essentially requires this too, a characteristic of both procedures is that the denominator of the Laplace transform of $e(t)$ or $f(t)$ be factored.

The object of this letter is to indicate that no factoring need be done, unless desired. We present an algebraic approach to the evaluation of

$$J_k = \int_0^\infty t^k e_1(t) e_2(t) dt \qquad k = 0, 1, 2 \ldots \qquad (3)$$

where e_1 and e_2 are functions with known rational Laplace transforms. Of course, by taking $e_1 = e_2$, one recovers the earlier results.

The procedure constitutes an extension of a technique due to Macfarlane[3] for the evaluation of the matrix

$$\int_0^\infty t^k e^{F't} Q e^{Ft} dt \qquad (4)$$

where F and Q are square matrices of the same order, and F has eigenvalues in the left-hand halfplane.

Denoting by $E_1(s)$ and $E_2(s)$ the Laplace transforms of $e_1(t)$ and $e_2(t)$, we begin by finding matrices F_1 and F_2, and vectors g_1, g_2, h_1 and h_2, such that

$$E_1(s) = h_1'(sI - F_1)^{-1} g_1 \qquad (5)$$

and $$E_2(s) = h_2'(sI - F_2)^{-1} g_2 \qquad (6)$$

It is important to realise that the desired matrices and vectors can be found by well known procedures (see, e.g., Reference 4) whether the denominators of $E_1(s)$ and $E_2(s)$ are factored or not. For simplicity, F_1 and F_2 should be taken to be of minimal dimension.

To guarantee that eqn. 3 is finite, we suppose that $E_1(s)$ and $E_2(s)$ have poles in the halfplane $\mathrm{Re}(s) < 0$. Then the eigenvalues of the matrices F_1 and F_2 all possess negative real parts.

Now, in eqn. 3, we have

$$J_k = \int_0^\infty t^k h_1' e^{F_1 t} g_1 h_2' e^{F_2 t} g_2 dt \qquad (7)$$

(where we are not assuming the availability of an explicit expression for $e^{F_1 t}$). Define the matrix

$$L_k = \int_0^\infty t^k e^{F_1 t} g_1 h_2' e^{F_2 t} dt \qquad (8)$$

so that

$$J_k = h_1' L_k g_2 \qquad (9)$$

We can give a recursion formula for L_k as follows. From eqn. 8, we have, for $k \geqslant 1$,

$$\begin{aligned} F_1 L_k + L_k F_2 &= \int_0^\infty t^k (F_1 e^{F_1 t} g_1 h_2' e^{F_2 t} + e^{F_1 t} g_1 h_2' e^{F_2 t} F_2) dt \\ &= \int_0^\infty t^k \frac{d}{dt}(e^{F_1 t} g_1 h_2' e^{F_2 t}) dt \\ &= k \int_0^\infty t^{k-1} e^{F_1 t} g_1 h_2' e^{F_2 t} dt \\ &= k L_{k-1} \qquad (10) \end{aligned}$$

Note that we have used the fact that the eigenvalues of F_1 and F_2 have negative real parts in the above sequence of equalities. For $k = 0$, it follows in a similar manner that

$$F_1 L_0 + L_0 F_2 = -g_1 h_2' \qquad (11)$$

Eqn. 11, and then eqn. 10 for $k = 1, 2, \ldots$ give successively $L_0, L_1, L_2 \ldots$ Note that the fact that all eigenvalues of F_1 and F_2 possess negative real part guarantees the solvability of these equations by standard procedures,[5] even when the dimensions of F_1 and F_2 differ.

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3rd April 1968

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p–n JUNCTION AS AN ULTRALINEAR CALCULABLE THERMOMETER

A mode of operation of a *p–n* junction is described which leads to an output voltage linearly related to temperature, independent of device geometry or semiconductor material, and determined only by fundamental physical constants and current.

The temperature dependence of the voltage across a forward-biased *p–n* junction offers the possibility of an electrical thermometer with relatively simple circuitry,[1] and seems to find wide application. At the cost of some circuit complexity,* its characteristics can be substantially improved, as will be shown. The collector current of an ideal junction transistor (there are sound reasons, also given later, why a transistor should be used rather than a diode) for instance, can be expressed as follows[2]:

$$I_c = KT^r \exp\left\{\frac{-q}{kT}(V_{g0} - V_{be})\right\} \qquad (1)$$

where K, r, V_{g0}, q and k are independent of temperature. V_{g0} is the extrapolated energy gap at $T = 0°$K ($V_{g0} = 1 \cdot 205$ V for Si). The constant K depends on geometrical factors and fixed physical constants, and r is determined by the temperature dependence of the diffusion constant of minority carriers in the base. In a silicon transistor, $r = 1 \cdot 5$ (*n–p–n*) or $r = 1 \cdot 3$ (*p–n–p*). q is the electron charge and k is Boltzmann's constant.

In terms of V_{be}, eqn. 1 can be rewritten

$$V_{be} = V_{g0} + \frac{kT}{q}(\ln I_c - \ln K - r \ln T) \qquad (2)$$

The temperature coefficient is obtained by differentiation:

$$\frac{\partial V_{be}}{\partial T} = \frac{k}{q}(\ln I_c - \ln K - r - r \ln T) \qquad (3)$$

Inspection of eqns. 2 and 3 leads to the following observations:

V_{be} is not a perfectly linear function of temperature (although very nearly so), and both V_{be} and its temperature coefficient depend on r as well as the geometry-sensitive factor K. In an accurately calibrated transistor thermometer circuit, the transistor transducer can thus only be interchanged with a carefully selected replacement; in general, readjustment of the circuit will be necessary.

If, instead of a constant collector current, the current is varied between a fairly high value, I_{c1}, and a fairly low value, I_{c2}, with corresponding base voltages V_{be1} and V_{be2}, it will be found that the base-voltage excursion is a linear function of temperature. With the aid of eqn. 2, this is found to be

$$\Delta V_{be} = V_{be1} - V_{be2} = \frac{kT}{q} \ln \frac{I_{c1}}{I_{c2}} \qquad (4)$$

Differentiation of this voltage excursion ΔV_{be} gives its temperature coefficient:

$$\frac{\partial}{\partial T}(\Delta V_{be}) = \frac{k}{q} \ln \frac{I_{c1}}{I_{c2}} = \text{constant} \qquad (5)$$

* For some of the features described in this letter, the necessary patent protection has been sought, and inquiries can be directed to the address at the end of the letter

Comparison of eqns. 4 and 5 with their steady-state counterparts, eqns. 2 and 3, shows a number of important advantages of this mode of operation:

(*a*) ΔV_{be} is exactly linear, and proportional to absolute temperature, without any offset.

(*b*) Both ΔV_{be} and its temperature coefficient are independent of any material or transistor parameter, resulting in complete interchangeability of transistor transducers, regardless of type number (this has been found to be largely true in practice).

(*c*) Within certain practical limits, the actual current levels I_{c1} and I_{c2} are immaterial, and they need not even be accurately stabilised, provided that their ratio is stable. In a practical circuit, a stable ratio can be more easily achieved, because the temperature coefficients of the two current-generating resistors could be matched, and even fairly large reference voltage fluctuations will have negligible effect on the current ratio.

(*d*) ΔV_{be} and $\frac{\partial}{\partial T}(\Delta V_{be})$ are exactly calculable (and within limits designable) without any previous transistor parameter measurements.

Part of the price to be paid for all these advantages will, of course, be greater circuit complexity. The most straightforward way would be to apply a square-wave collector current and amplify the resulting square-wave base voltage (amplitude ΔV_{be}) sufficiently in order to rectify it. If a digital voltmeter is used as an indicator, continuous presentation of the temperature range -100 to $+100°$C, resolved to within $0{\cdot}01°$C, without scale change, is a possibility.

The sensitivity is also reduced, although it is not so low as to present serious problems. If, for instance, $I_{c1} = 100\mu$A and $I_{c2} = 0{\cdot}1\mu$A, eqn. 5 yields

$$\frac{\partial}{\partial T}(\Delta V_{be}) = 0{\cdot}6\text{mV}/°\text{C}.$$

which is about four times less than $\partial V_{be}/\partial T$ in the steady-state mode.

The most obvious *p–n* junction device to use as temperature sensor is the diode, but a simple expression like eqn. 1 does not represent its total current/voltage characteristic, owing to the presence of surface leakage across the junction and electron–hole recombination in the barrier layer. Silicon diodes (planar or other) seem to be particularly prone to these effects and were not further considered for this purpose. Silicon planar transistors, with collector–base short-circuited and operated as diodes, were found to exhibit an I_e/V_{be} characteristic very close to a single exponential (like eqn. 1), but the slight deviation was just about enough to spoil the attraction of this proposed new mode of operation.

On connecting the transistor across an operational amplifier as a 3-terminal feedback network, as proposed by Gibbons and Horn[3] for the purpose of extending the range of logarithmic transfer response of a circuit, the present difficulty can also be overcome. The unwanted components of the emitter current are simply drained away as base current, and, if the current generator across the input supplies a step change in collector-current level (Fig. 1), eqns. 4 and 5 will

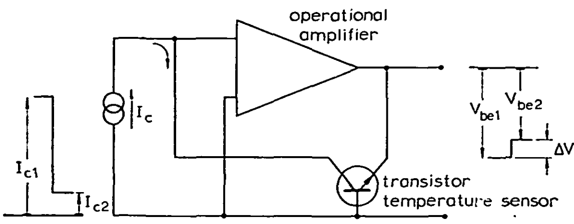

Fig. 1 *Basic circuit for making collector current the independent variable in practical operation, with the base–emitter voltage the dependent variable*

be satisfied nearly exactly. The only small residual error will be that due to voltage drops across the extrinsic resistances of the transistor (mostly $r_{bb'}$). For extreme accuracy, even these errors can be eliminated at the expense of further complication, with, for example, the introduction of additional current level(s).

The temperature range over which eqns. 4 and 5 are valid is illustrated in Fig. 2 for a representative silicon transistor.

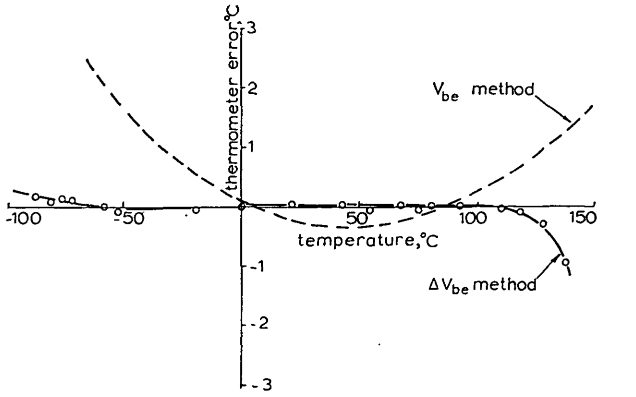

Fig. 2 *Calibration curves for current-step and constant-current operation of a transistor temperature sensor*

Positive and negative errors mean high and low readings, respectively
Transistor sensor, 2N1893
V_{be} method: $I_c = 57{\cdot}3\mu$A
ΔV_{be} method: $I_{c1} = 57{\cdot}3\mu$A, $I_{c2} = 5{\cdot}0\mu$A

With increasing temperature, a point is gradually approached where the emitter reverse leakage current I_{e0} is no longer insignificant relative to the lower current level I_{c2}. This imposes an upper temperature limit. At the low end of the scale, the deviation is much more gradual, being due to the increasing $I_b r_{bb'}$ voltage drop as the current gain h_{fe} falls.

Errors in temperature indication caused by random interchange of sensor transistors (without readjustment of circuit) have also experimentally been found to be reduced by at least an order of magnitude in the current-step mode, as compared with the conventional constant-current mode. More details will be published in a future paper.

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11th April 1968

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OPTIMALLY SENSITIVE NETWORKS

The constraints on *LC* transfer-function coefficients are investigated for networks to be optimally sensitive according to Schoeffler's criterion considered by Leeds and Ugron.

In a paper by Leeds and Ugron,[1] consideration has been given to the minimisation of Schoeffler's performance criterion, namely the sum of the squared magnitudes of sensitivities. The absolute minimum of this sum of the magnitude-squared criterion would result when all parts of the sum are equal, and Leeds and Ugron suggest that a class of networks exists which exhibit this property. The purpose of this letter is to investigate the requirements for a network to be potentially optimally sensitive to Schoeffler's criterion.

For optimum sensitivity, all the magnitude-squared sensitivities must be the same:

i.e. $|S_1(p)|^2 = |S_2(p)|^2 = |S_3(p)|^2 \ldots$

where *p* is the complex frequency variable.