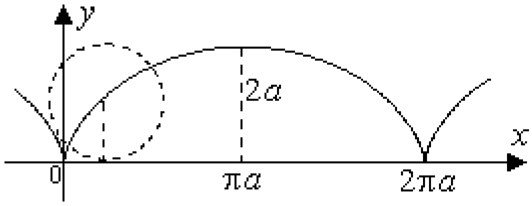
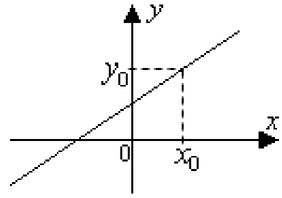
$$y = \psi(\Phi(x)).$$

It is said that the function $y = \psi(\Phi(x))$ is represented *parametrically* as $\begin{cases} x = \varphi(t), \\ y = \psi(t). \end{cases}$ The explicit expression of the dependence y on x, is obtained by eliminating t the parameter from equations $\begin{cases} x = \varphi(t), \\ y = \psi(t). \end{cases}$

Parametric equations are widely used in mechanics. If in the plane there is a certain material point in motion and if we know the laws of motion of the projections of this point on the coordinate axes, then

$$\begin{cases} x = \varphi(t), \\ y = \psi(t), \end{cases}$$

where the parameter t is the time. Then equations are parametric equations of the trajectory of moving point.

<table>
<tr><th colspan="3">Graphs of certain functions in parametric form.</th></tr>
<tr>
<td>
Circle
$$\begin{cases} x = a\cos t, \\ y = a\sin t, \end{cases} t \in [0, 2\pi]$$</td>
<td>
Cycloid
$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \end{cases} t \in [0, 2\pi]$$</td>
<td>
Straight line
$$y - y_0 = \frac{m}{l}(x - x_0),$$
$$\begin{cases} x = lt + x_0, \\ y = mt + y_0. \end{cases}$$</td>
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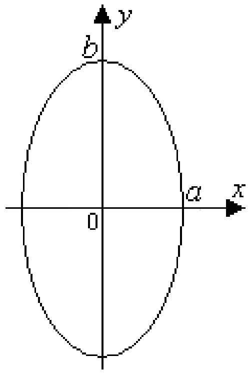
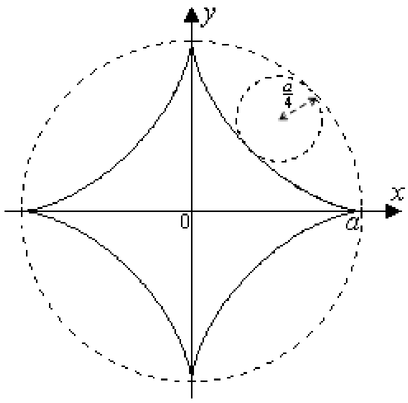
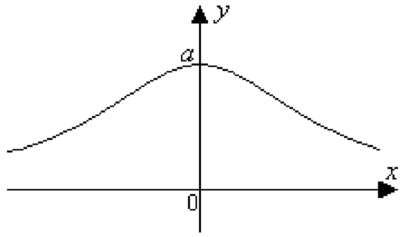
<tr>
<td>
Ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} t \in [0, 2\pi]$$</td>
<td>
Astroid
$$x^{2/3} + y^{2/3} = a^{2/3}$$
$$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t, \end{cases} t \in [0, 2\pi]$$</td>
<td>
Witch of Agnesi
$$y(x^2 + a^2) = a^3$$
$$\begin{cases} x = at, \\ y = \dfrac{a}{t^2 + 1}. \end{cases}$$</td>
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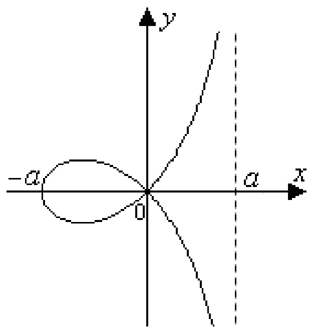
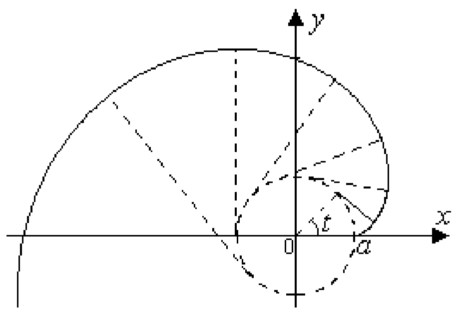
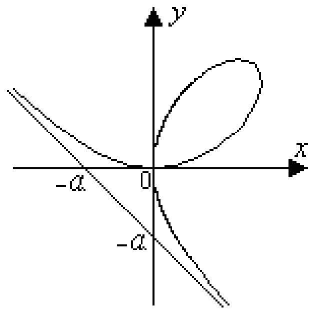
<td>
Strophoid
$$y^2(a - x) = x^2(a + x)$$
$$\begin{cases} x = a\dfrac{t^2 - 1}{t^2 + 1}, \\ y = at\dfrac{t^2 - 1}{t^2 + 1}. \end{cases}$$</td>
<td>
Involute of a circle
$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t), \end{cases} t \in [0, 2\pi]$$</td>
<td>
Folium of Decartes
$$y^3 + x^3 = 3axy$$
$$\begin{cases} x = \dfrac{3at}{t^3 + 1}, \\ y = \dfrac{3at^2}{t^3 + 1}. \end{cases}$$</td>
</tr>
</table>

REFERENCES

1. N.Piscunov Differential and Integral Calculus/ N.Piscunov - Mir Publisher, Moscow, 1966 - 895 p.

2. Smimov V. I. A Course of Higher Mathematics, Vo l.: Elementary Calculus/ Smimov V. I. - Oxford, Pergamon Press - 1964 – 558 p.

3. H. Jerome Keisler Elementary Calculus: an Infinitesimal Approach / H. Jerome Keisler - On-line Edition. 2000 - https://www.math.wisc.edu/~keisler/calc.html

4. http://tutorial.math.lamar.edu

5. Maron I.A. Problems in Calculus of One Variable/ Maron I.A.- Mir Publisher, Moscow, 1975 - 455 p.

6. Swokowski Earl William Calculus. 5th Edition / Swokowski Earl W., - Brooks/Cole, 1991 - 1152 p.