On an analytical method for calculating the bending of rectangular plates on an inhomogeneous elastic foundation

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Abstract

An analytical method for calculating the bending of a rectangular plate resting on an inhomogeneous elastic foundation and subjected to a continuously distributed variable transverse load is proposed. It is assumed that two parallel sides of the plate are hinged, while the other two are fixed in any manner. The foundation reaction is described by the Winkler model with a variable bedding coefficient.

Keywords: rectangular plate; inhomogeneous elastic foundation; Winkler hypothesis; variable bedding coefficient; analytical solution.

The problem of bending a rectangular plate ($0 \leqslant x \leqslant a$; $0 \leqslant y \leqslant b$) with constant cylindrical rigidity D is considered, where the sides $x = 0$; $x = a$ are fixed in any manner, and the sides $y = 0$; $y = b$ are hinged. The plate rests on an inhomogeneous elastic foundation and is subjected to a transverse continuously distributed load with intensity $q(x, y)$ (Fig. 1).

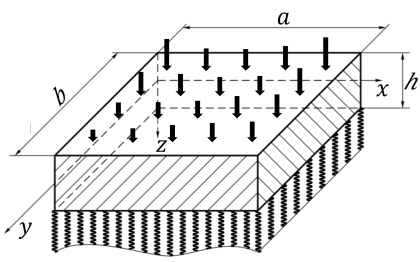

Figure 1: Plate on an elastic foundation under the action of a transverse load.

In the general case, the coefficient k is a function of the variables x and y. However, here it is assumed that the bedding coefficient continuously changes along the x-axis, while remaining constant along the y-axis, i.e., $k = k(x)$. For $k(x)$, the representation $k_0 B(x)$ is adopted, where k_0 is the value of the bedding coefficient at some characteristic point (for example, at the point $x = 0$), and $B(x)$ is a continuous dimensionless function that expresses the law of variation of the bedding coefficient over the segment $0 \leqslant x \leqslant a$.

The equation of the elastic surface of the plate in our case will have the form

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + k_0 B(x) w = q(x, y). \tag{1}$$

With the aim of satisfying the boundary conditions

$$w = 0, \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) = 0$$

at the edges of the plate $y = 0$ and $y = b$, the solution to the equation (1) is sought in the form

$$w(x, y) = \sum_{m=1}^{\infty} w_m(x) \sin \frac{m\pi y}{b}. \tag{2}$$

The given load is also approximated by a trigonometric series

$$q(x, y) = \sum_{m=1}^{\infty} q_m(x) \sin \frac{m\pi y}{b}. \tag{3}$$

For the function $q_m(x)$, we adopt the representation $q_m(x) = q_{m,0} C_m(x)$, where $q_{m,0}$ is a constant dimensional factor, and $C_m(x)$ is a dimensionless function.

Substituting the representations (2) and (3) into equation (1) and equating the terms of the series with the same indices, we obtain the ordinary nonhomogeneous fourth-order differential equation

$$w_m''''(x) - \frac{2\lambda_m^2}{a^2} w_m''(x) + \frac{1}{a^4}\left(\lambda_m^4 + K B(x)\right) w_m(x) = \frac{q_{m,0}}{D} C_m(x), \tag{4}$$

where

$$K = \frac{k_0 a^4}{D}, \ \lambda_m = \frac{m\pi a}{b}.$$

The solution of the equation (4) is found using the method developed in [Krutii, 2016], [Krutii, 2018], [Krutii, Suriyaninov, Vandynskyi 2018], [Krutii, Vandynskyi, 2019], [Krutii, Sur'yaninov, Karnaukhova, 2021]. The corresponding formulas, after approximating the functions $B(x)$ and $C_m(x)$ by Maclaurin series in powers of x/a, will have the form:

$$w_m(x) = w_m(0) X_{m,1}(x) + w_m'(0) a X_{m,2}(x) + w_m''(0) a^2 X_{m,3}(x) + w_m'''(0) a^3 X_{m,4}(x) + \frac{q_{m,0} a^4}{D} X_{m,5}(x); \tag{5}$$

$$X_{m,n}(x) = \alpha_{m,n,0}(x) - \alpha_{m,n,1}(x)K + \alpha_{m,n,2}(x)K^2 - \alpha_{m,n,3}(x)K^3 + \ldots \quad (n = 1,2,3,4,5); \tag{6}$$

$$\alpha_{m,n,k}(x) = \left(\frac{x}{a}\right)^{n+4k-1} \sum_{j=0}^{\infty} c_{m,n,k,j} \left(\frac{x}{a}\right)^j \quad (n = 1,2,3,4,5)(k = 0,1,2,\ldots); \tag{7}$$

$$c_{m,n,0,0} = \frac{1}{(n-1)!} \quad (n = 1,2,3,4); \tag{8}$$

$$c_{m,n,0,j} = 0 \quad (n = 1,2,3,4)(j = 1,3,5,\ldots); \tag{9}$$

$$c_{m,n,0,j} = \left(1 - \frac{j}{2}\right) \frac{\lambda_m^j}{(n+j-1)!} \quad (n = 1,2)(j = 2,4,6,\ldots); \tag{10}$$

$$c_{m,n,0,j} = \left(1 + \frac{j}{2}\right) \frac{\lambda_m^j}{(n+j-1)!} \quad (n = 3,4)(j = 2,4,6,\ldots); \tag{11}$$

$$c_{m,5,0,j} = \frac{C_{m,j}}{r_{5,0,j} r_{5,0,j+2}} \quad (j = 0,1); \tag{12}$$

$$c_{m,5,0,j} = \frac{1}{r_{5,0,j}\, r_{5,0,j+2}} \left(C_{m,j} + 2\lambda_m^2 r_{5,0,j}\, c_{m,5,0,j-2}\right) \quad (j = 2,3); \tag{13}$$

$$c_{m,5,0,j} = \frac{1}{r_{5,0,j}\, r_{5,0,j+2}} \left(C_{m,j} + 2\lambda_m^2 r_{5,0,j}\, c_{m,5,0,j-2} - \lambda_m^4 c_{m,5,0,j-4}\right) \quad (j = 4,5,6,\ldots); \tag{14}$$

$$c_{m,n,k,j} = \frac{1}{r_{n,k,j}\, r_{n,k,j+2}} \sum_{i=0}^{j} B_{j-i}\, c_{m,n,k-1,i} \quad (n = 1,2,3,4,5)(k = 1,2,3,\ldots)(j = 0,1); \tag{15}$$

$$c_{m,n,k,j} = \frac{1}{r_{n,k,j}\, r_{n,k,j+2}} \left(\sum_{i=0}^{j} B_{j-i}\, c_{m,n,k-1,i} + 2\lambda_m^2 r_{n,k,j}\, c_{m,n,k,j-2}\right) \quad (n = 1,2,3,4,5)(k = 1,2,3,\ldots)(j = 2,3); \tag{16}$$

$$c_{m,n,k,j} = \frac{1}{r_{n,k,j}\, r_{n,k,j+2}} \left(\sum_{i=0}^{j} B_{j-i}\, c_{m,n,k-1,i} + 2\lambda_m^2 r_{n,k,j}\, c_{m,n,k,j-2} - \lambda_m^4 c_{m,n,k,j-4}\right) \quad (n = 1,2,3,4,5)(k = 1,2,3,\ldots)(j = 4,5,6,\ldots), \tag{17}$$

where

$$r_{n,k,j} = (n+4k+j-4)(n+4k+j-3) \;\; (n = 1,2,3,4,5)(k = 0,1,2,\ldots)(j = 0,1,2,\ldots),$$

and B_j and $C_{m,j}$ are the corresponding coefficients of the Maclaurin series.

Thus, under the specified conditions, the formulas (5)–(17) effectively define an analytical method for calculating the bending of rectangular plates resting on an inhomogeneous Winkler elastic foundation.

References

Krutii, Y.S. (2016). *Development of a method for solving stability and vibration problems of deformable systems with varying continuous parameters. [in Ukrainian]* (Sc.D. diss.). Lutsk National Technical University, Lutsk.

Krutii, Y.S. (2018). *Construction of a solution of the problem of stability of a bar with arbitrary continuous parameters.* Journal of Mathematical Sciences, 231(5), 665–677. https://doi.org/10.1007/s10958-018-3843-8

Krutii, Y., Suriyaninov, M., Vandynskyi, V. (2018). *Development of the method for calculation of cantilever construction's oscillations taking into account own weight.* Eastern-European Journal of Enterprise Technologies, 3(7 (93)), 13–19. https://doi.org/10.15587/1729-4061.2018.131165

Krutii, Y., Vandynskyi, V. (2019). *Exact solution of buckling problem of the column loaded by self-weight.* IOP Conference Series: Materials Science and Engineering, 708, 012062. https://doi.org/10.1088/1757-899x/708/1/012062

Krutii, Y.S., Sur'yaninov, M.G., Karnaukhova, G.S. (2021). *Calculation method for axisymmetric bending of circular and annular plates on a changeable elastic bed. Part 1. Analytical relations.* Strength of Materials, 53(2), 247–257. https://doi.org/10.1007/s11223-021-00282-2