UDC 515

# ASYMPTOTICS OF GEOMETRIC CHARACTERISTICS OF SMOOTH MANIFOLDS IN THE PHASE FLOWS

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## Abstract

The work is devoted to the study of the asymptotics of the geometric characteristics of smooth manifolds in phase flows. One of the main such characteristics is the Richie curvature of the manifold with respect to the tangent vector.

*Keywords*: manifolds, curvature, metrics

## Introduction

In this paper we study asymptotic behaviour of the topological characteristic for riemannian manifolds in the smooth flows. It is necessary to determine how the Richie curvature changes for the movement along the trajectory of a smooth ergodic flow on the Riemannian manifold $M$.

## 1. Curvature of a Riemannian manifold

### 1.1. Riemannian manifols

Continuous coordinate system on an open subset $C$ of the space $R^n$ is a set of functions $\left\{x^1(y^1,...y^n),...x^n(y^1,..y^n)\right\}$, that define a one-to-one and mutually continuous mapping $\varphi : C \to A \subset R^n$.

Smooth tensor field of order $(p,q)$ on a manifold $M$ is a set of smooth functions $T_xM \to R^p \times R^q$, set in each local coordinate system $(x) = (x^1,...,x^n)$ $T^{i_1\dots i_p}_{j_1\dots j_q} = T^{i_1\dots i_p}_{j_1\dots j_q}(x)$ and when moving to a coordinate system $x^{\alpha'}$ change by law:

$$T^{i'_1\dots i'_p}_{j'_1\dots j'_q}(x^{\alpha'}) = \sum_{i_1\dots i_p}\sum_{j_1\dots j_q}\frac{\partial x^{i'_1}}{\partial x^{i_1}}\frac{\partial x^{i_p}}{\partial x^{i'_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\frac{\partial x^{j_q}}{\partial x^{j'_q}}T^{i_1\dots i_p}_{j_1\dots j_q}(x).$$

Manifolds are the natural higher-dimensional generalization of curves and surfaces. A Riemannian manifold is a manifold equipped with a way to measure the length of tangent vectors. The manifold $M$ is called Riemannian, if at any point $x_0 \in M$ in the tangent space $T_{x_0}$ a positive definite quadratic form is given (metric tensor of order $(0,2)$ $g = g_{ij}$, where $g_{ij}$ are the smooth functions of $x_0$). When switching from the coordinate system $x_\alpha$ to the coordinate system $y_{\alpha'}$ the components of the tensor should change as follows:

$$g_{i'j'} = \sum_{i,j}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}}g_{ij}.$$

Let $M-$ Riemannian manifold. Let $D \subset M--$ be an open subset of $M$ and there is a diffeomorphism $\varphi : D \to U \subset R^n$. $(\varphi, U)-$ is a map of $D$.
The volume of the area $D \subset M$ is the number:

$$vol(D) = \int\limits_U \sqrt{g(x)},$$

$$g(x) = det(g_{ij}(x)).$$

Let on a smooth manifold $M$ for every smooth atlas in every map there is a set of smooth functions $\Gamma^k_{ij}$ (Christoffel symbols) which change when transition to new coordinates as follows:

$$\Gamma^{i'}_{k'j'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^k_{ij} + \frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^i}{\partial x^{j'}\partial x^{k'}}.$$

A phase flow is a measurable one-parameter group of one-to-one mappings $T^t$, such that $T^t x : R \times S \to S$ is measurable mapping and for any t, any $A \in \Phi$

$$\mu((T^t)^{-1}A)) = \mu(A).$$

An example of a phase flow is the rotation group $T^t$ of the unit circle $S^2 : d\theta(r,\theta) = rdt$, with the Lebesgue measure on the circle, $(\theta, r)$ are polar coordinates. A geodesic is a curve in $M$ such that, for any two close enough points on the curve, the distance between these two points is realized by the curve. For example, the equator and meridians are geodesics of the sphere, but the parallels are not. Locally, geodesics between two points always exist [1].

**Theorem 1 (Birkhoff-Khinchin ergodic theorem)** $g^\tau$ *is a smooth flow, which preserves measure* $\mu$, $(S,\Phi,\mu)$ *is space with a normalized measure,* $f \in L^1(S,\Phi,\mu)$. *Then for almost all (in the sense of the measure* $\mu$*)* $x_0 \in S$*:*

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t f(g^\tau x_0)d\tau = M(f(x)|I),$$

*where I is a* $\sigma-$*algebra of invariants sets of flow* $g$.
*In case of the ergodic flow:*

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$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t f(g^\tau x_0)d\tau = \int\limits_S f(x)\mu(dx),$$

*and the limit does not depend on position* $x_0$ *[2].*

### 1.2. Curvature

Let $P, Q-$ 2 points on the manifold $M$, connected by a smooth curve $\gamma(t).\gamma(0) = P, \gamma(1) = Q$. $\gamma'-$field of velocities along $\gamma.\xi^k$ – components of this field. Differentiation operator along the curve $\gamma$:

$$\triangledown_{\gamma'} = \sum \xi^k \triangledown_k .$$

Then the curve $\gamma$ is called geodesic, if:

$$\triangledown_{\gamma'}\gamma' = 0.$$

Thus follows the Geodesic equation:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha k}\frac{dx^\alpha}{dt}\frac{dx^k}{dt} = 0.$$

The solution of such a system is determined by the conditions:

$$x^i(0) = P^i,$$

$$\frac{dx^i}{dt}(0) = a^i, a \in T_P.$$

For any $v \in T_P$ there exists unique geodesic $\gamma_v$ such that:

$$\gamma_v(0) = P,$$

$$\gamma_v'(0) = v.$$

Then $\gamma_v(1) = exp_P(v)$ is called exponential mapping in point $P$ [3].

$x \in M$. $T_x$ – is a tangent space in point $x$. $\{v, w\} \subset T_x.\varepsilon, \delta > 0$. $y$ – is the endpoint of the vector $v$. $d(x, y) = \delta$. $w'-$ is vector obtained by translation $w$ by $v$. Then

$$d(exp_x\varepsilon w, exp_y\varepsilon w') = \lim(\delta(1 - \frac{\varepsilon^2}{2}K(v, w)))$$

as $(\delta, \varepsilon) \to 0$. Where $K(v, w)$ is the sectional curvature in direction $(v, w)$ The Ricci curvature is the value:

$$Ric(v) = \frac{1}{vol(S)}\int_S K(v, w)dw,$$

where $s$ is the unit sphere in space $R^N$.

$K$ – is a compact set. $F_s : K \to K-$ a phase flow. Consider the case, when functional $\Phi$ is uniformly continuous on $K \times [0, 1]$ , and there is the limit:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(F_s, 0)ds,$$

then there is the limit:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Phi(F_s, e^{-s})ds.$$

**Proof:** $\forall\varepsilon > 0\exists\delta \in R\forall r', r'' \in K \times [0, 1] : |r' - r''| < \delta$

$$|\Phi(r') - \Phi(r'')| < \varepsilon$$

Therefore, if $e^{-s} < \delta$

$$\frac{1}{t}\left|\int_0^t \Phi(F_s, e^{-s})ds - \int_0^t \Phi(F_s, 0)ds\right| \le$$

$$\le \frac{1}{t}\int_0^t \left|\Phi(F_s, e^{-s}) - \Phi(F_s, 0)ds\right| ds < \varepsilon.$$

## Conclusion

In this study we used the main results of the theory of Ricci curvature of riemmanian manifolds to calculate the change of it in the smooth flows. It is established under what conditions there is a limit for its time averages.

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