

DIFFERENTIAL EQUATIONS INVARIANT WITH RESPECT TO THE ALGEBRA $\mathfrak{sl}(2, \mathbb{C})$ AND ITS CONTRACTIONS

M.O. NESTERENKO, M.S. STARYI

In 1953 [1] E. Inönü and E. Wigner showed that different physical theories are connected through contractions of their underlying symmetry algebras. The construction of contractions of realizations is a necessary step for establishing limiting relations between differential equations and physical theories. In this work, we explicitly construct the contractions between rank-three vector fields of the algebra $\mathfrak{sl}(2, \mathbb{C})$ and their differential invariants.

Let $\mathcal{L}_n$ be the set of all possible Lie brackets on V . We identify a Lie product $\mu \in \mathcal{L}_n$ with the Lie algebra $g = (V, \mu)$. $\mathcal{L}_n$ is an algebraic subset of the variety $V^* \otimes V^* \otimes V$ of bilinear maps from $V \times V$ to V .

The group $GL(V)$ acts on $\mathcal{L}_n$ in the following way:

$$(U \cdot \mu)(x, y) = U(\mu(U^{-1}x, U^{-1}y)) \quad \text{for all } U \in GL(V), \mu \in \mathcal{L}_n, x, y \in V.$$

Fixing the basis $\{e_1, \dots, e_n\}$ of V we can use the one-to-one correspondence between Lie bracket $\mu \in \mathcal{L}_n$ and a structure constant tensor (c_{ij}^k) . In this case the notion of a contraction matrix U_ε ($U: (0, 1] \rightarrow GL(V)$) and a parameterized family of new Lie brackets $[x, y]_\varepsilon$ is determined by

$$[x, y]_\varepsilon = U_\varepsilon^{-1}[U_\varepsilon x, U_\varepsilon y] \quad \text{for all } \varepsilon \in (0, 1], x, y \in V,$$

and the contraction is defined as follows.

If the limit $\lim_{\varepsilon \rightarrow +0} [x, y]_\varepsilon = \lim_{\varepsilon \rightarrow +0} U_\varepsilon^{-1}[U_\varepsilon x, U_\varepsilon y] =: [x, y]_0$ exists for any $x, y \in V$, then $[\cdot, \cdot]_0$ is a well-defined Lie bracket and $g_0 = (V, [\cdot, \cdot]_0)$ is called a *contraction* of the Lie algebra g .

Consider complex three-dimensional special linear Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with the basis elements $\{e_i : i = 1, \dots, 3\}$ and nonzero commutation relations

$$[e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 2e_2. \quad (1)$$

We also fix the representatives of the other necessary Lie algebras specifying their nonzero commutation relations

$$\mathfrak{g}_{3.4}^{-1}: [e_1, e_3] = e_1, [e_2, e_3] = -e_2; \quad \mathfrak{g}_{3.1}: [e_2, e_3] = e_1.$$

Theorem 1 (Nesterenko M., Popovych R. [2]). *The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ can be contracted only to the three-dimensional Poincaré algebra $p(1, 1) \sim \mathfrak{g}_{3.4}^{-1}$, to the Heisenberg algebra $\mathfrak{g}_{3.1}$, and to the three-dimensional abelian Lie algebra $3\mathfrak{g}_1$.*

This sequence of contractions is performed in the following diagram

$$\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{g}_{3.4}^{-1} \rightarrow \mathfrak{g}_{3.1} \rightarrow 3\mathfrak{g}_1.$$

The Inönü-Wigner contraction from $\mathfrak{sl}(2, \mathbb{C})$ to the Poincaré algebra $p(1, 1)$ is provided by the contraction matrix $U_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The structure constants c_{ij}^k of $\mathfrak{sl}(2, \mathbb{C})$ transform as $c_{ij}^k \rightarrow (c_\varepsilon)_{ij}^k \rightarrow \tilde{c}_{ij}^k$, where $\tilde{c}_{ij}^k$ are structure constants of $p(1, 1)$. The first arrow implies the application of the contraction matrix $U(\varepsilon)$ and the second arrow implies the limit process $\varepsilon \rightarrow 0$. More precisely, $\tilde{c}_{ij}^k(\varepsilon) = \lim_{\varepsilon \rightarrow 0} U_{\varepsilon i}^\alpha U_{\varepsilon j}^\beta c_{\alpha\beta}^\gamma (U_\varepsilon^{-1})_\gamma^k$, $i, j, k, \alpha, \beta, \gamma = 1, \dots, n$.

Let $M \subseteq \mathbb{C}^m$ be a complex domain and $\mathfrak{g}$ a Lie algebra. Denote by $\text{Vect}(M)$ the Lie algebra of analytic vector fields on M , endowed with the Lie bracket of vector fields $[\cdot, \cdot]$. A Lie algebra homomorphism $R: \mathfrak{g} \rightarrow \text{Vect}(M)$ is called a *realization* of the Lie algebra $\mathfrak{g}$.

To find the contraction of a realization, we need to construct a realization for the parameterized structure constants $(c_\varepsilon)_{ij}^k$ of $\mathfrak{sl}(2, \mathbb{C})$. For this, one can apply either the direct method from the previous section or the Shirokov's method. We constructed the maximal rank-three parameterized realization of the form (below $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$)

$$e_1 = \partial_1, \quad e_2 = \varepsilon x_1^2 \partial_1 + (1 - 2\varepsilon x_1 x_2) \partial_2 + 2\varepsilon x_1 \partial_3, \quad e_3 = x_1 \partial_1 - x_2 \partial_2 + \partial_3.$$

Taking the limit as ε tends to zero, we obtain a well-defined exact realization of rank three for the three-dimensional Poincaré algebra

$$e_1 = \partial_1, \quad e_2 = \partial_2, \quad e_3 = x_1 \partial_1 - x_2 \partial_2 + \partial_3.$$

Similarly, we can construct the Inönü-Wigner contraction from the Poincaré algebra to the Heisenberg algebra, as well as their composition.

Our interest in contractions of realizations of the algebra $\mathfrak{sl}(2, \mathbb{C})$ is motivated by the fact that it is the symmetry algebra of the Ermakov system, and to study its limiting cases we constructed differential invariants of the limiting realizations.

REFERENCES

- [1] Inonu E., Wigner E.P. *On the contraction of groups and their representations* // Proc. Nat. Acad. Sci. U.S.A.— 1953. —V. 39 — P. 510–524.
- [2] Nesterenko M., Popovych R. *Contractions of low-dimensional Lie algebras* // Journal of Mathematical Physics — 2006. — V. 47, 123515. — 45 p.

INSTITUTE OF MATHEMATICS, NAS OF UKRAINE, KYIV, UKRAINE AND UNIVERSITY "KYIV SCHOOL OF ECONOMICS", KYIV, UKRAINE

Email address: maryna.nesterenko@gmail.com

INSTITUTE OF MATHEMATICS, NAS OF UKRAINE, KYIV, UKRAINE

Email address: staryimath@gmail.com