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DIFFERENTIAL AND INTEGRAL EQUATIONS

Textbook

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This textbook is designed for students studying the discipline Differential and Integral Equations in technical university.

Each part contains basic mathematical conceptions and explains new mathematical terms. The most important concepts of the theory are explained and illustrated by figures and examples. Each topic is provided review questions and exercises.

The manual can be used to provide full-time, distance or mixed education.

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Introduction

Many physical phenomena and processes in nature and technology are described by ordinary differential equations and their systems. The classical theory of ordinary differential equations is a powerful tool for investigation and solving mathematical models of various applied problems.

The manual can be helpful for students who want to study the concepts of differential equations and methods of solving them.

The proposed manual consists of five main chapters. Each of them contains theoretical part and provides examples and figures to help clarify the theory and show some applications.

The first part deals with first-order differential equations: concept of the first-order differential equation, its solutions, initial value problem and its solution. Here the reader may find some practical problems arising in various fields of science.

The next part is concerned with the theory of higher-order differential equations. Here students could find the methods of solving and application of such kinds of equations. Special attention is paid to the linear homogeneous and nonhomogeneous differential equations with constant coefficients. In addition, the one can get acquainted with exploring mechanical vibrations by differential equations.

The third part is devoted to systems of differential equations. There are the basic definitions and theoretical principles that should be relied upon when solving problems of a given topic.

In the fifth chapter it is introduced the concept of stability theory. This question is one of basic in so-called qualitative theory of differential equations.

The last part deals with the basic consepts of integral equations. Here students may find out about Volterra and Fredholm equations, their properties and solving method.

Each topic is provided review questions and exercises, which will allow students to master the skills of solving practical problems and prepare for examination. Answers are given at the end of the manual.

This textbook is designed for students studying the discipline Differential and Integral Equations. The manual can be used to provide full-time, distance or mixed education.

1. First-Order Differential Equations

1.1 Basic Concepts and Definitions

I. General Definitions

Definition. The relationship which connects the independent variable x, the unknown function $y(x)$ and its derivatives (or differentials) is called *the nth-order differential equation with respect to function $y = y(x)$* and may be written as follows

$$F(x, y, y', y'', \dots, y^{(n)}) = 0.$$

Definition. *The order of a differential equation* is the order of the highest derivative (or differential) that appears in the equation.

If the unknown function $y(x)$ is a function of **one** variable, then the differential equation has ordinary derivatives and is called *ordinary differential equation*, abbreviated by **ODE**. Likewise, if the unknown function is a function of **several** variables (dependents upon two or more variables), then the differential equation has partial derivatives and is called *a partial differential equation*, abbreviated by **PDE**. The order of a differential equation does not depend on whether or not we've got ordinary or partial derivatives in the differential equation.

Example 1.

a) $y' + x^3 y = \sin xy$ - the first-order ODE with respect to function $y = y(x)$;

b) $y''' + e^x y'' = x + y$ - the third-order ODE with respect to function $y = y(x)$;

c) $\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = x^3 y$ - the first-order PDE with respect to function $u = u(x, y)$;

d) $\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$ - the second-order PDE with respect to function $u = u(x, y)$.

Further we will deal only with ordinary differential equations.

Definition. Any function $y = \phi(x)$ which satisfies the differential equation (when put into the equation, converts it into an identity) is called *the solution or integral* of a differential equation.

Definition. The process of searching the corresponding function $y = \phi(x)$ is called *an integration* of differential equation.

We say that we integrate (or solve) the differential equation when we discover the function $y(x)$ (or the set of functions $y(x)$).

II. The First-order ODE (General Concepts)

Definition. *The first-order differential equation* with respect to $y = y(x)$ is an ODE of the form

$$F(x, y, y') = 0$$

or (if it can be solved for the derivative $y' = \frac{dy}{dx}$)

$$y' = f(x, y). \qquad (1.1)$$

There exists yet another form of the first-order ODE:

$$P(x, y)dx + Q(x, y)dy = 0,$$

where $P(x, y)$ and $Q(x, y)$ are known functions. Here both variables x and y have the equal rights, that is any of them can be considered as a function of another.

Definition. Function $y = \phi(x, C)$ is called *the general solution* of the first-order differential equation if it satisfies the differential equation (1.1) for any specific value of the constant C. C is called *the constant of integration*. The relation $\Phi(x, y, C) = 0$, which contains implicitly the solution of differential equation, is called *the complete integral* of the equation (1.1).

Solving the relationship $\Phi(x, y, C) = 0$ for y, we obtain the general solution. Unfortunately, it is not always possible to express y in terms of elementary functions, and, in such cases, we used to leave the general solution in implicit form.

The geometric meaning of the integral curves of an equation $y' = f(x, y)$ is the following: it specifies a derivative (that is, a slope) at every point in the plane. The equation defines a direction field (or slope field) on the plane, that is, a field of directional vectors such that at each point (x, y) the tangent of the angle of inclination of the vector (tangent line) with the x-axis is equal to $f(x, y)$. A solution of the differential equation is a function whose graph has the given slope at every point it goes through. These curves nowhere intersect one another and are nowhere tangent to one another. These curves are called *integral curves* of the given differential equation.

Example 2.

Let us consider differential equation

$$\frac{dy}{dx} = \frac{x+2y}{x}.$$

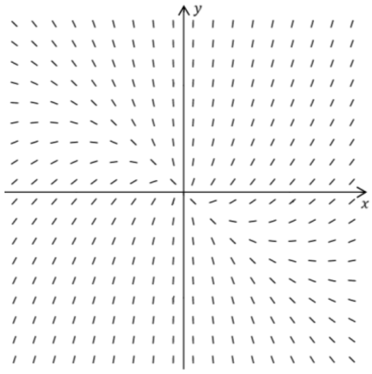

Figure 1.1

Thus, we get the slope field where the tangent of the angle of inclination is defined by

$$\tan\alpha = y' = \frac{x+2y}{x}.$$

At each point (x, y), we evaluate the slope

x	1	1	1	−1	−1	−1	...
y	0	1	−1	0	1	−1	...
$\tan\alpha$	1	3	−1	1	−1	3	...

and, then, draw a short line segment with the slope obtained (Fig.1.1).

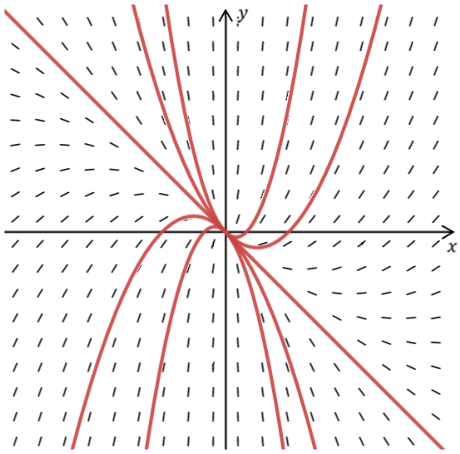

Figure 1.2

Since these segments are tangent lines to the solutions of the differential equation we use them as guides for sketching the integral curves of the given equation (Fig. 1.2).

Geometrically the general solution $y = \phi(x, C)$ or complete integral $\Phi(x, y, C) = 0$ is a **family of curves** in a coordinate plane, which depends on a single parameter C.

Hence, there are an infinite number of solutions of the differential equation.

Example 3.

Let us consider the differential equation

$$y' = 2x.$$

It is easy to see that the general solution is

$$y = x^2 + C,$$

where C is an arbitrary costant.

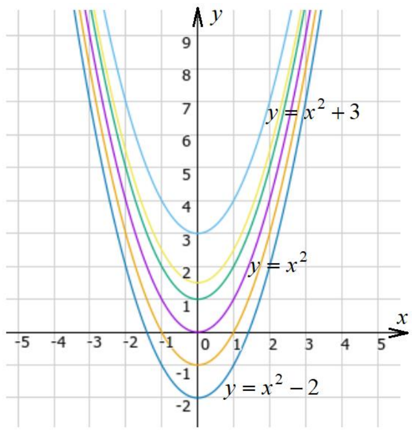

Figure 1.3

Thus, we obtain integral curves that depends on C (Fig. 1.3).

III.Initial Value Problem

Definition. Function $y = \phi(x, C_0)$ that is derived from the general solution by setting the constants C to particular values ($C = C_0$), is called *a particular solution.* In this case, the implicit function $\Phi(x, y, C_0) = 0$ is called *a particular integral* of the equation. A particular solution or integral is associated with only one curve of this family that passes through a **certain given point** on the plane.

Consider the following problem: among all the solutions of the differential equation $y' = f(x, y)$, it is necessary to find a solution that satisfies the condition: $y = y_0$ for $x = x_0$, where x_0 and y_0 are given numbers. This condition is called *the initial condition* and is written as follows:

$$y(x_0) = y_0 \text{ or } y|_{x=x_0} = y_0. \tag{1.2}$$

Hence, from the family of integral curves, which are determined by the general solution or general integral of the equation, we have to select the integral curve that passes through a given point $M_0(x_0; y_0)$.

Definition. A differential equation together with an initial condition, which specifies the value of the unknown function at the given point in the domain

$$y' = f(x, y), \tag{1.1}$$

$$y(x_0) = y_0 \tag{1.2}$$

is called *an initial value problem* or *Cauchy problem.*

To solve the Cauchy problem means to find the particular solution $y = \phi(x)$ of the differential equation (1.1), such that it satisfies the condition (1.2): $\phi(x_0) = y_0$.

Theorem.

Cauchy Theorem About Existence and Uniqueness.

If in the differential equation

$$y' = f(x, y)$$

function $f(x, y)$ and its partial derivative with respect to y, $\frac{\partial f}{\partial y}$, are continuous in some region D in an xy-plane containing some point $M_0(x_0; y_0)$, then there is only one solution of this equation $y = \phi(x)$ which satisfies the condition $\phi(x_0) = y_0$.

Geometrically that means, there exists one and only one such function $y = \phi(x)$, the graph of which passes through the point $M_0(x_0; y_0)$. No matter what the initial conditions $y = y_0$ for $x = x_0$, it is possible to find a value $C = C_0$ such that the function $y = \phi(x, C_0)$ satisfies the given initial condition. It is assumed here that the values x_0 and y_0 belongs to the range of the variables *x* and *y* in which the conditions of the existence theorem are fulfilled.

Example 4.

Let us consider the differential equation

$$y' = 2x \quad \text{when} \quad y(1) = 3.$$

Previously we have got the general solution of the equation:

$$y = x^2 + C.$$

Let us find the initial value problem ($y_0 = 3, x_0 = 1$)

$$y(1) = 1^2 + C = 3 \text{ and } C = 3 - 1 = 2.$$

Hence, the particular solution is

$$y = x^2 + 2,$$

the graph of which passes through the point $M_0(1; 3)$ (Fig. 1.4).

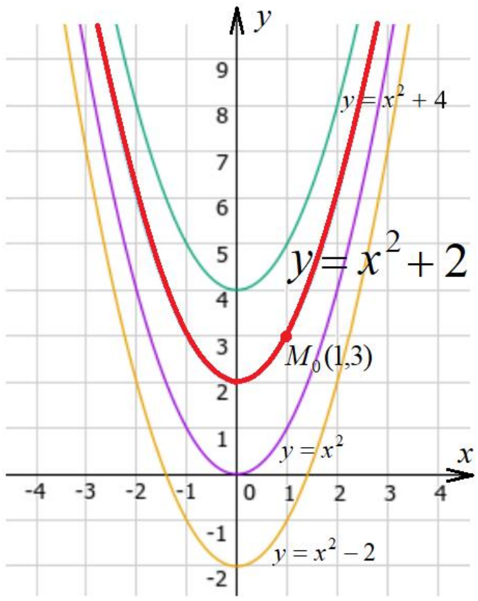

Figure 1.4

Definition. A function $y = \phi(x)$ is called *the singular solution* of the differential equation $y' = f(x, y)$, if uniqueness of solution is violated at some points of the domain of the equation.

Geometrically this means that more than one integral curve passes through the point $M_0(x_0; y_0)$. Such points are called *singular points*.

A singular solution of a differential equation is not described by the general integral, which is it cannot be obtained by assigning definite values to the arbitrary constants in the general solution.

Usually, singular solutions appear in differential equations when there is a need to divide in a term that might be equal to zero. Therefore, when one is solving a differential equation and using division one must check what happens if the term is equal to zero, and whether it leads to a singular solution. More about singular solutions you may find in [1,13,17].

Review Questions

1. What do you mean by differential equation?
2. What is the order of a differential equation?
3. What equation is called an ordinary differential equation?
4. What do you mean by the solution of a differential equation?
5. What do you mean by the general solution of a differential equation?
6. What equation is called the first-order differential equation?
7. What is the general solution and general integral of the first-order differential equation?
8. What is a particular solution and a particular integral of the first-order differential equation?
9. What is the geometric meaning of the solution of the differential equation?
10. What is the initial condition for the first-order differential equation? What is the geometric meaning of the initial condition for the first-order differential equation?
11. Formulate the Cauchy problem for the first-order differential equation.
12. Formulate the Existence and Uniqueness Theorem for the solution of the Cauchy problem of the first-order differential equation. Give a geometric interpretation of this theorem.
13. What solutions are called singular?

1.2 Separable Differential Equations

I. Method of Solving

Definition. A first-order differential equation $y' = f(x, y)$ is called *a separable equation* if the function $f(x, y)$ can be factored into the product of two functions of x and y:

$$y' = g(y) \cdot h(x) \tag{1.3}$$

where are $g(y)$ and $h(x)$ are continuous functions.

Let us consider the derivative y' as the ratio of two differentials $\frac{dy}{dx}$, then rewrite an equation (1.3) as

$$\frac{dy}{dx} = g(y) \cdot h(x).$$

Move dx to the right side of equation

$$dy = g(y) \cdot h(x)dx.$$

and divide both sides of equation by $g(y)$, assuming that $g(y) \neq 0$:

$$\frac{dy}{g(y)} = h(x)dx.$$

Note.

If there exists a number y_0 such that $g(y_0) = 0$, then this number is the **singular solution** of the differential equation.

Considering y as known function of x, the equation obtained may be regarded as the equality of two differentials. The primitives of them will differ by a constant C. Let us integrate the left side with respect to y and the right side with respect to x.

Thus,

$$\int \frac{dy}{g(y)} = \int h(x)dx + C,$$

where C is an integration constant.

Calculating the integrals, we get the general solution (complete integral) of the given equation.

Separable equation could be written in the form

$$P(x,y)dx + Q(x,y)dy = 0, \tag{1.4}$$

if $P(x,y) = M_1(x)N_1(y)$ and $Q(x,y) = M_2(x)N_2(y)$.

Divide both sides of equation (1.4) by $N_1(y)M_2(x)$, excluding the points at where $N_1(y) = 0$ and $M_2(x) = 0$:

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(y)}{N_1(y)}dy = 0.$$

or

$$\frac{M_1(x)}{M_2(x)}dx = -\frac{N_2(y)}{N_1(y)}dy.$$

Let us integrate the left side with respect to y and the right side with respect to x.

Thus

$$\int \frac{M_1(x)}{M_2(x)}dx = -\int \frac{N_2(y)}{N_1(y)}dy + C.$$

where C is an integration constant.

Example 1.

Solve the differential equation

$$x^2y^2y' + 1 = y.$$

It is clear, that this differential equation is separable. So, let us separate the differential equation.

Rewrite it as

$$x^2y^2dy = (y-1)dx.$$

Divide both sides of equation by $x^2(y-1)$:

$$\frac{y^2}{(y-1)}dy = \frac{1}{x^2}dx.$$

Integrate the left side with respect to y and the right side with respect to x

$$\int \frac{y^2}{(y-1)}dy = \int \frac{1}{x^2}dx + C.$$

Hence, we get the complete integral of the given equation.

$$\frac{y^2}{2} + y + \ln|y-1| = -\frac{1}{x} + C.$$

Since, after dividing by $x^2(y-1)$ we can lose the solutions $x=0$ and $(y-1)=0$, that is $y=1$. Let us see if $x=0$ and $y=1$ are solutions of the differential equation.

Substituting directly $x=0$ into the equation we obtain, that it is not the solution.

However, substitution of $y=1$ into the equation gives us $0\equiv 0$. Hence, it is the solution. The solution $y=1$ is not described by the general integral, thus, $y=1$ is a singular solution of a given differential equation.

Example 2.

Find the complete integral of the differential equation

$$(y-xy)dx+(x+xy)dy=0.$$

Factorize the expressions in the both parenthesis:

$$y(1-x)dx+x(1+y)dy=0.$$

Thus, this differential equation is separable. So, let us separate the differential equation dividing the equation by $xy\neq 0$:

$$\frac{1-x}{x}dx+\frac{1+y}{y}dy=0.$$

Integrate both sides

$$\int\frac{1-x}{x}dx+\int\frac{1+y}{y}dy=C,$$

$$\ln|x|-x+\ln|y|+y=C.$$

Since the constant of integration is arbitrary, we may represent the constant C as $C=\ln|C_1|$, where $C_1>0$. Then

$$\ln|xy|+\ln e^{y-x}=\ln|C_1|,$$

$$\ln(|xy|e^{y-x})=\ln|C_1|,\quad \Rightarrow |xy|e^{y-x}=|C_1|.$$

Hence the complete integral of the equation is

$$xye^{y-x}=\pm C_1$$

or

$$xye^{y-x}=C_2, C_2=\pm C_1.$$

Since we assume that $xy\neq 0$, then we have to check whether functions $x=0$ and $y=0$ are solutions of the differential equation.

Substituting directly $x=0$ and $y=0$ into the equation both are solutions.

On the other hand, they can be obtained from the general integral when $C = 0$. Thus, $x = 0$ and $y = 0$ are *particular solutions* of the equation.

Example 3.

Find the particular solution of differential equation (solve the Cauchy problem)

$$y' \cot x = -y, \qquad y\left(\frac{\pi}{3}\right) = -1.$$

Let $y' = \frac{dy}{dx}$, then

$$\frac{dy}{dx}\cot x = -y$$

and

$$ydx + \cot x\, dy = 0.$$

First, separate the equation

$$\frac{dy}{y} = -\frac{dx}{\cot x}.$$

Hence,

$$\frac{dy}{y} = -\tan x\, dx$$

and then integrate both sides

$$\int \frac{dy}{y} = -\int \tan x\, dx + \ln C\,,$$

$$\ln|y| = \ln|\cos x| + \ln C.$$

Thus, the complete integral of the equation is

$$\ln|y| = \ln C\,|\cos x|,$$

$$y = C \cdot \cos x.$$

Apply the initial condition $x_0 = \frac{\pi}{3}$, $y_0 = -1$ to get the value of C:

$$-1 = C \cdot \cos\frac{\pi}{3},$$

$$C = -2.$$

Thus, it gives a particular solution (the Cauchy problem solution)

$$y = -2\cos x.$$

II. Some Applications of Separable Differential Equations

In order to illustrate the applications of differential equations we consider the easiest mathematical models in different fields of science: biology, chemistry and physics.

1. Biology (Population Growth and Decay)

Let us consider some population (people in a country, bacteria in a laboratory culture, wild animals in a forest, ect.). Denote as $N(t)$ the total amount population at time t, and suppose that this function is real-valued, positive and differentiable (it is some kind of simplification of a process, since at any given time t is necessarily an integer). Hence, the rate of growth is regarded as a derivative with respect to t of the function $N(t)$.We assume that the rate of growth of population at a certain time is proportional to the total population at that time.

This assumption can be expressed as

$$\frac{dN(t)}{dt} = K \cdot N(t), \tag{1.5}$$

where $K, K > 0$ is called the growth constant or the decay constant (depending on the species and could be found experimentally). That constant is often described as the difference between the birth rate and the death rate.

The equation (1.5) is separable differential equation. Its solution is

$$N(t) = Ce^{Kt}. \tag{1.6}$$

If we know the population N_0 at an initial time ($t = 0$): $N(0) = N_0 \neq 0$, then we may find the law describing population of species

$$N(t) = N_0 e^{Kt}.$$

Such a model of growth for human population was invented by the English economist Thomas Malthus in 1798.

It is obvious that

- if $K > 0$, we have growth, the population grows and continues to expand to infinity when $t \to \infty$ (Fig. 1.5).

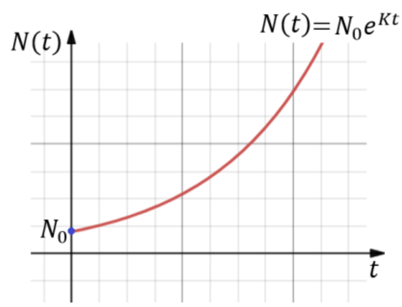

Figure 1.5

- if $K > 0$, we have decay, the population will shrink and tends to 0 (Fig. 1.6).

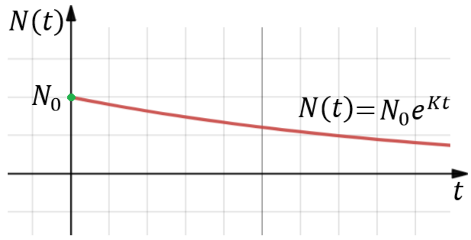

Figure 1.6

The Malthusian model is not precise and has many limitations. First, this model does not regard the limitations of space and resources (for example, food). Second, for human population, this hypothesis do not take into account some social factor, technological and economic changes, etc.

In any case, the Malthusian model was the first step to analyze mathematically the changes of population. This idea was developed in the works of Pierre Francois Verhulst, Raymond Pearl, Alfred James Lotka and Vito Volterra, etc.

2. Chemistry (Bi-molecular Reaction)

Let us consider a tank which contains the chemicals X and Y. Suppose that they are undergoing the chemical reaction $2X \xrightarrow{K} Y$ that involves the collision of two molecules of reactant X (so-called bi-molecular reaction). Here, the rate of decomposition of X is proportional (K is a coeffitient) to the square of the concentration of X. Assume that the initial $(t = 0)$ concentration of X is X_0, while the initial concentration of Y is zero.

In this case the mathematical model is

$$-\frac{1}{2} \cdot \frac{dX(t)}{dt} = K \cdot X^2(t), \qquad X(0) = X_0.$$

Let us solve this separable differential equation

$$-\int \frac{dX(t)}{X^2(t)} = 2K \int dt \quad \Rightarrow \quad \frac{1}{X(t)} = 2Kt + C.$$

General solution is

$$X(t) = \frac{1}{2Kt + C}.$$

Solving the initial value problem $X(0) = X_0$, we obtain

$$X(t) = \frac{X_0}{2KX_0 t + 1}.$$

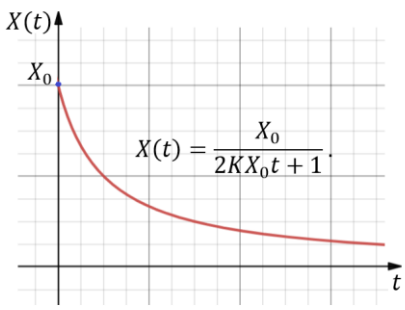

Figure 1.7

Thus, we get the function (Fig. 1.7) predicting the concentration of chemical X along with time.

3. Physics (Radioactive Decay)

Experimentally it was found that rate of radioactive decay is proportional to the number of atoms ($N(t)$) present. The proportionality constant λ is called the decay constant. Mathematical model of this statement could be written by the first-order differential equation

$$-\frac{dN(t)}{dt} = \lambda \cdot N(t).$$

The negative sign indicates that the amount of the radioactive material N decreases over time, as the decay events follow one after another. If $N(0) = N_0$, in which N_0 is the number of atoms present at the initial moment of time, then the solution is

$$N(t) = N_0 e^{-\lambda t}.$$

The half-life period T of a radioactive material is the time reguired to decay to one-half of the initial amount of the material. Thus, it follows from the given law that at the moment T

$$N(t) = \frac{N_0}{2} = N_0 e^{-\lambda T}.$$

Let us find the half-life period:

$$\frac{N_0}{2} = N_0 e^{-\lambda T} \quad \Rightarrow \quad \frac{1}{2} = e^{-\lambda T} \quad \Rightarrow \quad -\lambda T = \ln\frac{1}{2} \quad \Rightarrow \quad T = \frac{1}{\lambda}\ln 2\,.$$

The coefficient $\frac{1}{\lambda}$ is called the average lifetime of a radioactive atom and is denoted by τ.

Hence, the half-life T and the average lifetime τ are related to each other by the formula:

$$T = \tau \ln 2 \approx 0.693\tau.$$

These parameters vary widely for different substances. For example, the half-life of Polonium-212 is less than one microseconds, but the half-life of Thorium-232 is more than one billion years.

4. Physics (Newton's Law of Cooling)

Let us consider the solid of initial temperature T_0 is placed in the environment of the temperature T_e, $T_0 > T_e$ (the cup of hot tea in the cold room). Obviously, eventually the temperature of the solid must approach the temperature of the environment. Experiments showed that the cooling rate approximately proportional to the difference between the temperature of the solid and the temperature of the environment (*Newton's law of cooling*).

Thus, if $T = T(t)$ is the temperature of the solid at time t, then this Law could be written in the form of differential equation

$$\frac{dT(t)}{dt} = \frac{\alpha \cdot A}{C}\big(T_e - T(t)\big),$$

where C is the heat capacity of the body, A is the surface area of the body through which the heat is transferred, α is the heat transfer coefficient depending on the geometry of the solid, state of the surface and other factors.

Denoting $\frac{\alpha \cdot A}{C} = K$, we obtain

$$\frac{dT(t)}{dt} = K\big(T_e - T(t)\big), T(0) = T_0.$$

The given initial value problem has the solution

$$T(t) = T_e + (T_0 - T_e)e^{-Kt}$$

Thus, while cooling, the temperature of any solid exponentially approaches the temperature of the surrounding medium. The cooling rate depends on the parameter K. With increase of the parameter K, the cooling occurs faster (Fig. 1.8).

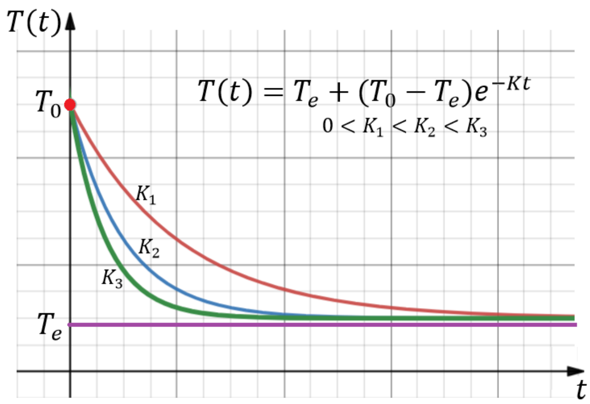

Figure 1.8

More examples of application of the first-order differential equations you may find in [1,12,14,15,17].

Review Questions

1. Give the definition of separable differential equation and formulate the method of its integration.

2. What are the forms separable differential equation? Write down examples of each form.

3. How do you find the singular solutions for separable differential equation?

4. Is it possible to claim that only one integral curve of the equation $y' = 3\sqrt[3]{y^2}$ passes through the point (0, 2)?

5. What is the order of the differential equation if its general integral has the form:

a) $\varphi(x, y, C) = 0$; b) $\varphi(x, y, C_1, C_2) = 0$?

Exercises 1.2

1-10. Find the general solution or integral of the differential equation:

1. $(y-1)^2dx + (1-x)^3dy = 0;$

2. $x\sqrt{9-y^2}dx - y(4+x^2)dy = 0;$

3. $\cos x \cos y\, dx - \sin x \sin y\, dy = 0;$

4. $\ln x \sin^3 y\, dx - x \cos y\, dy = 0;$

5. $(xy^2 - y^2)dx - (x^2y + x^2)dy = 0;$

6. $yy' + x = 1;$

7. $\left(\sqrt{xy} - 2\sqrt{x}\right)y' - y = 0;$

8. $(1+y')e^y + 1 = 0;$

9. $(1+x^2)y' = xy - y\sqrt{1+x^2};$

10. $y' = \frac{y \ln y}{\sqrt{1-x^2}\arcsin x};$

11-15. Find the particular solution or integral of the differential equation:

11. $3x\sqrt[3]{y}dx + (1-x^2)dy = 0, \quad y(0) = 0.$

12. $ydx - (4+x^2)\ln y\, dy = 0, \quad y(2) = 1.$

13. $\sin^2 x \cos^2 y\, dx - \cos^2 x\, dy = 0, \quad y(0) = \pi/4.$

14. $y'e^{-x} = x - 1, \quad y(1) = -e.$

15. $y'\left(x + \sqrt{x}\right) = \sqrt{1-y}, \quad y(0) = 1.$

Answers.

1.3 Homogeneous Differential Equations

I. Homogeneous Differential Equations

Definition. The function $f(x, y)$ is called *homogeneous function of the n-th degree* (n is natural) with respect to the variables x and y, if for any number λ the following identity is true:

$$f(\lambda x, \lambda y) \equiv \lambda^n f(x, y). \tag{1.7}$$

For example, function $f(x, y) = x^2 + y^2 - xy$ is homogeneous of the second degree, since

$$f(\lambda x, \lambda y) = (\lambda x)^2 + (\lambda y)^2 - (\lambda x)(\lambda y) = \lambda^2(x^2 + y^2 - xy) = \lambda^2 f(x, y).$$

Consider the equality (1.7) for $\lambda = \frac{1}{x}$

$$f\left(1, \frac{y}{x}\right) = \frac{1}{x^n} f(x, y) \Rightarrow f(x, y) = x^n f\left(1, \frac{y}{x}\right).$$

Hence, any homogeneous of the n-th degree function could be written in the form

$$f(x, y) = x^n \phi\left(\frac{y}{x}\right). \tag{1.8}$$

Definition. First-order differential equation of the form

$$y' = f(x, y) \tag{1.9}$$

is called *homogeneous* with respect to the variables x and y if the function $f(x, y)$ is homogeneous of **zero** degree with respect to the variables x and y.

Since the ratio of two homogeneous functions of the same degree is a homogeneous function of zero degree, we could consider alternative form of the equation.

Definition. First-order differential equation in the differential form

$$P(x, y)dx + Q(x, y)dy = 0 \tag{1.10}$$

is called *homogeneous* with respect to the variables x and y if functions $P(x, y)$ and $Q(x, y)$ are homogeneous functions of the **same** degree.

Using (1.8), homogeneous equations (1.9) or (1.10) can be transformed into an equation

$$y' = g\left(\frac{y}{x}\right). \tag{1.11}$$

where the right part of which is a function of the relationship $\frac{y}{x}$.

A homogeneous equation can be solved by substitution $\frac{y}{x} = u$, where $u = u(x)$ is new unknown function, which leads to a separable differential equation.

Thus

$$y = ux \Rightarrow y' = u'x + u.$$

Putting the expressions for y and y' into the equation (1.11), we get

$$u'x + u = g(u) \;\text{ or }\; x\frac{du}{dx} = g(u) - u.$$

This is the separable differential equation

$$\frac{du}{g(u) - u} = \frac{dx}{x}, \quad g(u) \neq u.$$

Integrating we obtain the general solution (complete integral) with respect to function u:

$$\int \frac{du}{g(u) - u} = \int \frac{dx}{x} + C.$$

Putting after integration the expression $\frac{y}{x}$ in place of u, we get the solution of the original differential equation.

Note.

It is not necessary to reduce homogeneous equations to the form (1.11) when solving. We may make a substitution $y = ux$ directly.

Example 1.

Find the general solution of the equation

$$y - xy' = y\ln\frac{y}{x}.$$

Rewrite the equation in the form:

$$xy' = y - y\ln\frac{y}{x},$$

$$y' = \frac{y}{x} - \frac{y}{x}\ln\frac{y}{x} = \phi\left(\frac{y}{x}\right).$$

Thus, this equation is homogeneous.

For solving let us make the substitution

$$\frac{y}{x} = u(x) \Rightarrow y = ux, \quad y' = u'x + u.$$

Substituting this expression into the equation gives:

$$u'x + u = u - u \ln u\,,$$

$$x \frac{du}{dx} = u \ln u.$$

Divide by $xu \ln u$ to obtain the separable equation

$$\frac{du}{u \ln u} = \frac{dx}{x}.$$

Let us integrate the left and the right side of the equation:

$$\int \frac{du}{u \ln u} = \int \frac{dx}{x} + \ln C,$$

$$\ln|\ln u| = \ln|x| + \ln C,$$

$$\ln|\ln u| = \ln C\,|x|.$$

Thus, by exponentiation we obtain solution with respect to u:

$$\ln u = \pm Cx,$$

$$u = e^{C_1 x}.$$

Returning to the old variable by substitution $u = \frac{y}{x}$, we get the general solution of the original equation

$$\frac{y}{x} = e^{C_1 x},$$

$$y = xe^{C_1 x}.$$

Example 2.

Solve the equation

$$(x^2 + y^2)dx - 2xydy = 0.$$

It is easy to see that the polynomials $P(x, y) = x^2 + y^2$ and $Q(x, y) = -2xy$ are homogeneous functions of the second order.

Therefore, the original differential equation is homogeneous.

Solving the equation with respect to derivative, we get

$$\frac{dy}{dx} = y' = \frac{x^2 + y^2}{2xy} = \frac{1 + \left(\frac{y}{x}\right)^2}{2\frac{y}{x}}.$$

Let us plug the substitution

$$\frac{y}{x} = u(x) \quad \Rightarrow \quad y = ux, \quad y' = u'x + u.$$

into the differential equation obtained.

Then

$$u'x + u = \frac{1 + u^2}{2u},$$

$$x\frac{du}{dx} = \frac{1 + u^2}{2u} - u,$$

$$xdu = \frac{1 - u^2}{2u}dx.$$

Separate the variables

$$\frac{2u}{1 - u^2}du = \frac{dx}{x}.$$

and integrate both sides

$$\int \frac{2u}{1 - u^2}du = \int \frac{dx}{x} - \ln C\,,$$

$$-\ln|1 - u^2| = \ln|x| - \ln C\,,$$

$$\ln|x(1 - u^2)| = \ln C.$$

Now exponentiate both sides and obtain

$$x(1 - u^2) = \pm C = C_1.$$

Returning to the old variable by $u = \frac{y}{x}$, we get the complete integral of the differential equation

$$x\left(1 - \frac{y^2}{x^2}\right) = C_1$$

or

$$x^2 - y^2 = xC_1.$$

Example 3.

Find the solution of the initial value problem:

$$xdy - ydx = ydy, \quad y(-1) = 1.$$

Let us transform the equation:

$$(x - y)dy - ydx = 0.$$

Then it is easy to see that the polynomials $P(x, y) = x - y$ and $Q(x, y) = -y$ are homogeneous functions of the first order. Thus, the differential equation is homogeneous.

Suppose that $y = ux$, where u is a new function depending on x.

Then

$$dy = xdu + udx.$$

Put the expressions into the equation and simplify

$$(x - ux)(xdu + udx) - uxdx = 0,$$

$$x^2\,(1 - u)du + (xu - u^2x - ux)dx = 0,$$

and obtain

$$x^2\,(1 - u)du - u^2xdx = 0.$$

Now let us rewrite the differential equation to get everything separated out:

$$\frac{1 - u}{u^2}du = \frac{dx}{x}.$$

Integrate the last expression to obtain

$$\int \frac{1 - u}{u^2}du = \int \frac{dx}{x} - C,$$

$$-\frac{1}{u} - \ln|u| = \ln|x| - C,$$

$$\frac{1}{u} + \ln|ux| = C.$$

Plugging the substitution back in, we get the complete integral

$$\frac{x}{y} + \ln\left|\frac{x}{y}x\right| = C$$

or

$$x = y(C - \ln|y|).$$

where C is an arbitrary real number.

Applying the initial condition and solving for C gives

$$y(-1) = 1,$$

$$-1 = 1(C - \ln 1),$$

$$C = -1.$$

Then the particular integral is

$$x = -y(1 + \ln|y|).$$

II. Equations Reducible to Homogeneous Equations

Let us consider the equation

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right), \tag{1.12}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are some real numbers, and $f(u)$ is an arbitrary continuous on some interval function. Equations of the form (1.12) could be reducible to homogeneous equation.

If $c_1 = c_2 = 0$ then equation (1.12) is obviously homogeneous.

$$y' = f\left(\frac{a_1x + b_1y}{a_2x + b_2y}\right)$$

Now let $c_1 \neq 0$ and(or) $c_2 \neq 0$.

1) if the determinant $\Delta = \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \neq 0$ then equation (1.12) is converted into a separable equation by substitution

$$x = t + x_0, \qquad y = z + y_0,$$

where (x_0, y_0) is the point of intersection of two straight lines

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0,$$

defined as solutions of a system of equations

$$\begin{cases} a_1x_0 + b_1y_0 + c_1 = 0, \\ a_2x_0 + b_2y_0 + c_2 = 0. \end{cases}$$

2) if the determinant $\Delta = \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = 0$ then the differential equation (1.12) is transformed into separable equation by using the change of variable:

$$a_1x + b_1y = z.$$

Example 4.

Solve the equation

$$y' = -\frac{2x + y + 1}{x + 2y - 1}.$$

Let us convert the equation into a homogeneous.

Calculate the determinant

$$\Delta = \begin{vmatrix} 21 \\ 12 \end{vmatrix} = 3 \neq 0.$$

Since the determinant is not equal to zero, we make the substitution

$$x = t + x_0, \quad y = z + y_0,$$

where (x_0, y_0) we get from the system

$$\begin{cases} 2x + y + 1 = 0, \\ x + 2y - 1 = 0; \end{cases} \Rightarrow \begin{cases} x_0 = -1, \\ y_0 = 1. \end{cases}$$

Thus,

$$x = t - 1, \quad y = z + 1 \Rightarrow dx = dt, \quad dy = dz \Rightarrow y' = \frac{dy}{dx} = \frac{dz}{dt}.$$

Plugging these expressions into the original equation, we obtain

$$\frac{dz}{dt} = -\frac{2t - 2 + z + 1 + 1}{t - 1 + 2z + 2 - 1} \Rightarrow \frac{dz}{dt} = -\frac{2t + z}{t + 2z}.$$

As a result we get the homogeneous equation with respect to t and z

$$z' = -\frac{2 + \frac{z}{t}}{1 + 2\frac{z}{t}}$$

which is solved by substitution

$$\frac{z}{t} = u \Rightarrow z(t) = ut \Rightarrow z' = u'\,t + u.$$

Then

$$u't + u = -\frac{2 + u}{1 + 2u} \Rightarrow u't = -\frac{2 + u}{1 + 2u} - u \Rightarrow u't = -2\frac{u^2 + u + 1}{1 + 2u}.$$

Let us separate the variables

$$-\frac{1}{2}\frac{1 + 2u}{u^2 + u + 1}du = \frac{dt}{t}$$

and integrate

$$-\frac{1}{2}\int \frac{1 + 2u}{u^2 + u + 1}du = \int \frac{dt}{t} + \ln|C_1|,$$

$$-\frac{1}{2}\int \frac{d(u^2 + u + 1)}{u^2 + u + 1} = \int \frac{dt}{t} - \ln|C_1|,$$

$$-\frac{1}{2}\ln|u^2 + u + 1| = \ln|t| - \ln|C_1|.$$

Transforming the expression obtained, we get the complete integral

$$t = \frac{C_2}{\sqrt{u^2 + u + 1}}, (C_2 = \pm C_1),$$

$$t\sqrt{u^2+u+1}=C_2.$$

Put $\frac{z}{t}=u$, then

$$\sqrt{z^2+zt+t^2}=C_2,$$

$$z^2+zt+t^2={C_2}^2,$$

Passing to original variables x and y by formulas

$$t=x+1, \quad z=y-1,$$

$$(y-1)^2+(y-1)(x+1)+(x+1)^2={C_2}^2,$$

we finally get the complete integral of the original differential equation

$$x^2+y^2+xy+x-y=C, \ \ (C={C_2}^2-1).$$

Example 5.

Solve the equation

$$(x+y+2)dx+(2x+2y-1)dy=0.$$

Since $\Delta=\begin{vmatrix}1 & 1\\ 2 & 2\end{vmatrix}=0$, we make a substitution

$$x+y=z \ \Rightarrow \ y=z-x, \quad dy=dz-dx.$$

Putting these expressions into the original equation, we get

$$(z+2)dx+(2z-1)(dz-dx)=0.$$

The equation is reduced to the separable differential equation

$$(3-z)dx+(2z-1)dz=0.$$

Solving it, we get

$$\frac{2z-1}{z-3}dz=dx.$$

Let us integrate the equation

$$\int\frac{2z-1}{z-3}dz=\int dx+C.$$

Then

$$2z+5\ln|z-3|=x+C.$$

Since $z=x+y$, we obtain the final solution in implicit form

$$x+2y+5\ln|x+y-3|=C.$$

Review Questions

1. What function is called homogeneous?

2. Are the following functions homogeneous? If so, specify the degree of homogeneity:

1) $\frac{4x^2+xy-y^2}{y-2x}$; 2) $\ln\frac{x+2y}{3y+5x}$; 3) $\frac{(\sqrt{xy}-2y)^5}{x^2+y^2}$; 4) $ye^{\frac{x}{y}}+4\frac{x^2}{y}$; 5) $x\sin\frac{y+1}{2x}-y$.

3. Formulate the definition of homogeneous differential equation.

4. What is the method of finding the general solution of a homogeneous differential equation?

5. The differential equation is written in the form: $P(x,y)dx+Q(x,y)dy=0$.

Under what condition will this equation be homogeneous? Separable?

6. What differential equations could be reduced to homogeneous? What is the method of their solving?

Exercises 1.3

1-12. Find the general solution or integral of the differential equation:

1. $y' = 2+\frac{y}{x}$.

2. $(x+y)dx+2xdy=0$.

3. $y-xy'=x+yy'$.

4. $ydy+(x-2y)dx=0$.

5. $ydx+(2\sqrt{xy}-x)dy=0$.

6. $y=x(y'-\sqrt[x]{e^y})$.

7. $x\sin\frac{y}{x}\cdot y'+x=y\sin\frac{y}{x}$.

8. $xy+y^2=(2x^2+xy)y'$.

9. $xy'\cdot\ln\frac{y}{x}=x+y\cdot\ln\frac{y}{x}$.

10. $(xy'-y)\sin\frac{y}{x}=x$.

11. $(x-2y+3)dy+(2x+y-1)dx=0$.

12. $(x-y+4)dy+(x+y-2)dx=0$.

13-18. Find the particular solution or integral of the differential equation:

13. $(2x-3y)dx+xdy=0,\ \ y(1)=-1$.

14. $(5\sqrt{xy}-y)dx+xdy=0,\ \ y(1)=25$.

15. $xy'-y=x\cos^2\frac{y}{x},\ \ y(3)=0$.

16. $xy'=y(3+\ln y-\ln x),\ \ y(1)=1/e$.

17. $xy'-y=x\cdot\tan\frac{y}{x},\ \ y(1)=\pi/2$.

18. $2(x+y)dy+(3x+3y-1)dx=0,\ \ y(0)=2$.

[Answers.](#)

1.4 First-Order Linear Differential Equations

I. Basic Definition

Definition. A differential equation of the form

$$y' + P(x)y = Q(x), \tag{1.13}$$

where $P(x)$ and $Q(x)$ are given continuous functions defined on some interval, is called *a first-order linear differential equation*.

Note that linear differential equation is **linear** in the unknown function y and its derivative y'.

Definition. If $Q(x) = 0$ then the equation

$$y' + P(x)y = 0, \tag{1.14}$$

is called *linear homogeneous differential equation* corresponding the nonhomogeneous equation (1.13).

The equation (1.14) is the separable differential equation

$$\frac{dy}{dx} = -P(x)y.$$

Let us separate the variables

$$\frac{dy}{y} = -P(x)dx$$

and integrate both sides of equation

$$\ln|y| = -\int P(x)dx + \ln|C|.$$

Finally, we obtain the general solution of (1.14)

$$y(x) = Ce^{-\int P(x)dx}. \tag{1.15}$$

Definition. If $Q(x) \neq 0$ then the equation (1.13) is called *a linear nonhomogeneous differential equation*.

Let us consider two basic methods of solving linear **nonhomogeneous** differential equations:

- Bernoulli's method;
- Method of variation of a constant (Lagrange's method).

II. Bernoulli's method

Bernoulli's method is based on the idea that the solution of the nonhomogeneous equation (1.13)

$$y' + P(x)y = Q(x), \qquad Q(x) \neq 0,$$

could be represented as the product of two functions

$$y = u \cdot v, \tag{1.16}$$

where $u = u(x)$ and $v = v(x)$ are unknown functions. One of them may be arbitrary (but unequal zero), while the other should be determined from the equation (1.13).

Let us differentiate the equality (1.16)

$$y' = u'v + uv'.$$

Plugging the expressions for y and y' into the equation (1.13)

$$u'v + uv' + P(x) \cdot uv = Q(x),$$

we get

$$u'v + u(v' + P(x)v) = Q(x). \tag{1.17}$$

Let us choose the function $v(x)$ such that the expression in parenthesis is equal to zero, that is

$$v' + P(x)v = 0.$$

That idea leads us to the system of equations (obtained from (1.17))

$$\begin{cases} v' + P(x)v = 0, \\ u'v = Q(x). \end{cases}$$

The first equation is linear homogeneous differential equation of the form (1.14) and its general solution is

$$v = C_1 e^{-\int P(x)dx}, \;\; C_1 \neq 0.$$

Since the function $v(x)$ is arbitrary, it is sufficient to choose any nonzero solution, let us put $C_1 = 1$. Then the particular solution is

$$v = e^{-\int P(x)dx}.$$

Substituting the value of the function $v(x)$ into the second equation of the system, we obtain the separable differential equation for $u(x)$:

$$\frac{du}{dx} e^{-\int P(x)dx} = Q(x).$$

Hence

$$du = Q(x)e^{\int P(x)dx}dx.$$

Integrate the both sides of equation

$$u = \int Q(x)e^{\int P(x)dx}\, dx + C.$$

Putting the expressions obtained for functions u and v into the equation (1.16), we finally obtain the general solution of the linear nonhomogeneous differential equation (1.13):

$$y(x) = u \cdot v = \left(\int Q(x)e^{\int P(x)dx}\, dx + C\right) \cdot e^{-\int P(x)dx}.$$

Example 1.

Find the general solution

$$y' - y \cot x = \frac{1}{\sin x}.$$

This equation is linear nonhomogeneous differential equation with respect to y and y', where $P(x) = \cot x$, $Q(x) = \frac{1}{\sin x}$.

Let

$$y = uv.$$

Then

$$y' = u'v + uv'.$$

Putting y and y' into the original equation

$$u'v + uv' - uv\text{cot}x = \frac{1}{\sin x},$$

$$u'v + u(v' - v\text{cot}x) = \frac{1}{\sin x}.$$

To determine the unknown functions $u(x)$ and $v(x)$ we get the system of equations

$$\begin{cases} v' - v \cot x = 0, \\ u'v = \dfrac{1}{\sin x}. \end{cases}$$

Solve the first equation by separating the variables

$$\frac{dv}{dx} = v \cot x$$

and integrating both sides of the equation

$$\int \frac{dv}{v} = \int \cot x \, dx + \ln|C_1|,$$

$$\ln|v| = \ln|\sin x| + \ln|C_1|.$$

Hence,

$$\ln|v| = \ln|C_1 \sin x|$$

and

$$v = C \sin x\,, \quad \text{where} \ \ C = \pm C_1.$$

Since the function $v(x)$ is arbitrary, it is sufficient to choose the particular solution

$$v = \sin x.$$

for $C_1 = 1$.

Plug the function $v(x) = \sin x$ into second equation of the system

$$\frac{du}{dx} \sin x = \frac{1}{\sin x} \quad \Rightarrow \quad du = \frac{dx}{\sin^2 x}.$$

Find the function $u(x)$ by integration

$$\int du = \int \frac{dx}{\sin^2 x} + C,$$

whence

$$u = -\cot x + C.$$

Substituting the functions u and v into $y = uv$

$$y = (-\cot x + C) \sin x,$$

we get the general solution of the original equation

$$y = -\cos x + C \sin x.$$

III. Method of Variation of a Constant (Lagrange's Method)

The idea of Lagrange's method is that in order to obtain the general solution of nonhomogeneous DE (1.13) we use the general solution (1.15) of homogeneous DE (1.14) where the arbitrary constant is replaced by the function of an independent variable.This function must be chosen such that the nonhomogeneous DE (1.13) is fulfilled.

Let us look through the algorithm of the method.

First, we write down the homogeneous DE (1.14) corresponding to nonhomogeneous DE (1.13)

$$y' + P(x)y = 0,$$

Then we integrate it and obtain the general solution in the form (1.15):

$$y(x) = Ce^{-\int P(x)dx}.$$

Next, we search the solution of nonhomogeneous DE (1.13) in the form (1.15) considering C as some undetermined function $C(x)$

$$y(x) = C(x)e^{-\int P(x)dx}. \tag{1.18}$$

We have to differentiate it using the product and chain rules

$$y'(x) = C'(x)e^{-\int P(x)dx} + C(x)e^{-\int P(x)dx}(-P(x))$$

and substitute the expressions for $y(x)$ and $y'(x)$ into (4.1).

Hence,

$$C'(x)e^{-\int P(x)dx} + C(x)e^{-\int P(x)dx}\big(-P(x)\big) + P(x)C(x)e^{-\int P(x)dx} = Q(x).$$

Thus, we obtain the separable equation with respect to unknown function $C(x)$:

$$C'(x)e^{-\int P(x)dx} = Q(x).$$

Integrating it, we get

$$C(x) = \int Q(x)\, e^{\int P(x)dx} dx + \tilde{C}.$$

Putting the function obtained of $C(x)$ into (1.18), we find a general solution of the nonhomogeneous DE (1.13)

$$y(x) = \left(\int Q(x)e^{\int P(x)dx}\, dx + \tilde{C}\right) \cdot e^{-\int P(x)dx}. \tag{1.19}$$

Note.

1. The solution (1.19) has the **same form** as the solution obtained by Bernoulli's method.

2. Let us rewrite (1.19) as

$$y(x) = \tilde{C}e^{-\int P(x)dx} + e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}\, dx.$$

It is easy to see, that the general solution of nonhomogeneous DE (1.13) is equal the sum of the general solution y_{gh} of corresponding homogeneous DE (1.14) and of the particular solution y_{pn} of nonhomogeneous DE (1.13): $y = y_{gh} + y_{pn}$.

Example 2.

Solve the initial value problem

$$y' + 2xy = 3x^2e^{-x^2}, \qquad y(0) = -1.$$

Let us solve this linear differential equationit by the method of variation of a constant.

Write down and solve the homogeneous DE corresponding to original nonhomogeneous DE

$$y' + 2xy = 0.$$

Separate the variables

$$\frac{dy}{dx} = -2xy,$$

$$\frac{dy}{y} = -2xdx$$

and integrate it

$$\int \frac{dy}{y} = -2\int xdx + \ln|C|.$$

Hence,

$$\ln|y| - \ln|C| = -x^2,$$

$$\ln\left|\frac{y}{C}\right| = -x^2,$$

$$\frac{y}{C} = e^{-x^2},$$

$$y = Ce^{-x^2}.$$

Thus, the solution of the nonhomogeneous DE is

$$y = C(x)e^{-x^2}, \qquad (*)$$

considering C as some undetermined function of variable x.

Differentiate the equality $(*)$

$$y' = C'(x)e^{-x^2} + C(x)e^{-x^2}(-2x).$$

Plugging y and y' into the original equation, we get

$$C'(x)e^{-x^2} - 2xC(x)e^{-x^2} + 2xC(x)e^{-x^2} = 3x^2e^{-x^2}.$$

Hence

$$C'(x)e^{-x^2} = 3x^2e^{-x^2}.$$

Then

$$C'(x) = 3x^2$$

and, upon integration,

$$C(x) = x^3 + \tilde{C}.$$

Substituting $C(x)$ into $(*)$, we obtain the general solution of the original equation

$$y = \left(x^3 + \tilde{C}\right)e^{-x^2} = \tilde{C}e^{-x^2} + x^3e^{-x^2}.$$

Note.

Here $y = y_{gh} + y_{pn}$, where $y_{gh} = \tilde{C}e^{-x^2}$ is general solution of corresponding homogeneous DE, while $y_{pn} = x^3e^{-x^2}$ is of the particular solution of nonhomogeneous DE.

Now, let us solve the Cauchy problem.

Apply the initial condition $y(0) = -1$ to find the value of $\tilde{C}$:

$$-1 = \left(0 + \tilde{C}\right)e^0 \;\; \Rightarrow \;\; \tilde{C} = -1.$$

The solution of the initial value problem is then

$$y = (x^3 - 1)e^{-x^2}.$$

Example 3.

Solve the equation

$$(x + y)y' = 1.$$

Since the equation contains the product yy', it is impossible to rewrite this equation in the form (1.13) and it is not linear with respect to function $y(x)$. However, it is linear with respect to function $x(y)$.

Indeed, transforming the equation

$$\frac{dy}{dx} = \frac{1}{(x + y)},$$

$$\frac{dx}{dy} = (x + y),$$

we obtain

$$x' - x = y$$

the linear nonhomogeneous differential equation with respect to function $x(y)$.

Let us solve it by Lagrange's method.

Find the general solution of corresponding homogeneous equation

$$x' - x = 0,$$

which can be solved by separating the variables

$$\frac{dx}{dy} = x,$$

$$\frac{dx}{x} = dy.$$

Integrating the equation obtained

$$\ln|x| = y + \ln C,$$

we get, upon potentiation,

$$x = Ce^y.$$

Hence, the general solution of the nonhomogeneous equation is in the form

$$x = C(y)e^y. \qquad (*)$$

Differentiate the equality $(*)$

$$x' = C'(y)e^y + C(y)e^y.$$

Substituting expressions for x and x' into initial equation, we get

$$C'(y)e^y + C(y)e^y - C(y)e^y = y$$

and

$$C'(y)e^y = y.$$

This yields

$$C'(y) = ye^{-y},$$

$$C(y) = \int ye^{-y}dy + C.$$

Integrating by parts, we obtain

$$C(y) = -e^{-y}(y+1) + C.$$

Plugging the function $C(y)$ into $(*)$, we get the general solution of the original equation

$$x = \left(C - e^{-y}(y+1)\right)e^y$$

or

$$x = Ce^y - (y+1).$$

IV. Some Applications of Linear Differential Equations

1. Physics (Series RL Circuit)

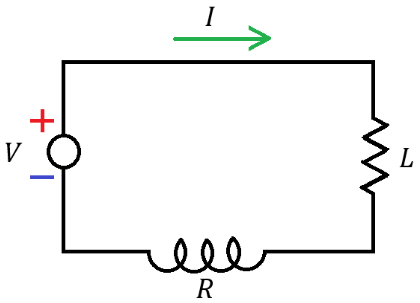

Figure 1.9

Let us consider a simple electrical circuit consists of a resistor R and an inductor L connected in series (Fig. 1.9), with a constant electromotive force V.

The voltage across the **resistor** is given by IR.

The voltage across the **inductor** is given by $L \cdot I'(t)$.

From Kirchhoff's Law for electrical circuits (the directed sum of the voltages around circuit must be zero) it follows that if $t > 0$, the current $I = I(t)$ satisfies the differential equation

$$L\frac{dI}{dt} + RI = V.$$

This differential equation may be written in the form

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L},$$

which is a first-order linear differential equation.

Applying the formula (1.19), we obtain

$$I(t) = \left(\int \frac{V}{L} e^{\int \frac{R}{L}dt}\, dt + \tilde{C}\right) \cdot e^{-\int \frac{R}{L}dt} = \left(\frac{V}{R} e^{\frac{R}{L}t} + \tilde{C}\right) \cdot e^{-\frac{R}{L}t}$$

Suppose that $I = 0$ when $t = 0$, then $\tilde{C} = -\frac{V}{R}$, and the solution of initial value problem is

$$I(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

Observe that as t infinitely increases, I approaches $\frac{V}{R}$, which is the current predicted by Ohm's law when no inductance is present.

Note.

In general electromotive force V is a function of t. Let the reader find the current $I(t)$ in the case when $V(t) = V_0 \sin 2\pi n t$, for natural n.

2. Physics (Newton's Law of Cooling)

Let us consider the solid of initial temperature T_0 is placed in the environment of the temperature $T_e(t)$, thus, here $T_e(t)$ is not costant (see page 18.), but it changes with time.

According to Newton's law of cooling, for the temperature of the solid at time $T = T(t)$, could be written the differential equation

$$\frac{dT(t)}{dt} = \frac{\alpha \cdot A}{C}\big(T_e(t) - T(t)\big),$$

where C is the heat capacity of the body, A is the surface area of the body through which the heat is transferred, α is the heat transfer coefficient.

Denoting $\frac{\alpha \cdot A}{C} = K$, we obtain the linear nonhomogeneous differential equation

$$\frac{dT(t)}{dt} + KT(t) = KT_e(t).$$

The given initial value problem has the solution

$$T(t) = T_e + (T_0 - T_e)e^{-Kt}.$$

The general solution of this equation (by (1.19))

$$T(t) = e^{-Kt}\left(\int KT_e(t)e^{Kt}dt + C\right),$$

where C is an arbitrary constant of integration, which may be determined from the initial condition $T(0) = T_0$.

3. Physics (Falling with Air Resistance)

Consider a body of mass m, which is dropped from some height. Let us find its velocity as a function of time $v(t)$, assuming that the air resistance is proportional to the velocity of the body (with constant of proportionality K).

Let force F is the force acting on the body in the direction of motion (downward force). This force is resultant of two forces: the force of gravity mg (g is the gravitational constant), and the force of air resistance $-Kv$ (we choose minus sign since it has opposite direction to that of the velocity).

Thus, the force F acting on a falling object of mass m is given by the difference

$$mg - Kv.$$

However, according to Newton's second law of motion, the net force F on a body is proportional to its acceleration $a = \frac{dv}{dt}$, provided the net force acting on the body is not zero, that is $F = ma$.

Hence

$$ma = mg - Kv$$

or

$$m\frac{dv}{dt} = mg - Kv.$$

Finally, we obtain the linear nonhomogeneous differential equation

$$\frac{dv}{dt} + \frac{K}{m}v = g.$$

General solution of this equation is

$$v(t) = Ce^{-\frac{K}{m}t} + \frac{mg}{K},$$

where C is a constant of integration.

Let us determine C from the initial condition $v(0) = v_0$ (initial velocity of the body). Substituting this condition into general solution, we get

$$v_0 = C + \frac{mg}{K} \quad \Rightarrow \quad C = v_0 - \frac{mg}{K}.$$

Thus,

$$v(t) = \left(v_0 - \frac{mg}{K}\right)e^{-\frac{K}{m}t} + \frac{mg}{K}.$$

Review Questions

1. What equation is called the first-order linear differential equation?
2. What equation is called nonhomogeneous first-order linear differential equation?
3. What equation is called linear homogeneous differential equation corresponding the nonhomogeneous equation?
4. What is the idea of Bernoulli’s method? What are the steps of Bernoulli’s method?
5. What is the idea of Lagrange’s method? What are the steps of Lagrange’s method?
6. What is the structure of the general solution of the first-order linear differential equation?

Exercises 1.4

1-12. Find the general solution or integral of the differential equation:

1. $y' - \frac{y}{x} = 3x.$

2. $y' + 4\frac{y}{x} + x = 0.$

3. $x^2y' + 2xy - 1 = 0.$

4. $y' - 7y = 8e^{3x}.$

5. $(x^2 + 1)y' - xy = x^3 + x.$

6. $\frac{dy}{dx} - \frac{y}{\sqrt{x}} - e^{2\sqrt{x}} = 0.$

7. $y' \cos x - y \sin x = \cos^2 x.$

8. $xy' \ln x = 5x - y.$

9. $\frac{dy}{dx} \cdot \cos^2 x + y = \tan x.$

10. $y'(x^2 + 4) - xy = \sqrt{x^2 + 4}.$

11. $y^2dx + (5xy - 4)dy = 0.$ (Accept $x(y)$ as an unknown function)

12. $\frac{dy}{dx} = \frac{y^2}{x+ye^{-1/y}}.$ (Accept $x(y)$ as an unknown function)

13-18. Find the particular solution or integral of the differential equation:

13. $y' + 3y = xe^{-3x}, \quad y(0) = 0.$

14. $y'(1 - x^2) = xy + 1, \quad y(\sqrt{3}/2) = 2\pi/3.$

15. $\frac{dy}{dx} + e^x y = e^{2x}, \quad y(0) = 1/e.$

16. $(1 - x^2)y' - 2xy = (1 - x^2)^2, \quad y(3) = 40.$

17. $x\frac{dy}{dx} - y = x^2 \sin x, \quad y(\pi/2) = \pi.$

18. $xy' \ln x = x + \ln x, \quad y(e^2) = 2\ln 2.$

Answers.

1.5 Bernoulli's Equation

Definition. The first-order differential equation of the form

$$y' + P(x)y = Q(x)y^n, \tag{1.20}$$

where $n \in R, n \neq 0, n \neq 1$, $P(x)$ and $Q(x)$ are continuous functions of x, is called *Bernoulli's equation.*

Note.

If $n = 0$ then equation (1.20) is a linear nonhomogeneous DE. If $n = 1$ then equation (1.20) is a separable DE.

For any $n \in R$ $(n \neq 0, n \neq 1)$ the Bernoulli's equation could be reduced to a linear equation by substitution.

First, we divide the equation (1.20) by y^n $(y \neq 0)$ and get

$$y^{-n}y' + P(x)y^{1-n} = Q(x). \tag{1.21}$$

Then, we use the substitution

$$z = y^{1-n},$$

where $z = z(x)$ is a new unknown function.

Differentiating the equality

$$z' = \frac{dz}{dx} = (1-n)y^{-n}y'$$

we obtain

$$y^{-n}y' = \frac{z'}{1-n}.$$

Substituting these expressions into (1.21), we get

$$\frac{1}{1-n}z' + P(x)z = Q(x),$$

$$z' + (1-n)P(x)z = (1-n)Q(x).$$

The equation obtained is a linear differential equation with respect to function $z(x)$. It could be solved by Bernoulli's or Lagrange's methods.

Note.

We may solve Bernoulli's equation by Bernoulli's method directly substituting $y = uv$. In this case, we do not need to reduce the equation to linear.

Example 1.

Find the general solution of the equation

$$y' + \frac{y}{x} = x^2y^2.$$

Since the left side of the equation is linear with respect to y and $y'$and the right side has a form x^2y^2, given equation is Bernoulli's equation.

Let us convert it to linear DE.

First, we divide both sides by y^2 ($y \neq 0$):

$$\frac{y'}{y^2} + \frac{1}{x} \cdot \frac{1}{y} = x^2,$$

$$y'y^{-2} + \frac{1}{x}y^{-1} = x^2. \qquad (*)$$

Second, we make a substitution

$$z = y^{1-n} = y^{1-2} = y^{-1}$$

and differentiate the function $z(x)$

$$z' = \frac{dz}{dx} = -y^{-2} \cdot y',$$

$$y'y^{-2} = -z'.$$

Putting the expressions for y^{-1} and $y'y^{-2}$ into ($*$), we get

$$-z' + \frac{1}{x}z = x^2$$

or

$$z' - \frac{1}{x}z = -x^2. \qquad (**)$$

The equation obtained is the first-order linear nonhomogeneous equation with respect to $z(x)$. To solve it, we use Bernoulli's method.

Let

$$z = uv$$

and

$$z' = u'v + uv'.$$

Plugging z and z' into ($**$), we get

$$u'v + uv' - \frac{1}{x}\,uv = -x^2$$

$$u'v + u\left(v' - \frac{1}{x}\,v\right) = -x^2.$$

According to Bernoulli's method, we have to consider the system of equations

$$\begin{cases} v' - \frac{1}{x}\,v = 0, \\ u'v = -x^2. \end{cases}$$

First equation is separable DE. Hence

$$\frac{dv}{v} = \frac{1}{x}\,dx,$$

$$\int \frac{dv}{v} = \int \frac{1}{x}\,dx,$$

$$\ln|v| = \ln C|x|\,,$$

$$v = x\ \ (C = 1).$$

Put $v = x$ into second equation

$$u'x = -x^2,$$

$$u' = -x.$$

Integrate it

$$du = -x\,dx,$$

$$\int du = -\int xdx + C,$$

$$u = -\frac{x^2}{2} + C.$$

Then the general solution of the equation $(**)$ is given by

$$z = uv = \left(C - \frac{x^2}{2}\right)x = Cx - \frac{x^3}{2}.$$

Returning to the function y by substituition $z = y^{-1}$, we have

$$\frac{1}{y} = \frac{2Cx - x^3}{2}$$

or

$$y = \frac{2}{2Cx - x^3}.$$

Since we have lost the solution $y = 0$ while dividing by y^2, the general solution is given by

$$\begin{cases} y = \dfrac{2}{2Cx - x^3}, \\ y = 0. \end{cases}$$

Example 2.

Solve the equation

$$x^2y^2y' + xy^3 = 1.$$

Dividing the equation by x^2y^2:

$$y' + \frac{y}{x} = \frac{1}{x^2y^2},$$

we obtain the Bernoulli's equation.

Let us solve it by Bernoulli's method directly.

Substituting

$$y = uv \text{ and } y' = u'v + uv'$$

into the equation, we obtain

$$u'v + uv' + \frac{uv}{x} = \frac{1}{x^2u^2v^2},$$

$$u'v + u\left(v' + \frac{v}{x}\right) = \frac{1}{x^2u^2v^2}.$$

Determine the unknown functions $u(x)$ and $v(x)$ from the system of equations

$$\begin{cases} v' + \dfrac{v}{x} = 0, \\ u'v = \dfrac{1}{x^2u^2v^2}. \end{cases}$$

Solving first separable equation, we get the particular solution

$$\frac{dv}{v} = -\frac{dx}{x},$$

$$\ln|v| = \ln|x^{-1}| \quad \Rightarrow \quad v = \frac{1}{x}.$$

Putting v into second equation, we obtain separable equation

$$\frac{du}{dx} \cdot \frac{1}{x} = \frac{x^2}{x^2u^2},$$

$$u^2 du = x dx.$$

Integrate it

$$\int u^2 du = \int x\,dx + C,$$

$$\frac{u^3}{3} = \frac{x^2}{2} + \frac{C}{3}.$$

Thus,

$$u^3 = \frac{3x^2}{2} + C,$$

$$u = \sqrt[3]{\frac{3}{2}x^2 + C}.$$

Hence, we get

$$y = uv = \left(\sqrt[3]{\frac{3}{2}x^2 + C}\right)\frac{1}{x}$$

Finally, the general solution of the original Bernoulli's equation is

$$y = \sqrt[3]{\frac{3}{2x} + \frac{C}{x^3}}.$$

Example 3.

Find the solution of Cauchy problem

$$(x^2 \ln y - x)y' = y, \qquad y\left(\frac{1}{2}\right) = 1.$$

Since the equation contains $\ln y$, this equation is not linear with respect to function $y(x)$. However, after the following transformations

$$(x^2 \ln y - x)\frac{dy}{dx} = y,$$

$$\frac{dy}{dx} = \frac{y}{x^2 \ln y - x},$$

$$\frac{dx}{dy} = \frac{x^2 \ln y - x}{y}$$

we obtain

$$\frac{dx}{dy} = \frac{x^2 \ln y}{y} - \frac{x}{y}.$$

Finally, the equation

$$x' + \frac{x}{y} = \frac{\ln y}{y} x^2$$

is Bernoulli's equation with respect to function $x(y)$.

Let us solve it by Bernoulli's method:

$$x = u(y) \cdot v(y) \quad \Rightarrow \quad x' = u'v + uv'.$$

Putting x and x' into the equation obtained, we have

$$u'v + uv' + \frac{uv}{y} = \frac{\ln y}{y} u^2 v^2,$$

$$u'v + u\left(v' + \frac{v}{y}\right) = \frac{\ln y}{y} u^2 v^2. \qquad (*)$$

Consider the equation

$$v' + \frac{v}{y} = 0,$$

separate variables

$$\frac{dv}{dy} = -\frac{v}{y},$$

$$\frac{dv}{v} = -\frac{dy}{y},$$

integrate both sides of equation

$$\int \frac{dv}{v} = -\int \frac{dy}{y}$$

and find the function $v(y)$

$$\ln|v| = -\ln|y| \;\Rightarrow\; v = \frac{1}{y}.$$

Substituting $v = \frac{1}{y}$ into $(*)$ and separating variables, we get

$$u' \frac{1}{y} = \frac{\ln y}{y} u^2 \frac{1}{y^2}$$

$$\frac{du}{dy} = \frac{\ln y}{y^2} u^2.$$

Integrate both sides of equation

$$\int \frac{du}{u^2} = \int \frac{\ln y}{y^2} dy,$$

$$\int \frac{\ln y}{y^2} dy = \left| \begin{matrix} u = \ln y, du = \frac{1}{y} dy \\ dv = \frac{dy}{y^2}, v = -\frac{1}{y} \end{matrix} \right| = -\frac{\ln y}{y} + \int \frac{dy}{y^2} + C = -\frac{\ln y}{y} - \frac{1}{y} + C.$$

This yields $u(y)$

$$-\frac{1}{u} = -\frac{\ln y}{y} - \frac{1}{y} + C,$$

$$u = \frac{y}{\ln y + 1 - Cy}.$$

Hence, general solution of the original equation is

$$x(y) = u(y) \cdot v(y) = \frac{1}{\ln y + 1 - Cy}.$$

Let us apply the initial condition $x = \frac{1}{2}, y = 1$:

$$\frac{1}{2} = \frac{1}{\ln 1 + 1 - C \cdot 1} \quad \Rightarrow \quad C = -1.$$

Thus the particular solution is

$$x = \frac{1}{\ln y + y + 1}.$$

Review Questions

1. What differential equation is called Bernoulli's equation?

2. What is the method of reduction the Bernoulli’s equation to a linear equation?

3. What is the difference between Bernoulli's equation and nonhomogeneous first-order linear differential equation?

4. What is the method of solving of Bernoulli's equation?

Exercises 1.5

1-10. Solve the differential equation:

1. $y' + y = x\sqrt{x}$.

2. $y' + 2\frac{y}{x} = 3x^2y^{4/3}$.

3. $y' - \frac{y}{x-1} = \frac{y^2}{x-1}$.

4. $y' + 2\frac{y}{x} = \frac{2\sqrt{y}}{\cos^2 x}$.

5. $4xy' + 3y = -e^x x^4 y^5$.

6. $y' + y = e^{\frac{1}{2}x}\sqrt{y} = 0, \quad y(0) = \frac{9}{4}$.

7. $y' + \frac{3x^2y}{x^3+1} = y^2(x^3+1)\sin x\,, \quad y(0) = 1$.

8. $ydx + (x + x^2y^2)dy = 0$. (Accept $x(y)$ as an unknown function)

9. $y' - 2y\tan x + y^2\sin^2 x = 0$.

10. $y'(y^2 + 2y + x^2) + 2x = 0, \quad y(1) = 0$. (Accept $x(y)$ as an unknown function)

Answers

1.6 Exact Differential Equations

I. The Concept of Exact DE

Definition. The first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1.22)$$

is called *an exact differential equation* if $M(x,y)$ and $N(x,y)$ are continuous differentiable functions of two variables x and y, such that there exist continuous partial derivatives $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ and the following equality is fulfilled

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (1.23)$$

Note.

Let us remind some facts from the **theory of line integrals of vector fields**.

Definition. Expression $P(x,y)dx + Q(x,y)dy$ is called *an exact differential of some function* $u = u(x,y)$ if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. It is denoted by

$$du(x,y) = P(x,y)dx + Q(x,y)dy.$$

In this case, the function $u = u(x,y)$ could be found by formula

$$u(x,y) = \int_{(x_0,y_0)}^{(x,y)} P(x,y)dx + Q(x,y)dy = \int_{x_0}^{x} P(x,y_0)\, dx + \int_{y_0}^{y} Q(x,y)\, dy.$$

It is easy to see that the left side of (1.22) under the condition (1.23) is exact differential of some function. That is, the equation (1.22) is an equation of the form

$$du(x,y) = 0$$

and, consequently, its complete integral is

$$u(x,y) = C.$$

We yields that the function $u(x,y)$ will have the form

$$u(x,y) = \int_{x_0}^{x} M(x,y_0)\, dx + \int_{y_0}^{y} N(x,y)\, dy,$$

where (x_0,y_0) is a fixed point in the neighborhood of which there is a solution of (1.22).

Finally, the complete integral of exact differential equation (1.22) is

$$\int_{x_0}^{x} M(x, y_0)\, dx + \int_{y_0}^{y} N(x, y)\, dy = C. \tag{1.24}$$

Example 1.

Solve the equation

$$(x + y - 1)dx + (x + e^y)dy = 0.$$

Here we have

$$M(x, y) = x + y - 1, \qquad N(x, y) = x + e^y.$$

First, we evaluate the partial derivatives of $M(x, y),\ N(x, y)$ to test for exactness of the equation

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = 1.$$

According to the test $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$, this differential equation is exact.

To solve it, we use formula (1.24) directly

$$\int_{x_0}^{x} (x + y_0 - 1)\, dx + \int_{y_0}^{y} (x + e^y)\, dy = C_1.$$

Thus,

$$\int_{x_0}^{x} (x + y_0 - 1)\, dx + \int_{y_0}^{y} (x + e^y)\, dy = \left(\frac{1}{2}x^2 + xy_0 - x\right)\Big|_{x_0}^{x} + (xy + e^y)|_{y_0}^{y} =$$

$$= \left(\frac{1}{2}x^2 + xy_0 - x\right) - \left(\frac{1}{2}{x_0}^2 + x_0 y_0 - x_0\right) + (xy + e^y) - (xy_0 + e^{y_0}) =$$

$$= \left(\frac{1}{2}x^2 + xy + e^y - x\right) - \left(\frac{1}{2}{x_0}^2 + x_0 y_0 - x_0 + e^{y_0}\right) = C_1.$$

Since (x_0, y_0) is a fixed point, the expression in the second parenthesis is some constant:

$$\frac{1}{2}{x_0}^2 + x_0 y_0 - x_0 + e^{y_0} = C_2.$$

Let $C = C_1 + C_2$.

Hence, the complete integral of the original equation is

$$\frac{1}{2}x^2 + xy + e^y - x = C.$$

II. Integrating Factor

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1.25)$$

where $M(x,y)$ and $N(x,y)$ are continuous differentiable functions of two variables x and y, such that there exist continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$, but

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Obvious, that the equation (1.25) is not **exact**. However, it is sometimes possible to convert the left side of equation into an exact differential by multiplying all terms of the equation by the special function $\mu(x,y)$ – *the integrating factor*. In this case, the general solution of the equation thus obtained coincides with the general solution of the original equation (1.25).

In order to find $\mu(x,y)$, we consider the equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0. \qquad (1.26)$$

Accorging to the idea of using integrating factor, it is exact equation and the following condition is fulfilled

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}.$$

Let us differentiate the functions:

$$\mu\frac{\partial M}{\partial y} + M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{\partial \mu}{\partial x}$$

or

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right). \qquad (1.27)$$

The equation obtained is the partial differential equation of first order that defines the integrating factor $\mu(x,y)$.

Unfortunately, in general case solving that equation is more complicated then integrating the original equation (1.25). However, there are some particular cases when this partial differential equation could be solved.

Let us consider the simplest cases of choosing the integrating factor $\mu(x, y)$.

1. Suppose that integrating factor is a function depends only of x, that is $\mu = \mu(x)$. Then the equation (1.27) is transformed into odinary differential equation

$$-N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

or

$$\frac{1}{\mu}\frac{\partial \mu}{\partial x} = \frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N}.$$

If the right side of this equation depends only on x $\left(\frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N} = F(x)\right)$, then there exists the integrating factor $\mu(x)$ that is expressed by integral

$$\ln \mu = \int \frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N} dx$$

or

$$\mu(x) = e^{\int \frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N} dx}.$$

2. Similarly, if integrating factor is a function depends only of y $\left(\mu = \mu(y)\right)$ and expression $\frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}$ is not dependent on x, then

$$\frac{1}{\mu}\frac{\partial \mu}{\partial y} = \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M}$$

and

$$\mu(y) = e^{\int \frac{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}{M} dx}.$$

Example 2.

Solve the equation

$$(y + \ln x)dx - xdy = 0$$

Here we have:

$$M(x, y) = y + \ln x, \quad N(x, y) = -x.$$

Let us test the equation for exactness

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = -1.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this differential equation is not exact.

Let us convert the equation into exact equation by using the integrating factor depending only of x, $\mu = \mu(x)$.

Evaluate

$$\frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N} = \frac{1+1}{-x} = -\frac{2}{x}.$$

Hence,

$$\mu(x) = e^{\int \frac{\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)}{N} dx} = e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = \frac{1}{x^2}.$$

Multiply all terms of the equation by the function $\mu(x)$

$$\frac{1}{x^2}(y + \ln x)dx - x \cdot \frac{1}{x^2} dy = 0,$$

$$\frac{1}{x^2}(y + \ln x)dx - \frac{1}{x} dy = 0.$$

Now we have $M_1(x,y) = \frac{1}{x^2}(y + \ln x)$, $N_1(x,y) = -\frac{1}{x}$. Let us test the equation for exactness

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial N_1}{\partial x} = \frac{1}{x^2}.$$

This differential equation is exact, since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$.

Solving it, we use formula (1.24) directly

$$\int_{x_0}^{x} \left(\frac{y_0}{x^2} + \frac{\ln x}{x^2}\right) dx - \int_{y_0}^{y} \frac{1}{x} dy = C_1.$$

Integrating by parts

$$\int \frac{\ln x}{x^2} dx = \begin{vmatrix} u = \ln x & du = \frac{1}{x} \\ dv = \frac{1}{x^2} & v = -\frac{1}{x} \end{vmatrix} = -\frac{1}{x}\ln x + \int \frac{1}{x^2} dx = -\frac{1}{x}\ln x - \frac{1}{x} + C,$$

we obtain

$$\int_{x_0}^{x} \left(\frac{y_0}{x^2} + \frac{\ln x}{x^2}\right) dx - \int_{y_0}^{y} \frac{1}{x} dy = \left(-\frac{y_0}{x} - \frac{1}{x}\ln x - \frac{1}{x}\right)\Bigg|_{x_0}^{x} - \left(\frac{y}{x}\right)\Bigg|_{y_0}^{y} =$$

$$= -\frac{y_0}{x} - \frac{1}{x}\ln x - \frac{1}{x} + \frac{y_0}{x_0} + \frac{1}{x_0}\ln x_0 + \frac{1}{x_0} - \frac{y}{x} + \frac{y_0}{x} =$$

$$= \left(-\frac{1}{x}\ln x - \frac{1}{x} - \frac{y}{x}\right) + \left(\frac{y_0}{x_0} + \frac{1}{x_0}\ln x_0 + \frac{1}{x_0}\right) = C_1.$$

Since (x_0, y_0) is a fixed point, the expression $\left(\frac{y_0}{x_0} + \frac{1}{x_0}\ln x_0 + \frac{1}{x_0}\right)$ is some constant.

Hence, the complete integral of the original equation is

$$\frac{1}{x}\ln x + \frac{1}{x} + \frac{y}{x} = C,$$

and the general solution is

$$y = Cx - \ln x - 1.$$

Example 3.

Find the complete integral of the equation

$$xdx + (y^2 + x^2 + y)dy = 0.$$

Since

$$\frac{\partial M}{\partial y} = \frac{\partial x}{\partial y} = 0 \neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y^2 + x^2 + y) = 2x,$$

this differential equation is not exact.

We may convert the equation into exact equation by multiplying by the integrating factor depending only of y:

$$\mu(y) = e^{\int \frac{\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right)}{M} dy} = e^{\int \frac{2x}{x} dy} = e^{\int 2dy} = e^{2y}.$$

Multiply all terms of the equation by the function $\mu(y) = e^{2y}$

$$xe^{2y}dx + (y^2 + x^2 + y)e^{2y}dy = 0,$$

Now we have $M_1(x, y) = \frac{1}{x^2}(y + \ln x)$, $N_1(x, y) = -\frac{1}{x}$. Let us test the equation for exactness.

This differential equation is exact, since

$$\begin{aligned} \frac{\partial M_1}{\partial y} &= \frac{\partial xe^{2y}}{\partial y} = 2xe^{2y} \\ \frac{\partial N_1}{\partial x} &= \frac{\partial}{\partial y}(y^2 + x^2 + y)e^{2y} = 2xe^{2y} \end{aligned} \quad \Longrightarrow \quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}.$$

Let us use formula (1.24) directly

$$\int_{x_0}^{x} xe^{2y_0}\,dx + \int_{y_0}^{y} (y^2 + x^2 + y)e^{2y}\,dy = C_1.$$

Then

$$\int_{x_0}^{x} xe^{2y_0}\,dx + \int_{y_0}^{y} x^2 e^{2y}\,dy + \int_{y_0}^{y} (y^2 + y)e^{2y}\,dy = C_1.$$

Solving the third integral by parts, we get

$$\int (y^2 + y)e^{2y}\,dy = \begin{vmatrix} u = y^2 + y & du = 2y + 1 \\ dv = e^{2y} & v = \dfrac{e^{2y}}{2} \end{vmatrix} = (y^2 + y)\frac{e^{2y}}{2} - \int (2y + 1)\frac{e^{2y}}{2}\,dy =$$

$$= \begin{vmatrix} u = 2y + 1 & du = 2 \\ dv = \dfrac{e^{2y}}{2} & v = \dfrac{e^{2y}}{4} \end{vmatrix} = (y^2 + y)\frac{e^{2y}}{2} - (2y + 1)\frac{e^{2y}}{4} + \int \frac{e^{2y}}{2}\,dy =$$

$$= (y^2 + y)\frac{e^{2y}}{2} - (2y + 1)\frac{e^{2y}}{4} + \frac{e^{2y}}{4} + C = y^2\frac{e^{2y}}{2} + C.$$

Hence,

$$\left.\left(\frac{x^2}{2}e^{2y_0}\right)\right|_{x_0}^{x} + \left.\left(x^2\frac{e^{2y}}{2} + y^2\frac{e^{2y}}{2}\right)\right|_{y_0}^{y} =$$

$$= \frac{x^2}{2}e^{2y_0} - \frac{{x_0}^2}{2}e^{2y_0} + x^2\frac{e^{2y}}{2} + y^2\frac{e^{2y}}{2} - x^2\frac{e^{2y_0}}{2} - {y_0}^2\frac{e^{2y_0}}{2} =$$

$$= \left(x^2\frac{e^{2y}}{2} + y^2\frac{e^{2y}}{2}\right) - \left(\frac{{x_0}^2}{2}e^{2y_0} + {y_0}^2\frac{e^{2y_0}}{2}\right) = C_1.$$

Since the expression $\left(\frac{{x_0}^2}{2}e^{2y_0} + {y_0}^2\frac{e^{2y_0}}{2}\right)$ is some constant, the complete integral of the original equation is

$$x^2\frac{e^{2y}}{2} + y^2\frac{e^{2y}}{2} = C$$

or

$$(x^2 + y^2)e^{2y} = C.$$

Review Questions

1. What equation is called exact differential equation?
2. The differential equation is written in the form: $P(x,y)dx + Q(x,y)dy = 0$.
   Under what condition will this equation be exact?
3. What is the method of solving of exact differential equation?
4. What is the integrating factor? Why do we need the integrating factor?
5. What are the methods of finding of the integrating factor?

Exercises 1.6

1-6. Solve the differential equation:

1. $(x + \ln|y|)dx + \left(1 + \frac{x}{y} + \sin y\right) dy = 0.$

2. $(2y - 3)dx + (2x + 3y^2)dy = 0.$

3. $2x\cos^2 y\, dx + (2y - x^2 \sin 2y)dy = 0,\ y(0) = 0.$

4. $3x^2e^y + (x^3e^y - 1)y' = 0,\ y(0) = 1.$

5. $e^{-y}dx + (1 - xe^{-y})dy = 0.$

6. $(y^3 - 4y^2x + 12x^3)dx + (3y^2x - 4yx^2)dy = 0,\ y(1) = 0.$

7-9. Solve the differential equation by using the integrating factor depending only of x, $\mu = \mu(x)$.

7. $(x^2y^2 - 1)dx + 2x^3ydy = 0.$

8. $(x^2 - 3y^2)dx + 2xydy = 0$

9. $\cos y\, dx + (\sin y + e^x)dy = 0,\ y(0) = 0.$

10-12. Solve the differential equation by using the integrating factor depending only of y, $\mu = \mu(y)$.

10. $(1 + 3x^2 \sin y)dx - x\cot y\, dy = 0.$

11. $xdx + (e^{2y} - x^2)dy = 0,\ y(0) = 0.$

12. $(y^2 \cos x + y\ln y)dx + (y\sin x + x)dy = 0.$

Answers.

2. Higher-Order Differential Equations

2.1 Basic Concepts and Definitions

I. Basic Concepts and Definitions

Definition. *The n-th order differential equation* has a form

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{2.1}$$

or, if it can be solved for n-th derivative,

$$y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right), \tag{2.2}$$

where functions F and f are continuous functions of the specified arguments.

Definition. *The solution or integral* of a differential equation (2.2) is any n-times continuously differentiable function $y = \phi(x)$ ($\phi(x)$, $\phi'(x)$, $\phi''(x)$, … , $\phi^{(n-1)}(x)$ are continuous functions) such that when putting into (2.2), converts the equation into an identity

Definition. If the function $y = \phi(x, C_1, C_2, \ldots, C_n)$, where $C_1, C_2, \ldots, C_n$ are arbitrary constants satisfies the equation (2.2) for any values of the constants, then the function is called *the general solution* of the n-th order differential equation.

Definition. If the implicit function $\Phi(x, y, C_1, C_2, \ldots, C_n) = 0$ satisfies the equation for any values of the arbitrary constants $C_1, C_2, \ldots, C_n$, then the function is called *the complete integral* of the n-th order differential equation.

Definition. A differential equation (2.2) together with an initial condition

$$y^{(n)} = f\left(x, y, y', y'', \ldots, y^{(n-1)}\right), \tag{2.2}$$

$$y(x_0) = y_0, y'(x_0) = y_0', \; \ldots, \; y^{(n-1)}(x_0) = y_0 \tag{2.3}$$

is called *an initial value problem* or *Cauchy problem* for n-th order differential equation.

For this problem, we have a theorem on the existence and uniqueness of solution, similar to the corresponding theorem on the solution of the first-order equations.

Theorem.

Cauchy Theorem About Existence and Uniqueness.

If in the differential equation

$$y^{(n)} = f\left(x, y, y', y'', \dots, y^{(n-1)}\right)$$

function $f(x, y, y', y'', \dots, y^{(n-1)})$ *and its partial derivative with respect to* $\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}, \dots, \frac{\partial f}{\partial y^{(n-1)}}$ *are continuous in some region D containing some point* $M_0(x_0; y_0; y'_0; \dots; y_0^{(n-1)})$, *then there is only one solution to this equation* $y = \phi(x)$, *which satisfies the condition* $y(x_0) = y_0, y'(x_0) = y'_0$, ..., $y^{(n-1)}(x_0) = y_0$.

Thus, for any initial conditions there are unique values of constants $C_1 = C_1^0, C_2 = C_2^0, \dots, C_n = C_n^0$ such that function $y = \phi(x, C_1^0, C_2^0, \dots, C_n^0)$ is a solution of the equation and satisfies the conditions (2.3).

Definition. The solution $y = \phi(x, C_1^0, C_2^0, \dots, C_n^0)$ is called *the particular solution* of (2.2). The function $\Phi(x, y, C_1^0, C_2^0, \dots, C_n^0) = 0$, which implicitly defines the particular solution, is called *the particular integral* of the equation (2.2).

Definition. The graph of any solution of n-th order differential equation is called *integral curve*. General solution determines a set of integral curves, but the particular solution corresponds to only one integral curve.

II. Some Problems that Leads to Higher-Order Differential Equations

1. Physics (Newton's Second Law of Motion)

It is well-known that the relationship of acceleration a of a body of constant mass m and the force F acting on the body is written by the equation $F = ma$. Let $s = s(t)$ be the displacement of the body from some original point, measured positive upward, then $a = \frac{d^2s}{dt^2}$. Assume that the motion of the body is along a vertical line.

Let us consider several cases.

- If the force F depends only on time t, then the equation has a form

$$m\frac{d^2s}{dt^2} = F(t).$$

- If the force F is proportional to the velocity $v(t) = \frac{ds}{dt}$ (for example, if a body moves in a liquid or gaseous environment it experiences a frictional force), that is

$$F = -Kv(t) = -K\frac{ds}{dt},$$

where K in turn is proportional to the viscosity.

Then the equation has a form

$$m\frac{d^2s}{dt^2} = -K\frac{ds}{dt}$$

or

$$m\frac{d^2s}{dt^2} + K\frac{ds}{dt} = 0.$$

- If the force F depends on the position s (for example, elastic force $F = -ks$, force of gravitational attraction $F = -G\frac{m_1m_2}{s^2}$), then the equation has a form

$$m\frac{d^2s}{dt^2} = F(s).$$

2. Physics (Series RLC Circuit)

Let us consider a simple electrical circuit consists of a resistor R, an inductor L and a capacitor C connected in series, with a constant electromotive force V.

From Kirchhoff's Law for electrical circuits the current $I = I(t)$ satisfies the differential equation

$$L\frac{dI}{dt} + RI(t) + \frac{1}{C}q(t) = E(t),$$

where $q(t)$ denotes electrical charge.

Since $I(t) = \frac{dq}{dt}$, we could write the equation in terms of q and obtain a second-order linear differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q(t) = E(t).$$

Review Questions

1. What is the n-th order differential equation?

2. What is general solution of the n-th order differential equation?

3. What is particular solution of the n-th order differential equation?

5. What is general and particular integral of the n-th order differential equation?

6. How many arbitrary constants has the solution of the second-order differential equation?

7. Formulate the Cauchy problem for the n-th order differential equation and for the second-order differential equation.

8. What is geometric meaning of the particular integral of the n-th order differential equation?

9. Formulate the Existence and Uniqueness Theorem for the solution of the Cauchy problem of the n-th order differential equation.

2.2 Higher-Order Differential Equations: Reduction of Order

Sometimes n-th order differential equations can be reduced to the lower-order (for example first order) equations followed by usage of solving methods for these types of equations. Let us consider three particular cases of reduction of order.

I. Differential equation of the form $y^{(n)} = f(x)$

Let us consider the DE

$$y^{(n)} = f(x), \tag{2.4}$$

where the right side of the equation depends only on the variable x.

Let us multiply both sides of equation by dx and integrate the equation obtained

$$\int y^{(n)}\, dx = \int f(x)\, dx.$$

Since $y^{(n)} = \left(y^{(n-1)}\right)'$, we get

$$y^{(n-1)} = \phi_1(x) + C_1,$$

where $\phi_1(x)$ is primitive of $f(x)$, and C_1 is a constant of integration. As a result, we obtain $(n-1)$-th order differential equations.

Integrating once more, we obtain

$$\int y^{(n-1)}\, dx = \int \phi_1(x)\, dx + \int C_1 dx$$

or

$$y^{(n-2)} = \phi_2(x) + C_1 x + C_2,$$

where $\phi_2(x)$ is primitive of $\phi_1(x)$, and C_1, C_2 are constants of integration.

Continuing the process of integration (n times), we finally obtain the general solution of (2.4):

$$y = \phi_n(x) + \frac{C_1}{(n-1)!}x^{n-1} + \frac{C_2}{(n-2)!}x^{n-2} + \ldots + C_{n-1}x + C_n,$$

where $C_1, C_2, \ldots, C_n$ are constants of integration.

Example 1.

Find the general solution of the equation

$$y^{(4)} = \sin 3x.$$

This is forth-order differential equation of the form (2.4), if $n = 4$. To solve such an equation we have to integrate with respect to x four times:

$$\int y^{(4)}(x)dx = \int \sin 3x\,dx \Rightarrow y''' = -\frac{1}{3}\cos 3x + C_1;$$

$$\int y'''\,dx = -\frac{1}{3}\int \cos 3x\,dx + \int C_1 dx \Rightarrow y'' = -\frac{1}{9}\sin 3x + C_1 x + C_2;$$

$$\int y''\,dx = -\frac{1}{9}\int \sin 3x\,dx + \int C_1 x dx + \int C_2 dx$$

$$\Rightarrow \quad y' = \frac{1}{27}\cos 3x + C_1\frac{x^2}{2} + C_2 x + C_3.$$

$$\int y'\,dx = \frac{1}{27}\int \cos 3x\,dx + \int C_1\frac{x^2}{2}dx + \int C_2 x dx + \int C_3 dx$$

Finally, we get the general solution of the equation

$$y' = \frac{1}{81}\sin 3x + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4.$$

II. Differential equation of the form $F\left(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}\right) = 0.$

1. Let us consider the second-order differential equation that does not contain the original function. That is

$$y'' = f(x, y') \text{ or } F(x, y', y'') = 0. \qquad (2.5)$$

It is possible to reduce the equation to the first-order DE by replacing

$$y' = p(x),$$

where $p = p(x)$ is a new unknown function.

Then

$$y'' = p'(x)$$

and the equation (2.5) is reduced to the first-order differential equation with respect to function $p(x)$ of the form

$$F(x, p, p') = 0. \qquad (2.6)$$

Suppose, the general solution of the equation (2.6) is $p = \phi(x, C_1)$. Then, returning to original function $y(x)$, we get

$$p = y'.$$

As a result, we obtain first-order differential equation with respect to function $y(x)$:

$$y' = \phi(x, C_1).$$

Integrating the equation, we get the general solution of (2.5)

$$y = \int \phi(x, C_1)\, dx + C_2.$$

Note.

Particular case of (2.5) is an equation of the form

$$y'' = f(y') \tag{2.7}$$

that does not contain the original function $y(x)$ and the independent variable x.

Using the same substitution

$$y' = p(x) \Rightarrow y'' = p'(x),$$

we obtain separable differential equation

$$p' = f(p).$$

Integrating, we find the function $p(x)$, and then the function $y(x)$.

2. Let us consider the equation

$$F\left(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}\right) = 0 \tag{2.8}$$

that does not contain the original function and its $(k-1)$ first derivatives.

The order of this equation could be reduced by k units by substitution

$$y^{(k)} = p(x).$$

Thus,

$$y^{(k+1)} = p'(x), y^{(k+2)} = p''(x), \dots, \; y^{(n)} = p^{(n-k)}(x)$$

As a result, the equation (2.8) converts into $(n-k)$-th order differential equations

$$F(x, p, p', \dots, p^{(n-k)}) = 0.$$

If the equation obtained is solvable, then we determine the function

$$y^{(k)} = p(x).$$

Integrating k times, we find the function $y(x)$.

Note.

The particular case of the equation (2.8) is

$$F\left(x, y^{(n-1)}, y^{(n)}\right) = 0 \tag{2.9}$$

that contains only two derivatives $y^{(n-1)}$ and $y^{(n)}$ of the original function.

By substitution $y^{(n-1)} = p(x),\ y^{(n)} = p'(x)$, (2.9) is reduced to the equation

$$F(x, p, p') = 0.$$

Example 2.

Solve the equation

$$y'' + \frac{2}{x}y' = x.$$

Since this second-order differential equation does not contain function y, let us reduce the order by substitution

$$y' = p(x) \Rightarrow y'' = p'(x).$$

Thus, we get the first-order differential equation with respect to $p(x)$:

$$p' + \frac{2}{x}p = x.$$

This is nonhomogeneous linear differential equation. Let us solve this equation by Bernoulli's method.

$$p = u(x)v(x) \Rightarrow p' = u'v + uv'.$$

Hence,

$$u'v + uv' + \frac{2}{x}uv = x,$$

$$u'v + u\left(v' + \frac{2}{x}v\right) = x.$$

To determine the unknown functions $u(x)$ and $v(x)$ we solve the system of equations

$$\begin{cases} v' + \frac{2}{x}v = 0, \\ u'v = x. \end{cases}$$

Integrating the first equation

$$\frac{dv}{dx} = -\frac{2}{x}v,$$

$$\int \frac{dv}{v} = -\int \frac{2dx}{x},$$

$$\ln|v| = -2\ln|x|,$$

we get

$$v(x) = \frac{1}{x^2}.$$

Then, we solve the second equation

$$u'\frac{1}{x^2} = x,$$

$$u' = x^3,$$

$$\int du = \int x^3 dx,$$

$$u(x) = \frac{x^4}{4} + C_1.$$

Thus, the function $p(x)$ is

$$p = u(x)v(x) = \left(\frac{x^4}{4} + C_1\right)\frac{1}{x^2} = \frac{x^2}{4} + \frac{C_1}{x^2}.$$

Now we recall that $p = y'$ and obtain the first-order differential equation

$$y' = \frac{x^2}{4} + \frac{C_1}{x^2}.$$

Separate the variables

$$dy = \left(\frac{x^2}{4} + \frac{C_1}{x^2}\right)dx$$

and integrate

$$\int dy = \int \left(\frac{x^2}{4} + \frac{C_1}{x^2}\right)dx.$$

Hence, we get the general solution of the original second-order differential equation

$$y = \frac{x^3}{12} - \frac{C_1}{x} + C_2.$$

Example 3.

Find the general solution of equation

$$y^{(4)}x = y'''.$$

Since this equation has a form $F(x, y^{(n-1)}, y^{(n)}) = 0$ (2.9), let us reduce the order by substitution

$$y''' = p(x).$$

Then $y^{(4)} = p'(x)$ and we rewrite the equation as follows

$$p'x = p.$$

Separate variables

$$\frac{dp}{dx}x = p,$$

$$\frac{dp}{p} = \frac{dx}{x}$$

and integrate both sides of equation

$$\int \frac{dp}{p} = \int \frac{dx}{x},$$

$$\ln|p| = \ln|x| + \ln C_1.$$

Thus,

$$p = C_1 x.$$

Replacing p by y''', we obtain

$$y''' = C_1 x.$$

This is the equation of the form (2.4). Integrating it three times, we get

$$\int y'''\,dx = \int C_1 x\,dx \Rightarrow y'' = C_1\frac{x^2}{2} + C_2,$$

$$\int y''\,dx = \int C_1\frac{x^2}{2}dx + \int C_2 dx \Rightarrow y' = C_1\frac{x^3}{6} + C_2 x + C_3,$$

$$\int y'\,dx = \int C_1\frac{x^3}{6}dx + \int C_2 x dx + \int C_3 dx \Rightarrow y = C_1\frac{x^4}{24} + C_2\frac{x^2}{2} + C_3 x + C_4,$$

and, finally,

$$y = \tilde{C}_1 x^4 + \tilde{C}_2 x^2 + C_3 x + C_4,$$

where $\tilde{C}_1 = \frac{C_1}{24}$, $\tilde{C}_2 = \frac{C_2}{2}$.

III. Differential equation of the form $F\left(y, y', y'', \ldots, y^{(n)}\right) = 0$.

1. Let us consider the second-order differential equation

$$y'' = f(y, y') \text{ or } F(y, y', y'') = 0 \tag{2.10}$$

that does not contain the independent variable explicitly.

It is possible to reduce the order of equation by substitution

$$y' = p(y),$$

where $p = p(y)$ is new unknown function. Then

$$y''(x) = p'_y(y) \cdot y'_x = p'_y p.$$

Plugging derivatives y' and y'' into the original equation (2.10), we get a new first-order differential equation with respect to $p(y)$

$$F\left(y, p, p'_y\right) = 0. \tag{2.11}$$

Solving this equation, we may determine the function $p(y) = \phi(y, C_1)$ and then, while returning to function $y(x)$ by $p = y'$, we obtain

$$y' = \phi(y, C_1).$$

This is separable differential equation with respect to function y

$$\frac{dy}{dx} = \phi(y, C_1).$$

Separating the variables

$$\frac{dy}{\phi(y, C_1)} = dx$$

and integrating the equation

$$\int \frac{dy}{\phi(y, C_1)} = \int dx,$$

we get the complete integral of the original equation (2.10)

$$\int \frac{dy}{\phi(y, C_1)} = x + C_2.$$

Note.

Particular case of (2.10) is

$$y'' = f(y) \tag{2.12}$$

that does not contain the independent variable and first derivative of y.

By the same substitution

$$y' = p(y), \quad y'' = p'_y p,$$

we convert the equation (2.12) into separable differential equation

$$p'_y p = f(y).$$

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2. The general case of the equation (2.10) is the equation of the form

$$F\left(y, y', y'', \dots, y^{(n)}\right) = 0 \tag{2.13}$$

that does not contain the independent variable explicitly.

Reduction the order of equation we provide by substitution

$$y' = p(y),$$

$$y'' = p'_y p,$$

$$y''' = \left(p'_y p\right)_x' = \left(p'_y p\right)_x' \cdot y'_x = \left(p''_{yy} p + p'_y p'_y p\right) \cdot p = p''_{yy} p^2 + \left(p'_y\right)^2 p^2, \dots$$

As a result, the equation (2.13) is converted into

$$F\left(y, p, p'_y p, p''_{yy} p^2 + \left(p'_y\right)^2 p^2, \dots\right) = 0.$$

Simplifying and solving this equation, we determine the function $p(y)$ and then find $y(x)$ from the equation

$$y' = \phi(y, C_1).$$

Example 4.

Find the complete integral of the equation

$$y'' \cot y = 2(y')^2.$$

This equation does not contain independent variable x, thus, it is the equation of the form (2.10).

Let us make a substitution

$$y' = p(y) \Rightarrow y'' = p' p.$$

Putting y' and y'' into the equation, we get the equation with respect to $p(y)$:

$$p' p \cot y = p^2,$$

$$\frac{dp}{dy} \cot y = p.$$

Separating variables

$$\int \frac{dp}{p} = \int \tan y \, dy \, ,$$

we obtain

$$\ln|p| = -\ln|\cos y| + \ln C_1,$$

$$p(y) = \frac{C_1}{\cos y}.$$

Since $p(y) = y'$, we get

$$y' = \frac{C_1}{cos\, y}.$$

Separate the variables

$$\frac{dy}{dx} = \frac{C_1}{cos\, y}$$

and integrate the equality

$$\int \cos y \, dy = C_1 \int dx \, .$$

Finally, we obtain the complete integral of the original equation

$$\sin y = C_1 x + C_2.$$

The general solution of the equation is

$$y = \arcsin(C_1 x + C_2).$$

Example 5.

Solve the Cauchy problem

$$y'' + y - 1 = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

Here we have the equation of the form (2.12), and it does not contain independent variable x. Let

$$y' = p(y), \quad y'' = p'p.$$

Then the equation can be written as

$$p'p + y - 1 = 0$$

or

$$p'p = 1 - y.$$

Separate the variables and integrate

$$\int p\,dp = \int (1-y)dy\,,$$

$$\frac{p^2}{2} = y - \frac{y^2}{2} + \frac{C_1}{2} \quad \Rightarrow \quad p^2 = 2y - y^2 + C_1.$$

Hence, the function $p(y)$ is defined as

$$p(y) = \pm\sqrt{2y - y^2 + C_1}.$$

Returning to the original variable $y(x)$ by $p(y) = y'$, we obtain differential equation:

$$y' = \pm\sqrt{2y - y^2 + C_1}.$$

Let us find constant C_1, applying the initial condition. Since $y = 1$ and $y' = 1$ when $x = 0$, we have

$$1 = \pm\sqrt{2 - 1 + C_1} \Rightarrow 1 = \pm\sqrt{1 + C_1}\,.$$

Since $y'(0) = 1 > 0$, we choose «+» sign for square root and, finally, get

$$C_1 = 0.$$

Thus, we obtain

$$y' = \pm\sqrt{2y - y^2}.$$

Separating and integrating this equation

$$\int \frac{dy}{\sqrt{2y - y^2}} = \int dx\,,$$

$$\int \frac{dy}{\sqrt{1 - (y-1)^2}} = x + C_2,$$

gives the function

$$\arcsin(y - 1) = x + C_2.$$

Determine C_2 from the initial conditions. Since $y = 1, x = 0,$ we have

$$\arcsin(1 - 1) = 0 + C_2 \Rightarrow C_2 = 0.$$

Consequently, the solution of Cauchy problem is

$$\arcsin(y - 1) = x,$$

$$y - 1 = \sin x$$

or

$$y = \sin x + 1.$$

Review Questions

1. What is the method of solving of the equation $y^{(n)} = f(x)$?
2. What is the method of solving of the equation $F(x, y', y'') = 0$?
3. What is the method of solving of the equation $F(y, y', y'') = 0$?
4. For what kind of the differential equations we use substitution $y' = p(x)$?
5. For what kind of the differential equations we use substitution $y' = p(y)$?
6. How can we reduce the order of the differential equation $y'' = f(y, y')$?
7. How can we reduce the order of the differential equation $F\left(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}\right) = 0$?
8. How can we reduce the order of the differential equation $F\left(x, y^{(k-1)}, y^{k}\right) = 0$?
9. How can we reduce the order of the differential equation $y'' = f(y)$?
10. How can we reduce the order of the differential equation $F\left(y, y', y'', \dots, y^{(n)}\right) = 0$?

Exercises 2.2

1-14. Find the general solution or integral of the differential equation:

1. $y'' = \sin 2x$.

2. $y''' = e^{-x/4}$.

3. $y'' = \ln x$.

4. $x^2 y'' = 2$.

5. $xy'' - y' = 0$.

6. $y''(e^x + 1) + y' = 0$.

7. $x^2 y'' + xy' = 1$.

8. $y'' + y' \tan 2x = \sin 2x$.

9. $2xy'y'' = (y')^2 + 1$.

10. $2yy'' = (y')^2$.

11. $y''(3y + 4) - 3(y')^2 = 0$.

12. $yy'' = y^2 y' + (y')^2$.

13. $y'' = 8/y^3$.

14. $yy'' + (y')^2 = 1$.

15-20. Find the particular solution or integral of the differential equation:

15. $y'' = \sin 3x, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 4/9$.

16. $xy'' = 1, \quad y(1) = 0, \quad y'(1) = 2$.

17. $y'' - \frac{1}{x-1} y' = x(x-1), \quad y(0) = 0, \quad y'(0) = -1$.

18. $yy'' + (y')^3 - (y')^2 = 0, \quad y(0) = 1, \quad y'(0) = 1/2$.

19. $y''(x^2 + 1) = 2xy', \quad y(1) = 1/3, \quad y'(1) = 2$.

20. $2yy'' = (y')^2 - y^2, \quad y(0) = 1, \quad y'(0) = 1$.

Answers.

2.3 Linear Higher-Order Differential Equations. Basic Concepts

I. Basic Concepts and Definitions

Definition. *The linear n-th order differential equation* is the equation of the form

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = f(x), \qquad (2.14)$$

where coefficients $a_1, \ldots, a_n$ and f are given continuous functions of variable x or constants. The function $f(x)$ is called *the right-handed member of the equation.*

Obvious, that the equation (2.14) is **linear** with respect to unknown function $y = y(x)$ and its derivatives $y', y'', \ldots, y^{(n)}$.

Definition. If $f(x) \not\equiv 0$ then the equation (2.14) is called *nonhomogeneous linear*.

If $f(x) \equiv 0$ then the equation (2.14) is called *homogeneous linear* and is written as

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0. \qquad (2.15)$$

Let us consider some basic concepts and properties of this kind of the equations using the example of the second order linear homogeneous differential equations, that is for

$$y'' + a_1 y' + a_2 y = 0, \qquad (2.16)$$

where a_1, a_2 are given continuous functions of variable x or constants.

Theorem.

Principle of Superposition

If $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are particular solutions of the equation (2.16) then function $Ay_1(x) + By_2(x)$, where A, B are real numbers, is also a solution of this equation.

Proof. Since y_1 and y_2 are solutions of the equation (2.16), then

$$y_1'' + a_1 y_1' + a_2 y_1 = 0$$

and

$$y_2'' + a_1 y_2' + a_2 y_2 = 0.$$

Putting the sum $Ay_1 + By_2$ directly into the equation, we get

$$(Ay_1 + By_2)'' + a_1(Ay_1 + By_2)' + a_2(Ay_1 + By_2) =$$
$$= A(y_1'' + a_1y_1' + a_2y_1) + B(y_2'' + a_1y_2' + a_2y_2) = A \cdot 0 + B \cdot 0 = 0.$$

Hence, $Ay_1(x) + By_2(x)$ is a solution of the equation (2.16).

Note.

The same property holds true for the equation (2.15) too.

II. Linear Independence of Functions

Consider the system of differentiable on the interval (a, b) functions $\{y_1, y_2, \dots y_n\}$.

Definition. The expression $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x)$, where $\alpha_1, \alpha_2, \dots \alpha_n$ are real numbers, is called *the linear combination of the functions* $y_1, y_2, \dots y_n$.

Definition. Functions $y_1, y_2, \dots y_n$ are called *linearly dependent* on an interval (a, b), if there exist **nonzero** constants $\alpha_1, \alpha_2, \dots \alpha_n$ such that

$$\alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n = 0$$

for all $x \in (a, b)$.

Otherwise, they are called *linearly independent*. That is $\alpha_1 y_1 + \alpha_2 y_2 + \dots$ $\dots + \alpha_n y_n = 0$ if and only if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Note.

Obvious, that two functions y_1 and $y_2$are linearly dependent if their ratio on the interval (a, b) are proportional, that is

$$\frac{y_1}{y_2} = \lambda \text{ or } y_1 = \lambda y_2, \quad \text{where } \lambda = const.$$

Hence, if $\frac{y_1}{y_2} \neq \lambda$, then y_1 and y_2 are linearly independent.

Example 1.

a) Functions e^{2x} and e^x are linearly independent on $(-\infty, +\infty)$, since the ratio $\frac{e^{2x}}{e^x} = e^x$ does not remain constant as x varies.

b) Functions $2e^x$ and e^x are linearly dependent, because the ratio $\frac{2e^x}{e^x} = 2 = const.$

General method to investigate the linear independence of the system of functions $\{y_1, y_2, \dots y_n\}$ is so-called ***Wronskian method***.

Definition. *Wronskian or determinant of Wronski* of the system of functions $\{y_1, y_2, \dots y_n\}$ is the determinant of the form

$$W(y_1, y_2, \dots y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}.$$

Theorem

If the functions $y_1, y_2, \dots y_n$ are linearly dependent then, since differentiation is a linear operation, the columns of the Wronskian are linearly dependent too and $W(y_1, y_2, \dots y_n) = 0$.

Note.

In general, the converse is not true: if Wronskian vanish, this does not imply that the functions are linearly dependent. However, the converse is true in many special cases. For example, if the functions are polynomials, trigonometrical or exponential and Wronskian vanish, then the functions are linearly dependent.

Thus, the Wronskian can be used to show that a set of differentiable functions is linearly independent on an interval by showing that $W(y_1, y_2, \dots y_n) \not\equiv 0$. However, it may vanish at some isolated points.

Example 2.

Investigate if functions $y_1 = 1,\ y_2 = x,\ y_3 = x^2$ are linear dependent.

Let us evaluate the Wronskian

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0.$$

Thus, functions $y_1 = 1,\ y_2 = x,\ y_3 = x^2$ are linear independent on $(-\infty, +\infty)$.

III. Structure of the Solution of Linear Homogeneous Differential Equation

Consider the equation of the form (2.15):

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0.$$

Let system of particular solutions of the equation (2.14) $y_1 = y_1(x)$, $y_2 = y_2(x), \ldots$, $y_n = y_n(x)$ is linear independent, that is

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} \not\equiv 0.$$

Definition. The linear independent system of n particular solutions $y_1,\ y_2, \ldots,\ y_n$ of the n-th order linear homogeneous differential equation is called *the fundamental system of solutions* on the interval (a, b).

Theorem

General solution of n-th order linear homogeneous differential equation is written in the form of linear combination

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n, \qquad (2.16)$$

where C_i $(i = 1,2, \ldots, n)$ are arbitrary constants, $y_1, y_2, \ldots, y_n$ is the fundamental system of solutions of equation.

IV. Solution of the Second-Order Linear Homogeneous Differential Equation

Let us consider differential equations

$$y'' + a_1 y' + a_2 y = 0, \qquad (2.17)$$

where a_1, a_2 are given continuous functions of variable x or constants.

Suppose, that it is known one of the particular solutions of the equation $y_1 = y_1(x)$. Another particular solution $y_2 = y_2(x)$, so that y_1 and y_2 are linearly independent, could be found by formula

$$y_2 = y_1 \int \frac{e^{-\int a_1(x)dx}}{{y_1}^2} dx. \qquad (2.18)$$

Thus, the general solution of the equation (2.17) is of the form

$$y = C_1 y_1 + C_2 y_1 \int \frac{e^{-\int a_1(x)dx}}{{y_1}^2} dx, \tag{2.19}$$

where C_1, C_2 are arbitrary constants.

Example 3.

Solve the equation

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

It is easy to verify that $y = x$ is a particular solution of the equation. Plugging the solution directly into equation, we get

$$(1 - x^2)(x)'' - 2x(x)' + 2x = -2x + 2x \equiv 0.$$

Let us find the second particular solution by formula (2.18), noting, that $a_1(x) = \frac{-2x}{1-x^2}$.

$$y_2 = x \int \frac{e^{\int \frac{2x}{1-x^2} dx}}{x^2} dx = x \int \frac{e^{-\ln|1-x^2|}}{x^2} dx = x \int \frac{1}{x^2(1-x^2)} dx =$$

$$= x \int \left(\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}\right) dx = x\left(-\frac{1}{x} + \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\right)$$

$$= -1 + \frac{1}{2x}\ln|1+x| - \frac{1}{2x}\ln|1-x|.$$

Consequently, the general solution of the equation is the linear combination of y_1 and y_2:

$$y = C_1 x + C_2 \left(\frac{1}{2x}\ln|1+x| - \frac{1}{2x}\ln|1-x| - 1\right),$$

where C_1, C_2 are arbitrary constants.

Review Questions

1. What is the form of the linear n-th order differential equation?

3. Formulate the definition of the linear dependence and independence of the system of functions.

4. What determinant is called Wronskian?

5. How could we investigate the linear independence of the system of functions by Wronskian?

6. What conclusions can we make about the functions y_1 and y_2 defined on the interval (a, b), if the Wronskian $W(y_1, y_2)$ is zero?

7. Which two particular solutions of a second-order linear homogeneous equation are called linearly independent?

8. Which of the given pairs of functions are linearly independent in for any x:

1) $y_1 = e^{3x}, y_2 = 2e^{3x}$;

2) $y_1 = \cos x\,, y_2 = \sin^2 x\,$;

3) $y_1 = x^k, y_2 = x^m \ (k \neq m)$;

4) $y_1 = 3\cos x\,, y_2 = 5\sin x$?

9. What is called a fundamental system of solutions of a linear homogeneous equation?

10. Formulate the theorem about the structure of general solution of the second order linear homogeneous differential equation *(Principle of Superposition).*

11. What is the form of particular solutions of a linear homogeneous differential equation?

Exersises 2.3

1-4. Investigate if functions $y_1,\ y_2,\ y_3$ are linear independent on (a, b):

1. $y_1 = e^x,\ y_2 = xe^x,\ y_3 = x^2e^x$ on $(-\infty, +\infty)$;

2. $y_1 = 1,\ y_2 = \arcsin 2x\,,\ y_3 = \arccos 2x$ on $(-1,1)$;

3. $y_1 = \ln(5x)\,,\ y_2 = \ln(3x),\ y_3 = \ln(9x)$ on $(0, +\infty)$;

4. $y_1 = \sqrt{x},\ y_2 = \sqrt{x+1},\ y_3 = \sqrt{x+2}$ on $[0, +\infty)$;

5-8. Solve the equation if the particular solution y_1 is known.

5. $y'' + \frac{2}{x}y' + y = 0,\ \ y_1 = \frac{\sin x}{x}$;

6. $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0,\ \ y_1 = x$;

7. $y'' - \tan x\, y' + 2y = 0,\ \ y_1 = \sin x\,$;

8. $(1 + x^2)y'' - 2xy' + 2y = 0,\ \ y_1 = x$.

Answers.

2.4 Linear Homogeneous Differential Equations with Constant Coefficients

Definition. *Linear homogeneous differential equations with constant coefficients* is the equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0, \tag{2.20}$$

where $a_1, \ldots, a_n$ are some given real numbers.

I. Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

Consider second-order linear homogeneous differential equations with constant coefficients

$$y'' + a_1 y' + a_2 y = 0, \tag{2.21}$$

where a_1 and a_2 are real numbers.

According to the previous chapter 2.3, in order to find the general solution of the equation (2.21), it is sufficient to find two linear independent particular solutions (the fundamental system of solutions).

Let us find the particular solutions. We may get some solutions analyzing the form of equation: $y'' + a_1 y' + a_2 y = 0$. Since a_1 and a_2 are real numbers, we need particular solutions as functions whose first and second derivatives are similar to the original function. One of the functions that comes back to itself after two derivatives is an exponential function.

Hence, we assume that the solutions of (2.21) have a form

$$y = e^{\lambda x},$$

where λ is some number (real or complex). Let us find λ.

Plug the solution and its derivatives into (2.21). Since

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x},$$

we get

$$\lambda^2 e^{\lambda x} + a_1 \lambda e^{\lambda x} + a_2 e^{\lambda x} = 0,$$

$$e^{\lambda x}(\lambda^2 + a_1 \lambda + a_2) = 0.$$

Whereas $e^{\lambda x} \neq 0$, we obtain

$$\lambda^2 + a_1\lambda + a_2 = 0. \tag{2.22}$$

Definition. The equation (2.22) is called *a characteristic or auxiliary equation* with respect to equation (2.21).

Equation (2.22) is a quadratic equation and so we obtain two roots: λ_1 and λ_2. Once we have these two roots we have two particular solutions of the differential equation (2.21):

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x}.$$

Let us look through the three cases of the solutions of the equation (2.22).

Case 1. The roots of the characteristic equation (2.22) are **real and distinct**:

$$\lambda_1 \neq \lambda_2 \in R.$$

Then the particular solutions are $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$. These solutions are linearly independent for $x \in (-\infty, +\infty)$ because

$$\frac{y_1}{y_2} = e^{(\lambda_1 - \lambda_2)x} \neq const,$$

or by Wronskian method

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = e^{(\lambda_1 + \lambda_2)x}(\lambda_2 - \lambda_1) \neq 0.$$

Hence, $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ is the fundamental system of solutions of (2.21).

According to the formula (2.16), the general solution of (2.21) is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{2.23}$$

Case 2. The roots of the characteristic equation (2.22) are **real and equal**:

$$\lambda_1 = \lambda_2 = \lambda = -\frac{a_1}{2} \in R.$$

Then the particular solutions are $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$.

It is easy to verify that $xe^{\lambda x}$ is a solution of (2.21):

$$(xe^{\lambda x})'' + a_1(xe^{\lambda x})' + a_2 xe^{\lambda x} = (2\lambda e^{\lambda x} + \lambda^2 xe^{\lambda x}) + a_1(e^{\lambda x} + \lambda xe^{\lambda x}) + a_2 xe^{\lambda x} =$$

$$= \left(\underbrace{2\lambda + a_1}_{=-2\cdot\frac{a_1}{2}+a_1=0}\right)e^{\lambda x} + \left(\underbrace{\lambda^2 + a_1\lambda + a_2}_{=0}\right)xe^{\lambda x} \equiv 0.$$

These two solutions are linearly independent for $x \in (-\infty, +\infty)$,since

$$\frac{y_1}{y_2} = \frac{1}{x} \neq const.$$

Thus, $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$ is the fundamental system of solutions of (2.21), and the general solution is

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}. \tag{2.24}$$

Case 3. The roots of the characteristic equation (2.22) are **complex**:

$$\lambda_1 = a + ib,\ \lambda_2 = a - ib,\ a, b \in R \text{ and } b \neq 0.$$

The particular solutions may be written in the form

$$y_1 = e^{(a+ib)x}, \qquad y_2 = e^{(a-ib)x}.$$

These are complex functions of a real argument that satisfy the differential equation (2.21). Let us rewrite the complex solutions in the form

$$y_1 = e^{(a+ib)x} = e^{ax}\cos b\,x + ie^{ax}\sin b\,x,$$

$$y_2 = e^{(a-ib)x} = e^{ax}\cos b\,x - ie^{ax}\sin b\,x.$$

Hence, we have that the particular solutions are $y_1 = e^{ax}\cos b\,x$, $y_2 = e^{ax}\sin b\,x$.

Since

$$\frac{y_1}{y_2} = \frac{e^{ax}\cos b\,x}{e^{ax}\sin b\,x} = \cot bx \neq const,$$

$y_1 = e^{ax}\cos b\,x$ and $y_2 = e^{ax}\sin b\,x$ form the fundamental system of solutions of (2.21), the general solution is

$$y = e^{ax}(C_1\cos b\,x + C_2 \sin b\,x). \tag{2.25}$$

Note.

We do not need integration for solving the linear equation (2.21).

The process of solving consists of two steps:

- finding the roots of the characteristic equation (2.22)
- applying formulas (2.23) - (2.25).

Example 1.

Find the solution of the equation

$$y'' + 5y' - 6y = 0.$$

It is the second-order linear homogeneous differential equations with constant coefficients of the form (2.21).

Let us write the corresponding characteristic equation

$$\lambda^2 + 5\lambda - 6 = 0$$

and find its roots

$$\lambda_1 = 1, \qquad \lambda_2 = -6.$$

Since the roots are real and distinct $\lambda_1 \neq \lambda_2 \in R$, we have two linear independent particular solutions:

$$y_1 = e^x \text{ and } y_2 = e^{-6x}.$$

According to (2.23), general solution of the given equation is

$$y = C_1 e^x + C_2 e^{-6x}.$$

Example 2.

Solve the equation

$$y'' + 6y' + 9y = 0.$$

Write the characteristic equation

$$\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0.$$

Thus, we have two equal real roots

$$\lambda_{1,2} = -3 \in R.$$

Then the particular solutions are

$$y_1 = e^{-3x} \text{ and } y_2 = xe^{-3x}.$$

By formula (2.24), we write the general solution of the original equation

$$y = C_1 e^{-3x} + C_2 x e^{-3x}.$$

Example 3.

Find the solution

$$y'' - 4y' + 5y = 0.$$

Let us solve the characteristic equation

$$\lambda^2 - 4\lambda + 5 = 0,$$

$$D = (-4)^2 - 4 \cdot 5 = -4 < 0 \implies \sqrt{D} = \sqrt{-4} = \pm 2i\,,$$

$$\lambda_{1,2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

Hence, we obtain complex roots, where real part is $a = 2$ and imaginary part is $b = 1$.

Then the particular solutions are

$$y_1 = e^{2x} \cos x\,, \qquad y_2 = e^{2x} \sin x.$$

According to formula (2.25), the general solution of a given equation is

$$y = e^{2x}(C_1 \cos x + C_2 \sin x).$$

Example 4.

Solve the initial value problem

$$y'' + 4y = 0, \qquad y(0) = 1, y'(0) = 4.$$

First, we write the corresponding characteristic equation

$$\lambda^2 + 4 = 0.$$

This equation has pure imaginary roots

$$\lambda_{1,2} = \pm 2i.$$

Then the linear independent particular solutions are

$$y_1(x) = \cos 2x, \quad y_2(x) = \sin 2x,$$

and the general solution of the given homogeneous equation is

$$y = C_1 \cos 2x + C_2 \sin 2x.$$

Now we find the particular solution that satisfies the given initial conditions and determine the corresponding values of C_1 and C_2.

Plug

$$y = C_1 \cos 2x + C_2 \sin 2x \text{ and } y' = -2C_1 \sin 2x + 2C_2 \cos 2x$$

into the initial conditions to get the following system of equations with respect to C_1 and C_2:

$$\begin{cases} C_1 + C_2 \cdot 0 = 1, \\ -2C_1 \cdot 0 + 2C_2 = 4; \end{cases} \Rightarrow \left[\begin{array}{l} C_1 = 1, \\ C_2 = 2. \end{array}\right.$$

Finally, the solution of Cauchy problem is

$$y = \cos 2x + 2 \sin 2x.$$

II. Higher-Order Linear Homogeneous Differential Equations with Constant Coefficients

Similar method could be applied to the higher-order linear homogeneous differential equations with constant coefficients (2.20).

Consider

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0,$$

where $a_1, a_2, \ldots\ a_n$ are real numbers.

The particular solutions of (2.20) have a form

$$y = e^{\lambda x},$$

where λ is some number (real or complex).

To find λ we use the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0. \qquad (2.26)$$

This equation is n-th order algebraic equation and it has n solutions real or complex (including repeated roots). Let denote these roots by $\lambda_1, \lambda_2, \ldots, \lambda_n$.

1. The roots of the characteristic equation (2.26) are **real and distinct**.

Then the particular linearly independent solutions are

$$e^{\lambda_1 x}, \quad e^{\lambda_2 x}, \quad \ldots, \quad e^{\lambda_n x}.$$

2. The roots of the characteristic equation (2.26) are **real and λ is a root of multiplicity $k > 1$, that is λ occurs k times in the list of roots:**

Then, the particular solution $e^{\lambda_i x}$corresponds to each distinct root λ_i and for the root λ of multiplicity $k > 1$ we have the following k particularsolutions

$$e^{\lambda x}, \quad x e^{\lambda x}, \quad x^2 e^{\lambda x}, \ldots, \quad x^{k-1} e^{\lambda x}.$$

3. The roots of the characteristic equation (2.26) are **complex**.

If $a \pm ib$ occurs only once in the list of roots then the particular solutions are

$$e^{ax} \cos b\,x \ \text{ and } e^{ax} \sin b\,x.$$

If $a \pm ib$ has a multiplicity of $k > 1$ then we obtain the following set of $2k$ particular solutions of the form

$$e^{ax} \cos b\,x, \quad x e^{ax} \cos b\,x, \quad \ldots, \quad x^{k-1} e^{ax} \cos b\,x;$$

$$e^{ax} \sin b\,x, \quad x e^{ax} \sin b\,x, \quad \ldots, \quad x^{k-1} e^{ax} \sin b\,x.$$

Thus, let us denote corresponding particular solutions by $y_1, y_2, \ldots, y_n$. It is the fundamental system of solutions of differential equation (2.20).

According to the formula (2.16), the general solution of (2.20) is

$$y = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n.$$

Example 5.

Solve the equation

$$y''' - 2y'' - 3y' = 0.$$

Here we have the third-order linear homogeneous differential equation of the form (2.20). Write the corresponding characteristic equation

$$\lambda^3 - 2\lambda^2 - 3\lambda = 0$$

and solve it

$$\lambda(\lambda^2 - 2\lambda - 3) = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 0, \\ \lambda_2 = -1, \\ \lambda_3 = 3. \end{array}\right.$$

Since all roots are real and distinct, they correspond to particular solutions

$$y_1(x) = 1, \qquad y_2(x) = e^{-x}, \qquad y_3(x) = e^{3x}.$$

Thus, the general solution of the original equation is

$$y = C_1 + C_2 e^{-x} + C_3 e^{3x}.$$

Example 6.

Find general solution of equation

$$y^{IV} + 2y''' - 2y' - y = 0.$$

It is forth-order linear homogeneous differential equation of the form (2.20).

Let us solve the corresponding characteristic equation

$$\lambda^4 + 2\lambda^3 - 2\lambda - 1 = 0.$$

It is easy to see, that $\lambda_1 = 1$ is one of the solutions. Then

$$(\lambda - 1)(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = 0 \Rightarrow (\lambda - 1)(\lambda + 1)^3 = 0.$$

Finally, we have

$$\lambda_1 = 1, \ \lambda_2 = \lambda_3 = \lambda_4 = -1.$$

Thus, we have two real roots here, $\lambda_1 = 1$ and $\lambda_{2,3,4} = -1$ which is multiplicity of 3.

The fundamental system of solutions is

$$y_1(x) = e^x, \qquad y_2(x) = e^{-x}, \qquad y_3(x) = xe^{-x}, \qquad y_4(x) = x^2e^{-x}.$$

The linear combination of these particular solutions is a general solution of the equation

$$y = C_1e^x + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x}$$

or

$$y = C_1e^x + e^{-x}(C_2 + C_3x + C_4x^2).$$

Example 7.

Solve the equation

$$y^V + y^{IV} + 2y''' + 2y'' + y' + y = 0.$$

Write the corresponding characteristic equation

$$\lambda^5 + \lambda^4 + 2\lambda^3 + 2\lambda^2 + \lambda + 1 = 0,$$

$$(\lambda + 1)(\lambda^4 + 2\lambda^2 + 1) = 0$$

$$(\lambda + 1)(\lambda^2 + 1)^2 = 0.$$

Here we have one of the solutions $\lambda_1 = -1$. Another four solution are complex each with multiplicity 2

$$\lambda_{2,3} = \pm i, \qquad \lambda_{4,5} = \pm i.$$

Hence, the particular solutions are

$$y_1(x) = e^{-x},$$

$$y_2(x) = \cos x, \quad y_3(x) = \sin x,$$

$$y_4(x) = x\cos x, \quad y_5(x) = x\sin x.$$

The linear combination of the fundamental system of solutions give us a general solution of the equation

$$y = C_1e^{-x} + C_2\cos x + C_3\sin x + C_4x\cos x + C_5x\sin x$$

or

$$y = C_1e^{-x} + (C_2 + C_4x)\cos x + (C_3 + C_5x)\sin x.$$

Review Questions

1. What is the form of the linear n-th order differential equation with constant coefficients?

2. What is the form of the second-order linear homogeneous differential equation with constant coefficients?

3. What is the form of particular solutions of a linear homogeneous differential equation with constant coefficients?

4. Can the function $y = \ln x$ be a solution to the equation $y'' + py' + qy = 0,\ p, q = const$?

5. What equation is called characteristic of second-order linear homogeneous equation with constant coefficients?

6. What is the form of general solution of second-order linear homogeneous equation with constant coefficients, if roots of the characteristic equation are:

1) real and distinct, $\lambda_1 \neq \lambda_2$;

2) real and equal, $\lambda_1 = \lambda_2$;

3) pure imaginary, $\lambda_{1,2} = \pm ib$;

4) complex, $\lambda_{1,2} = a \pm ib$?

7. Formulate an algorithm for solving a linear homogeneous differential equation with constant coefficients.

Exercises 2.4

1-16. Find general solution:

1. $y'' - y' - 12y = 0.$

2. $y'' + 7y' + 6y = 0.$

3. $y'' + 4y' + 4y = 0.$

4. $y'' - 4y' + 13y = 0.$

5. $y'' + 8y' = 0.$

6. $y'' + 25y = 0.$

7. $y''' - 2y'' - y' + 2y = 0.$

8. $y''' + 2y'' - 15y' = 0.$

9. $y''' - 8y'' + 16y' = 0.$

10. $y''' - 3y' - 2y = 0.$

11. $y''' - 3y'' + 3y' - y = 0.$

12. $y''' + 64y' = 0.$

13. $y^{IV} - y''' + y'' - y' = 0.$

14. $y^{IV} + 18y'' + 81y = 0.$

15. $y^{V} - 4y''' = 0.$

16. $y^{V} - 81y' = 0.$

17-21. Solve the initial value problem:

17. $y'' + 5y' + 6y = 0,\ \ y(0) = 1,\ \ y'(0) = -6.$

18. $y'' - 10y' + 25y = 0,\ \ y(0) = 0,\ \ y'(0) = 1.$

19. $y'' - 2y' + 10y = 0,\ \ y(\pi/6) = 0,\ \ y'(\pi/6) = e^{\pi/6}.$

20. $9y'' + y = 0,\ \ y(3\pi/2) = 2,\ \ y'(3\pi/2) = 0.$

21. $y'' + 3y' = 0,\ \ y(0) = 1,\ \ y'(0) = 2.$

Answers.

2.5 Linear Nonhomogeneous Differential Equations with Constant Coefficients. General Method

I. Basic Concepts

Definition. *The linear nonhomogeneous n-th order differential equation with constant coefficients* is the equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = f(x), \qquad (2.27)$$

where coefficients $a_1, a_2, \ldots, a_n$ are real numbers and $f(x) \neq 0$ is given continuous function of variable $x \in (a, b)$.

The function $f(x)$ is called *the right-sided member of the equation*.

Definition. The equation of the form (if $f(x) = 0$)

$$y^{(n)} + a_1 y^{(n-1)} + a_1 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0 \qquad (2.28)$$

is called *the related homogeneous or complementary equation*.

Theorem.

About the structure of the general solution of the linear nonhomogeneous differential equation

The general solution y_g of a nonhomogeneous equation (2.27) is the sum of the general solution y_h of the related homogeneous equation (2.28) and a particular solution y_p of the nonhomogeneous equation (2.27), that is

$$y_g = y_h + y_p. \qquad (2.29)$$

II. Method of Variation of Constants (Lagrange Method)

Let us consider second-order linear nonhomogeneous differential equation with constant coefficients

$$y'' + a_1 y' + a_2 y = f(x). \qquad (2.30)$$

Then the related homogeneous equation has a form

$$y'' + a_1 y' + a_2 y = 0. \tag{2.31}$$

General solution of the equation (2.30) is written by formula (2.29), where

$$y_h = C_1 y_1(x) + C_2 y_2(x) \tag{2.32}$$

is a general solution of the related homogeneous equation. It has one of the form (2.23), (2.24) or (2.25).

Let us find general solution y_g of a nonhomogeneous equation (2.30) in the form (2.32)

$$y_g = C_1(x) y_1(x) + C_2(x) y_2(x), \tag{2.33}$$

considering $C_1$and C_2 as some undetermined functions of x.

Differentiate y_g

$$y_g' = C_1' y_1 + C_2' y_2 + C_1 y_1' + C_2 y_2',$$

and choose the functions C_1 and C_2 so that

$$C_1' y_1 + C_2' y_2 = 0.$$

Then

$$y_g' = C_1 y_1' + C_2 y_2'.$$

Differentiate the expression obtained

$$y_g'' = C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2'.$$

Let us put y_g, y_g' and y_g'' into the equation (2.30).

Hence

$$y'' + a_1 y' + a_2 y =$$

$$= C_1 y_1'' + C_2 y_2'' + C_1' y_1' + C_2' y_2' + a_1(C_1 y_1' + C_2 y_2') + a_2(C_1 y_1 + C_2 y_2) =$$

$$= C_1(y_1'' + a_1 y_1' + a_2 y_1) + C_2(y_2'' + a_1 y_2' + a_2 y_2) + C_1' y_1' + C_2' y_2' = f(x).$$

Since y_1 and y_2 are the solutions of (2.31), the expressions in the first two parentheses are zero.

Thus,

$$C_1' y_1' + C_2' y_2' = f(x). \tag{2.34}$$

Finally we have, that the function (2.26) will be the solution of (2.30) provided the functions $C_1(x)$ and $C_2(x)$ satisfy the system of

$$\begin{cases} C_1'(x) y_1(x) + C_2'(x) y_2(x) = 0, \\ C_1'(x) y_1'(x) + C_2'(x) y_2'(x) = f(x). \end{cases} \tag{2.35}$$

System (2.35) is linear nonhomogeneous system of equations with respect to $C_1'(x)$ and $C_2'(x)$. Since the determinant of this system $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian for the linear independent partial solutions of (2.31) ($y_1(x)$ and $y_2(x)$), it is not equal to zero:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

Hence, the system (2.35) has the unique solution. It could be solved by Cramer's method (or any other method):

$$C_1'(x) = -\frac{y_2(x) \cdot f(x)}{W(y_1, y_2)},$$

$$C_2'(x) = \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)}.$$

Integrating, we obtain

$$C_1(x) = -\int \frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} dx + \tilde{C}_1,$$

$$C_2(x) = \int \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} dx + \tilde{C}_2,$$

where $\tilde{C}_1$and $\tilde{C}_2$ are constants of integration.

Substituiting $C_1(x)$ and $C_2(x)$ into (2.33), we get the general solution of (2.30):

$$y_g = \tilde{C}_1 y_1(x) + \tilde{C}_2 y_2(x) -$$

$$-\int \frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} dx \cdot y_1(x) + \int \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} dx \cdot y_2(x). \qquad (2.36)$$

The solution (2.36) satisfies the theorem about the structure of the general solution of the linear nonhomogeneous equation, that is

$$y_g = y_h + y_p,$$

where

$$y_h = \tilde{C}_1 y_1(x) + \tilde{C}_2 y_2(x)$$

is the general solution of the linear homogeneous equation (2.31) and

$$y_p = -\int \frac{y_2(x) \cdot f(x)}{W(y_1, y_2)} dx \cdot y_1(x) + \int \frac{y_1(x) \cdot f(x)}{W(y_1, y_2)} dx \cdot y_2(x)$$

is the particular solution of the linear nonhomogeneous equation (2.30).

Note.

Lagrange Method could be applied for solving n-th order linear nonhomogeneous differential equation (2.27). In this case, the functions

$$y_g = C_1(x)y_1(x) + C_2(x)y_2(x) + \cdots + C_n(x)y_n(x),$$

where $y_1(x)$, $y_2(x), \ldots\ y_n(x)$ are linear independent particular solutions of (2.28), and functions $C_1(x), C_2(x), \ldots C_n(x)$ are determined from the system

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) + \ldots + C_n'(x)y_n(x) = 0, \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) + \ldots + C_n'(x)y_n'(x) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1'(x)y_1^{(n-2)}(x) + C_2'(x)y_2^{(n-2)}(x) + \ldots + C_n'(x)y_n^{(n-2)}(x) = 0, \\ C_1'(x)y_1^{(n-1)}(x) + C_2'(x)y_2^{(n-1)}(x) + \ldots + C_n'(x)y_n^{(n-1)}(x) = f(x). \end{cases}$$

Example 1.

Solve the equation

$$y'' - y' = \frac{1}{1+e^x}.$$

Here we have the second-order linear nonhomogeneous differential equation.

First, let us find the general solution of corresponding homogeneous equation

$$y'' - y' = 0.$$

Solve the characteristic equation

$$\lambda^2 - \lambda = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 0, \\ \lambda_2 = 1. \end{array}\right.$$

Thus, we have two distinct real roots and the particular solutions are $y_1(x) = e^{0x} = 1$ and $y_2(x) = e^x$.

By formula (2.23), we write the general solution of the homogeneous equation

$$y_h = C_1 y_1(x) + C_2 y_2(x) = C_1 + C_2 e^x.$$

Second, let us find the general solution of nonhomogeneous equation in the form

$$y_g = C_1(x) + C_2(x)e^x, (*)$$

where $C_1(x)$ and $C_2(x)$ are unknown functions determined from the system (2.35).

Since $y_1(x) = 1$, $y_2(x) = e^x$ and $y_1'(x) = 0$, $y_2'(x) = e^x$,we write down the system with respect to functions $C_1'(x)$ and $C_2'(x)$:

$$\begin{cases} C_1'(x)\cdot 1 + C_2'(x)e^x = 0, \\ C_2'(x)e^x = \dfrac{1}{e^x+1}. \end{cases}$$

Solve the system

$$C_2'(x) = \frac{1}{e^x(e^x+1)}, \qquad C_1'(x) = -\frac{1}{e^x+1}.$$

Integrate the equalities obtained

$$C_1(x) = -\int \frac{1}{e^x+1}dx = -\int \frac{1+e^x-e^x}{e^x+1}dx = -\int \frac{1+e^x}{e^x+1}dx + \int \frac{e^x}{e^x+1}dx =$$

$$= -\int dx + \int \frac{d(e^x+1)}{e^x+1} = -x + \ln|e^x+1| + \tilde{C}_1,$$

$$C_2(x) = \int \frac{dx}{e^x(e^x+1)} = \int \frac{dx}{e^{2x}(1+e^{-x})} = \int \frac{e^{-2x}dx}{e^{-x}+1} = \int \frac{e^{-2x}-1+1}{e^{-x}+1}dx =$$

$$= \int \left(e^{-x} - 1 + \frac{e^x}{e^x(e^{-x}+1)}\right)dx = \int (e^{-x}-1)dx + \int \frac{d(e^x+1)}{e^x+1} =$$

$$= -e^{-x} - x + \ln|e^x+1| + \tilde{C}_2.$$

Plugging functions $C_1(x)$ and $C_2(x)$ into $(*)$, we obtain the general solution of the original equation

$$y_g = -x + \ln|e^x+1| + \tilde{C}_1 + \left(-e^{-x} - x + \ln|e^x+1| + \tilde{C}_2\right)e^x =$$

$$= \tilde{C}_1 + \tilde{C}_2 e^x - x + \ln(e^x+1) - 1 - xe^x + e^x\ln(e^x+1) =$$

$$= \tilde{C}_1 + \tilde{C}_2 e^x + (e^x+1)(\ln(e^x+1) - x) - 1.$$

Example 2.

Find the general solution of the equation

$$y''' + y' = \frac{1}{\cos^2 x}\ .$$

We first need the general solution for the corresponding homogeneous differential equation

$$y''' + y' = 0.$$

The characteristic equation

$$\lambda^3 + \lambda = 0$$

has three roots $\lambda_1 = 0,\ \lambda_{2,3} = \pm i$.

Write down the fundamental system of solutions

$$y_1(x) = 1, \qquad y_2(x) = \cos x, \qquad y_3(x) = \sin x$$

and the general solution for the homogeneous equation

$$y_h = C_1 + C_2 \cos x + C_3 \sin x.$$

Thus, the solution of nonhomogeneous equation has the form

$$y_g = C_1(x) + C_2(x) \cos x + C_3(x) \sin x.$$

The system with respect to $C_1'(x)$, $C_2'(x)$, $C_3'(x)$ is

$$\begin{cases} C_1'(x) \cdot 1 + C_2'(x) \cos x + C_3'(x) \sin x = 0, \\ C_1'(x) \cdot 0 - C_2'(x) \sin x + C_3'(x) \cos x = 0, \\ C_1'(x) \cdot 0 - C_2'(x) \cos x - C_3'(x) \sin x = \dfrac{1}{\cos^2 x}. \end{cases}$$

Solve the system by Cramer's rule.

The main determinant (Wronskian) is

$$\Delta = W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1.$$

Let us calculate the auxiliary determinants

$$\Delta_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \dfrac{1}{\cos^2 x} & -\cos x & -\sin x \end{vmatrix} = \frac{1}{\cos^2 x} \cdot \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \frac{1}{\cos^2 x},$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \dfrac{1}{\cos^2 x} & -\sin x \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \dfrac{1}{\cos^2 x} & -\sin x \end{vmatrix} = -\frac{1}{\cos x},$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \dfrac{1}{\cos^2 x} \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \dfrac{1}{\cos^2 x} \end{vmatrix} = -\frac{\sin x}{\cos^2 x}.$$

As a result, we have

$$C_1'(x) = \frac{1}{\cos^2 x} \quad \Rightarrow \quad C_1(x) = \int \frac{dx}{\cos^2 x} = \tan x + \tilde{C}_1;$$

$$C_2'(x) = -\frac{1}{\cos x} \quad \Rightarrow \quad C_2(x) = -\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + \tilde{C}_2;$$

$$C_3'(x) = -\frac{\sin x}{\cos^2 x} \quad \Rightarrow \quad C_3(x) = -\int \frac{\sin x}{\cos^2 x} dx = \int \frac{d(\cos x)}{\cos^2 x} = -\frac{1}{\cos x} + \tilde{C}_3.$$

Putting $C_1(x)$, $C_2(x)$, $C_3(x)$ into y_g, we obtain the general solution of the initial nonhomogeneous equation

$$y_g = \tilde{C}_1 + \tan x + \left(\tilde{C}_2 + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\right)\cos x + \left(\tilde{C}_3 - \frac{1}{\cos x}\right)\sin x =$$

$$= \tilde{C}_1 + \tilde{C}_2\cos x + \tilde{C}_3\sin x + \tan x + \cos x\cdot\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| - \tan x =$$

$$= \tilde{C}_1 + \tilde{C}_2\cos x + \tilde{C}_3\sin x + \cos x\cdot\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|.$$

Review Questions

1. What is the form of the n-th order linear nonhomogeneous differential equation with constant coefficients?
2. What is the form of the second-order linear nonhomogeneous differential equation with constant coefficients?
3. What equation is called the related homogeneous or complementary?
4. Formulate the theorem about the structure of the general solution of the linear nonhomogeneous differential equation.
5. What is the idea of the method of variation of constants (Lagrange Method) for solving second-order linear nonhomogeneous differential equation?

Exercises 2.5

1-10. Solve by Lagrange Method:

1. $y'' + y = \frac{1}{\cos x}$.

2. $y'' + y = \cot x$.

3. $y'' - y = \frac{e^x}{e^x+1}$.

4. $y'' - 2y' + y = \frac{e^x}{x}$.

5. $y'' + 9y = \frac{1}{\sin 3x}$.

6. $y'' + 2y' + y = \frac{1}{xe^x}$.

7. $y'' + 4y = \frac{1}{\sin^2 x}$.

8. $y'' + y = \frac{1}{\cos^3 x}$.

9. $y'' + y = \tan x$, $y(0) = 0$, $y'(0) = 1$.

10. $y''' + y' = \frac{1}{\cos x}$.

Answers.

2.6 Linear Nonhomogeneous Differential Equations with Constant Coefficients. Method of undetermined coefficients

I. Basic Concepts

Consider the linear nonhomogeneous differential equation with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = f(x), \qquad (2.37)$$

where the right-sided function could be written in the form

$$f(x) = e^{\alpha x}(P_n(x)\cos\beta x + Q_m(x)\sin\beta x) \qquad (2.38)$$

where α, β are real numbers, $P_n(x)$ and $Q_m(x)$ are the polynomials of degree n and m, respectively.

In this case, solving the equation by Lagrange method leads to complicated and cumbersome operations while integrating. Thus, we seek for the general solution as a sum of the general solution y_h of the related homogeneous equation and a particular solution y_p of the nonhomogeneous equation. To find a particular solution y_p of (2.37) it is better to use *the method of undetermined coefficients*.

The idea of the method is seeking a particular solution y_p in the form corresponding to the structure of the right-sided function (2.38) of the nonhomogeneous equation (2.37):

$$y_p = x^r e^{\alpha x}\left(\tilde{P}_l(x)\cos\beta x + \tilde{Q}_l(x)\sin\beta x\right), \qquad (2.39)$$

where r is the multiplicity of the root $\lambda = \alpha + \beta i$ of the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0; \qquad (2.40)$$

$\tilde{P}_l(x)$ and $\tilde{Q}_l(x)$ are polynomials of degree l, $l = max\{n, m\}$ with unknown (*undetermined*) coefficients. The coefficients of the polynomials $\tilde{P}_l(x)$ and $\tilde{Q}_l(x)$ are determined by direct substitution of the trial solution y_p into the nonhomogeneous differential equation (2.37).

Note.

1. If $f(x)$ has a form of an exponential, polynomial or trigonometric function or a combination of these functions, then it corresponds to the form (2.38).

2. The numbers α and β are the same in (2.38) and (2.39).

3. If the number $\alpha + \beta i$ (from 2.38) does not coincide with a root of the characteristic equation (2.40) then $r = 0$.

4. If the expression (2.38) contains at least one of the functions $\cos \beta x$ or $\sin \beta x$ then the expected expression y_p (2.39) includes both functions.

Let us consider some cases of applying the method of undetermined coefficients.

Case 1. $f(x) = P_n(x)$

The right-sided function (2.38) of the equation (2.37) has the form

$$f(x) = P_n(x),$$

where $P_n(x)$ is a polynomial of degree n, $\alpha = 0$ and $\beta = 0$.

Then

a) if $\alpha + \beta i = 0 \neq \lambda$ then $\lambda = 0$ is not the root of characteristic equation (2.40) and $r = 0$, hence, the particular solution of (2.37) has a form

$$y_p = \tilde{P}_n(x);$$

b) if $\alpha + \beta i = 0 = \lambda_{1,2,\dots,r}$ then $\lambda = 0$ is a root of the multiplicity r and the particular solution of (2.37) has a form

$$y_p = x^r \tilde{P}_n(x).$$

Here

$$\tilde{P}_n(x) = A_0 x^n + A_1 x^{n-1} + \cdots + A_n$$

is a polynomial of degree n, where $A_0,\ A_1, \dots,\ A_n$ are unknown coefficients.

To determine coefficients $A_0,\ A_1, \dots,\ A_n$ we have to find $y_p', y_p'', \dots, y_p^{(n)}$ and, then, put them into the equation (2.37) and compare the coefficients with the like powers of x in the right and left sides. These coefficients must be identical, that leads us to the system $(n+1)$ linear equations with respect to $A_0,\ A_1, \dots,\ A_n$.

Example 1.

Solve the equation

$$y'' + 3y' - 4y = 4x^2 + 3.$$

The solution of the equation is written in the form

$$y_g = y_h + y_p\,.$$

First, we need the general solution for the corresponding homogeneous differential equation

$$y'' + 3y' - 4y = 0.$$

The characteristic equation

$$\lambda^2 + 3\lambda - 4 = 0$$

has two distinct real roots $\lambda_1 = 1$ and $\lambda_2 = -4$.

The general solution for the homogeneous equation is

$$y_h = C_1 e^x + C_2 e^{-4x}.$$

Since the right-sided function has the form of polynomial

$$P_n(x) = 4x^2 + 3, (n = 2),$$

then $\alpha = 0$ and $\beta = 0$, that is $\alpha + \beta i = 0 \neq \lambda_{1,2}$ and $r = 0$.

Hence,

$$y_p = Ax^2 + Bx + C.$$

To determine $A,\ B,\ C$ put y_p into the original equation

$$y_p' = 2Ax + B, \qquad y_p'' = 2A.$$

Thus

$$2A + 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2 + 3,$$

$$-4Ax^2 + (6A - 4B)x + 2A + 3B - 4C = 4x^2 + 3.$$

Comparing the coefficients with the like powers of x in the right and left sides, we obtain

$$\begin{array}{c|c} x^2 & -4A = 4, \\ x^1 & 6A - 4B = 0, \\ x^0 & 2A + 3B - 4C = 3. \end{array}$$

This gives us

$$A = -1, \qquad B = -3, \qquad C = -4.$$

Consequently

$$y_p = -x^2 - 3x - 4.$$

Finally, the general solution is

$$y_g = C_1 e^x + C_2 e^{-4x} - x^2 - 3x - 4.$$

Example 2.

Find the general solution of the equation

$$y''' - y'' = x - 1.$$

Let us solve the corresponding homogeneous equation

$$y''' - y'' = 0.$$

Since the characteristic equation

$$\lambda^3 - \lambda^2 = 0$$

has the roots $\lambda_{1,2} = 0, \lambda_3 = 1$, the solution of homogeneous equation is

$$y_h = C_1 + C_2 x + C_3 e^x.$$

The right-sided function of the original equation contains only a polynomial

$$P_n(x) = x - 1, (n = 1),$$

and $\alpha + \beta i = 0 + 0i = 0 = \lambda_{1,2}, r = 2.$

Therefore, we seek a particular solution in the form

$$y_p = x^2(Ax + B) = Ax^3 + Bx^2.$$

Differentiate it

$$y_p' = 3Ax^2 + 2Bx, \qquad y_p'' = 6Ax + 2B, \qquad y_p''' = 6A.$$

Plugging into the original equation, we get

$$6A - (6Ax + 2B) = x - 1,$$

$$-6Ax + 6A + 2B = x - 1.$$

Comparing the coefficients with the like powers of$x$in the right and left sides, we obtain

$$\begin{array}{l|r} x^1 & -6A = 1, \\ x^0 & 6A + 2B = -1. \end{array}$$

Determine the coefficients:

$$A = -\frac{1}{6}, \qquad B = 1.$$

Hence, the particular solution is

$$y_p = -\frac{1}{6}x^3 + x^2.$$

Thus, the general solution is given by

$$y_g = C_1 + C_2 x + C_3 e^x - \frac{1}{6}x^3 + x^2.$$

Case 2. $f(x) = P_n(x)e^{\alpha x}$

The right-sided function (2.38) of the equation (2.37) has the form

$$f(x) = P_n(x)e^{\alpha x},$$

where $\alpha \neq 0$ is real number, $\beta = 0$ and $P_n(x)$ is a polynomial of degree n.

Then

a) if $\alpha + \beta i = \alpha \neq \lambda$ then $\lambda = \alpha$ is not the root of characteristic equation (2.40) and $r = 0$, hence, the particular solution of (2.37) has a form

$$y_p = e^{\alpha x}\tilde{P}_n(x);$$

b) if $\alpha + \beta i = \alpha = \lambda_{1,2,\dots,r}$ then $\lambda = \alpha$ is a root of the multiplicity r and the particular solution of (2.37) has a form

$$y_p = x^r e^{\alpha x}\tilde{P}_n(x).$$

Here

$$\tilde{P}_n(x) = A_0 x^n + A_1 x^{n-1} + \dots + A_n$$

is a polynomial of degree n, where $A_0, A_1, \dots, A_n$ are unknown coefficients.

Example 3.

Solve the initial value problem (Cauchy problem)

$$y'' + y' - 2y = 3xe^x, \qquad y(0) = 3, \qquad y'(0) = -\frac{1}{3}.$$

To solve the Cauchy problem we have to find general solution of the equation and then use the initial condition to find constants for particular solution.

Write the characteristic equation for corresponding homogeneous equation and find its roots

$$\lambda^2 + \lambda - 2 = 0 \Rightarrow \left[\begin{array}{l}\lambda_1 = 1,\\ \lambda_2 = -2.\end{array}\right.$$

Since roots are real and distinct, the general solution of homogeneous equation is

$$y_h = C_1 e^x + C_2 e^{-2x}.$$

The right-sided function of the original equation has a form $f(x) = 3xe^x$, where

$$P_n(x) = 3x, (n = 1),$$

and $\alpha + \beta i = 1 + 0i = 1 = \lambda_1$ (is a solution of the characteristic equation), $r = 1$.

Therefore, we seek a particular solution in the form

$$y_p = x(Ax + B)e^x = (Ax^2 + Bx)e^x.$$

Differentiating twice, we have

$$y_p' = (2Ax + B)e^x + (Ax^2 + Bx)e^x = (Ax^2 + (2A + B)x + B)e^x,$$

$$y_p'' = (2Ax + 2A + B)e^x + (Ax^2 + (2A + B)x + B)e^x =$$

$$= (Ax^2 + (4A + B)x + 2A + 2B)e^x.$$

Substitution into the original equation gives us

$$(Ax^2 + (4A + B)x + 2A + 2B)e^x + (Ax^2 + (2A + B)x + B)e^x - 2(Ax^2 + Bx)e^x =$$

$$= 3xe^x,$$

which reduces to

$$6Ax + 2A + 3B = 3x.$$

Comparing the coefficients with the like powers of x in the right and left sides, we get for $A$and B:

$$\begin{array}{c|c} x^1 & 6A = 3, \\ x^0 & 2A + 3B = 0. \end{array}$$

Therefore,

$$A = \frac{1}{2}, \quad B = -\frac{1}{3},$$

and the particular solution is

$$y_p = \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x.$$

Thus, the general solution of the given equation is

$$y_g = C_1e^x + C_2e^{-2x} + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x.$$

To solve the Cauchy problem we have to use initial conditions

$$y(0) = 3, \qquad y'(0) = -\frac{1}{3}.$$

Differentiate the general solution

$$y_g' = C_1e^x - 2C_2e^{-2x} + \left(x - \frac{1}{3}\right)e^x + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x.$$

Hence,

$$y(0) = \left(C_1e^x + C_2e^{-2x} + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x\right)\Big|_{x=0} = C_1 + C_2 = 3,$$

$$y'(0) = \left(C_1e^x - 2C_2e^{-2x} + \left(x - \frac{1}{3}\right)e^x + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x\right)\Big|_{x=0} = C_1 - 2C_2 = -\frac{1}{3}.$$

Therefore,

$$\begin{cases} C_1 + C_2 = 3, \\ C_1 - 2C_2 = -\dfrac{1}{3}, \end{cases} \Rightarrow \begin{bmatrix} C_1 = 2, \\ C_2 = 1. \end{bmatrix}$$

Finally, we get the solution of the Cauchy problem

$$y = 2e^x + e^{-2x} + \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x.$$

Case 3. $f(x) = P_n(x)\cos\beta x + Q_m(x)\sin\beta x$

The right-sided function (2.38) of the equation (2.37) has the form

$$f(x) = P_n(x)\cos\beta x + Q_m(x)\sin\beta x,$$

where $\beta \neq 0$ is real number, $\alpha = 0$ and $P_n(x)$, $Q_m(x)$ are polynomials of degree n and m respectively.

Then

a) if $\alpha + \beta i = \beta i \neq \lambda$ then $\lambda = \beta i$ is not the root of characteristic equation (2.40) and $r = 0$, hence, the particular solution of (2.37) has a form

$$y_p = \tilde{P}_l(x)\cos\beta x + \tilde{Q}_l(x)\sin\beta x\,;$$

b) if $\alpha + \beta i = \beta i = \lambda_{1,2,\dots,r}$ then $\lambda = \beta i$ is a root of the multiplicity r and the particular solution of (2.37) has a form

$$y_p = x^r\left(\tilde{P}_l(x)\cos\beta x + \tilde{Q}_l(x)\sin\beta x\right),$$

where

$$\tilde{P}_l(x) = A_0x^n + A_1x^{n-1} + \dots + A_n, \qquad \tilde{Q}_l(x) = B_0x^n + B_1x^{n-1} + \dots + B_n$$

are polynomials of degree $l = max\{n, m\}$, where A_0, $A_1, \dots,$ A_l, B_0, $B_1, \dots,$ B_l are unknown coefficients.

Example 4.

Find the general solution of equation

$$y'' + 3y' + 2y = 2\cos 3x + 4\sin 3x.$$

First, we need the general solution for the corresponding homogeneous differential equation

$$y'' + 3y' + 2y = 0.$$

The characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0$$

has two distinct real roots $\lambda_1 = -1$ and $\lambda_2 = -2$.

Then the general solution of homogeneous equation is

$$y_h = C_1 e^{-x} + C_2 e^{-2x}.$$

The right-sided function of the original equation has a form

$$f(x) = 2\cos 3x + 4\sin 3x,$$

where

$$P_n(x) = 2, (n = 0), Q_m(x) = 4, (m = 0), \text{ and } \alpha + \beta i = 3i \neq \lambda_{1,2}, r = 0.$$

Therefore, a particular solution has the form

$$y_p = A\cos 3x + B\sin 3x.$$

The derivatives for y_p have the form

$$y_p' = -3A\sin 3x + 3B\cos 3x,$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x.$$

Substitute this into the given equation

$$-9A\cos 3x - 9B\sin 3x + 3(-3A\sin 3x + 3B\cos 3x) + 2(A\cos 3x + B\sin 3x) =$$
$$= 2\cos 3x + 4\sin 3x$$

and simplify

$$(-9A + 9B + 2A)\cos 3x + (-9B - 9A + 2B)\sin 3x = 2\cos 3x + 4\sin 3x,$$

$$(-7A + 9B)\cos 3x + (-7B - 9A)\sin 3x = 2\cos 3x + 4\sin 3x.$$

Equating the coefficients of $\cos 3x$ and $\sin 3x$, we get the system for determining coefficients A and B:

$$\begin{array}{l|l} \cos 3x & -7A + 9B = 2, \\ \sin 3x & -9A - 7B = 4. \end{array}$$

whence $A = -\frac{5}{13}, B = -\frac{1}{13}$.

As a result, the particular solution is written as

$$y_p = -\frac{5}{13}\cos 3x - \frac{1}{13}\sin 3x.$$

Finally, the general solution is given by

$$y_g = C_1 e^{-x} + C_2 e^{-2x} - \frac{5}{13}\cos 3x - \frac{1}{13}\sin 3x.$$

Example 5.

Solve the equation

$$y'' + y = x \sin x.$$

Let us find the solution of corresponding homogeneous equation

$$y'' + y = 0.$$

Since the characteristic equation

$$\lambda^2 + 1 = 0$$

has complex roots

$$\lambda_{1,2} = \pm i,$$

It follows that the general solution of homogeneous equation is

$$y_h = C_1 \cos x + C_2 \sin x.$$

The right-sided function of the original equation has a form

$$f(x) = x \sin x,$$

where

$$P_n(x) = 0, (n = 0), \qquad Q_m(x) = x, (m = 1), \qquad l = max\{0,1\} = 1,$$

and $\alpha + \beta i = 0 + i = \lambda_1, r = 1$.

Hence, a particular solution has the form

$$y_p = x \cdot \big((Ax + B)\cos x + (Cx + D)\sin x\big) = (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x.$$

Differentiate it

$$y_p' = (2Ax + B + Cx^2 + Dx)\cos x + (2Cx + D - Ax^2 - Bx)\sin x,$$

$$y_p'' = (2D + 2A - Ax^2 - Bx + 4Cx)\cos x + (2C - 2B - Cx^2 - Dx - 4Ax)\sin x,$$

and plug derivatives into the original equation

$$(2D + 2A - Ax^2 - Bx + 4Cx)\cos x + (2C - 2B - Cx^2 - Dx - 4Ax)\sin x +$$
$$+(Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x = x\sin x,$$
$$(2D + 2A - Ax^2 - Bx + 4Cx + Ax^2 + Bx)\cos x +$$
$$+(2C - 2B - Cx^2 - Dx - 4Ax + Cx^2 + Dx)\sin x = x\sin x,$$
$$(2D + 2A + 4Cx)\cos x + (2C - 2B - 4Ax)\sin x = x\sin x.$$

Equating the coefficients of $x\cos x$, $x\sin x$, $\cos x$ and $\sin x$, we get the system for determining unknown coefficients:

$$\begin{array}{l|l} x\cos x & \quad 4C = 0, \\ x\sin x & \;\; -4A = 1, \\ \cos x & 2D + 2A = 0, \\ \sin x & 2C - 2B = 0. \end{array}$$

This gives us $C = 0,\ A = -\frac{1}{4},\ D = \frac{1}{4},\ B = 0.$

Consequently,

$$y_p = -\frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x.$$

Hence, the general solution is

$$y_g = C_1\cos x + C_2\sin x - \frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x.$$

Case 4. $f(x) = e^{\alpha x}(P_n(x)\cos\beta\, x + Q_m(x)\sin\beta\, x)$

The right-sided function (2.38) of the equation (2.37) has the form

$$f(x) = e^{\alpha x}(P_n(x)\cos\beta\, x + Q_m(x)\sin\beta\, x),$$

where $\alpha \neq 0$, $\beta \neq 0$ are real numbers and $P_n(x)$, $Q_m(x)$ are polynomials of degree n and m respectively.

a) if $\alpha + \beta i \neq \lambda$ then $\lambda = \alpha + \beta i$ is not the root of characteristic equation (2.40) and $r = 0$, hence, the particular solution of (2.37) has a form

$$y_p = e^{\alpha x}\left(\tilde{P}_l(x)\cos\beta x + \tilde{Q}_l(x)\sin\beta x\right);$$

b) if $\alpha + \beta i = \lambda_{1,2,\dots,r}$ then $\lambda = \alpha + \beta i$ is a root of the multiplicity r and the particular solution of (2.37) has a form

$$y_p = x^r e^{\alpha x}\left(\tilde{P}_l(x)\cos\beta x + \tilde{Q}_l(x)\sin\beta x\right),$$

where

$$\tilde{P}_l(x) = A_0x^n + A_1x^{n-1} + \cdots + A_n,$$
$$\tilde{Q}_l(x) = B_0x^n + B_1x^{n-1} + \ldots + B_n$$

are polynomials of degree $l = max\{n, m\}$, where A_0, $A_1, \ldots,$ A_l, B_0, $B_1, \ldots,$ B_l are unknown coefficients.

Example 6.

Find the solution of the equation

$$y'' - 4y' + 13y = e^x \cos x.$$

To find the solution of the corresponding homogeneous equation

$$y'' - 4y' + 13y = 0,$$

we use the characteristic equation

$$\lambda^2 - 4\lambda + 13 = 0.$$

Since $\lambda_{1,2} = 2 \pm 3i$ are complex, then

$$y_h = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

The right-sided function of the original equation has a form

$$f(x) = e^x \cos x,$$

where

$$P_n(x) = 1, (n = 0), \quad Q_m(x) = 0, (m = 0), \quad l = max\{0,0\} = 0,$$

and $\alpha + \beta i = 1 + i \neq \lambda_{1,2}, r = 0$.

Thus, a particular solution is written in the form

$$y_p = e^x(A \cos x + B \sin x).$$

Plug function y_p and its derivatives

$$y_p' = e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) =$$
$$= e^x\big((A + B) \cos x + (B - A) \sin x\big),$$
$$y_p'' = e^x\big((A + B) \cos x + (B - A) \sin x\big) + e^x(-(A + B) \sin x + (B - A) \cos x) =$$
$$= e^x(2B \cos x - 2A \sin x)$$

into the original equation

$$e^x(2B \cos x - 2A \sin x) - 4e^x\big((A + B) \cos x + (B - A) \sin x\big) +$$
$$+13e^x(A \cos x + B \sin x) = e^x \cos x,$$

which reduces to

$$(9A - 2B)\cos x + (2A + 9B)\sin x = \cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$, we get the system for determining coefficients A and B:

$$\begin{array}{r|l} \cos x & 9A - 2B = 1, \\ \sin x & 2A + 9B = 0. \end{array}$$

Thus, we get $A = \frac{9}{13}$ and $B = -\frac{2}{13}$.

The particular solution is

$$y_p = e^x\left(\frac{9}{13}\cos x - \frac{2}{13}\sin x\right).$$

Consequently, the general solution of the nonhomogeneous equation is

$$y_g = e^{2x}(C_1\cos 3x + C_2\sin 3x) + e^x\left(\frac{9}{13}\cos x - \frac{2}{13}\sin x\right).$$

II. The Superposition Principle

If the right-sided function of a linear nonhomogeneous differential equation with constant coefficients (2.37) is a sum of several functions of the form (2.38) then the partial solution of such an equation could be found by using **the superposition principle**. We formulate it for the case of the second-order differential equation.

Theorem.

Principle of Superposition

Let the function $y_{1p}(x)$ be a particular solution of the equation

$$y'' + a_1 y' + a_2 y = f_1(x)$$

and the function $y_{2p}(x)$ be a particular solution of the equation

$$y'' + a_1 y' + a_2 y = f_2(x).$$

Then the sum of the functions

$$y_p(x) = y_{1p}(x) + y_{2p}(x)$$

is a solution of the linear nonhomogeneous differential equation

$$y'' + a_1 y' + a_2 y = f_1(x) + f_2(x).$$

Proof. Let us prove the statement by direct substitution of $y_{1p} + y_{2p}$ into the equation

$$(y_{1p} + y_{2p})'' + a_1(y_{1p} + y_{2p})' + a_2(y_{1p} + y_{2p}) = f_1(x) + f_2(x).$$

Hence,

$$(y_{1p}'' + a_1 y_{1p}' + a_2 y_{1p}) + (y_{2p}'' + a_1 y_{2p}' + a_2 y_{2p}) = f_1(x) + f_2(x).$$

Since $y_{1p}'' + a_1 y_{1p}' + a_2 y_{1p} = f_1(x)$ and $y_{2p}'' + a_1 y_{2p}' + a_2 y_{2p} = f_2(x)$, it follows that we have an identity and the theorem is proved.

Example 7.

Solve the equation

$$y'' + 9y = \cosh x.$$

The right-sided function could be written as a sum of functions of the form (2.38)

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = f_1(x) + f_2(x).$$

According to the superposition principle, the general solution of the equation has a form

$$y_g = y_h + y_{1p} + y_{2p}.$$

First, we have to solve the corresponding homogeneous equation

$$y'' + 9y = 0.$$

Since the characteristic equation $\lambda^2 + 9 = 0$ has complex roots $\lambda_{1,2} = \pm 3i$, the general solution of homogeneous equation is

$$y_h = C_1 \cos 3x + C_2 \sin 3x.$$

Particular solution y_{1p}, corresponding the first addend $f_1(x) = \frac{1}{2}e^x$ of the right-sided function, has a form

$$y_{1p} = Ae^x,$$

since $P_n(x) = \frac{1}{2}$, $(n = 0)$ and $\alpha + \beta i = 1 \neq \lambda_{1,2}$, $(r = 0)$.

Let us differentiate y_{1h}

$$y_{1p}' = Ae^x, y_{1p}'' = Ae^x$$

and plug into the equation

$$y'' + 9y = \frac{1}{2}e^x.$$

Thus

$$Ae^x + 9Ae^x = \frac{1}{2}e^x,$$

which reduces to

$$10Ae^x = \frac{1}{2}e^x \Rightarrow A = \frac{1}{20}.$$

Therefore, the first particular solution is

$$y_{1p} = \frac{1}{20}e^x.$$

Second particular solution y_{2p} has a form

$$y_{2p} = Be^{-x},$$

since for $f_2(x) = \frac{1}{2}e^{-x}$ we have $P_n(x) = \frac{1}{2}, (n = 0)$ and $\alpha + \beta i = -1 \neq \lambda_{1,2}, (r = 0)$.

Differentiate y_{2p}

$$y'_{2p} = -Be^{-x}, y''_{2p} = Be^{-x}$$

and plug into the equation

$$y'' + 9y = \frac{1}{2}e^{-x}.$$

Hence,

$$Be^{-x} + 9Be^{-x} = \frac{1}{2}e^{-x},$$

and that leads us to

$$10Be^x = \frac{1}{2}e^x \Rightarrow B = \frac{1}{20}.$$

As a result, we have

$$y_{2p} = \frac{1}{20}e^{-x}.$$

Finally, we have the general solution of a given equation

$$y_g = y_h + y_{1p} + y_{2p} = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{20}e^x + \frac{1}{20}e^{-x}$$

or

$$y_g = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{10}\cosh x.$$

Review Questions

1. What form should the right-sided function have for applying the method of undetermined coefficients?

2. What is the idea of the method of undetermined coefficients for solving linear nonhomogeneous differential equation?

3. Is it possible to apply the method of undetermined coefficients for solving **any** linear nonhomogeneous differential equation?

4. Is there a relationship between the roots of the characteristic equation and the form of the particular solution of the nonhomogeneous equation? If so, what is this relationship?

5. Determine the form of the particular solution of a linear nonhomogeneous equation, if the roots of its characteristic equation and the right-sided function are known:

1) $\lambda_1 = 1,$ $\lambda_2 = 0,$ $f(x) = ax^2 + bx + c;$

2) $\lambda_1 = -1,$ $\lambda_2 = 1,$ $f(x) = e^{-x}(ax + b);$

3) $\lambda_1 = -1,$ $\lambda_2 = -1,$ $f(x) = e^{-x}(ax + b);$

4) $\lambda_1 = 2i,$ $\lambda_2 = -2i,$ $f(x) = a\sin 2x + b\cos 2x\,;$

5) $\lambda_1 = 2i,$ $\lambda_2 = -2i,$ $f(x) = e^{x}(\cos 2x + \sin 2x);$

6) $\lambda_1 = -1,$ $\lambda_2 = 0,$ $f(x) = ae^{-x} + bx + c;$

7) $\lambda_1 = 2,$ $\lambda_2 = 1,$ $f(x) = e^{2x} + e^{x};$

8) $\lambda_1 = 2 - i,$ $\lambda_2 = 2 + i,$ $f(x) = e^{2x} + \sin x.$

6. Formulate the Superposition Principle for second order linear nonhomogeneous differential equation.

7. What method is general: the method of variation of constants or the method of undetermined coefficients?

8. What are advantages of the method of variation of constants? What are the disadvantages?

9. What are advantages of the method of undetermined coefficients?

Exercises 2.6

1-10. For the given equation, write down the general solution and the particular solution (do not evaluate undetermined coefficients):

1. $y'' + 2y' + 5y = e^{-2x}(x^2 - 7x + 2)$; **2.** $y'' + 4y' + 3y = xe^{-3x}$;

3. $y'' - 8y' = x^3 - 2x$; **4.** $y'' - 10y' + 25y = e^{5x}(1 - x^2)$.

5. $y'' - 2y' + 10y = xe^x \cos 2x$. **6.** $y'' + 16y = (x^2 - 7)\sin 4x$.

7. $y'' - 6y' + 13y = e^{3x}\sin 2x$. **8.** $y'' + 10y' = x^2 + xe^{-10x}\sin x$.

9. $y'' - 36y = xe^{-6x} - e^{6x} + \sin 6x$.

10. $y'' + 9y' + 25y = e^{-3x}\cos 4x + x\sin 4x$.

11-11. Find general solution by method of undetermined coefficients:

11. $y'' + 3y' - 10y = 10x^2 + 4x - 5$. **12.** $y'' - 4y' - 5y = (27x - 39)e^{-4x}$.

13. $y'' - 4y' + 3y = 10e^{3x}$. **14.** $y'' + 4y' = -2xe^{-4x}$.

15. $y'' + 16y = (34x + 13)e^{-x}$. **16.** $y'' + 4y' + 4y = 3xe^{-2x}$.

17. $y'' + 5y' = 50\cos 5x$. **18.** $y'' - 4y' + 5y = 2\cos x + 6\sin x$.

19. $y'' + 4y = 10\cos 2x - 6\sin 2x$. **20.** $y'' - 4y = e^{2x}\sin 2x$.

21. $y'' - 2y' + 2y = e^x \sin x$. **22.** $y'' - 3y' = e^{3x} + 12x - 7$.

23-24. Find the particular solution

23. $y'' + 4y = (6x + 5)e^{-2x}$, $y(0) = 0$, $y'(0) = 3/4$.

24. $y'' + 2y' - 8y = (12x + 20)e^{2x}$, $y(0) = 0$, $y'(0) = 1$.

25. $y'' - 2y' + 10y = 74\sin 3x$, $y(0) = 6$, $y'(0) = 3$.

26. $y'' + y = -8\sin x - 6\cos x$, $y(\pi/2) = -\pi/2$, $y'(\pi/2) = -2\pi$.

27. $y'' - 4y' + 13y = e^{2x}\cos 3x$, $y(0) = 1$, $y'(0) = -1$.

Answers.

2.7 Exploring Mechanical Vibrations by Differential Equations

Let us look now through one of applications of second-order differential equations – the simplest case of so-called *mechanical vibrations*. We determine “Vibrations” as some oscillations in mechanical system. Oscillations could be periodic (the motion of a pendulum) or random (balloon swaying in the wind), desirable (the mobile-phone vibration for call) or undesirable (large amplitude oscillations of an aircraft wing).

The mechanical vibration analysis is a powerful mathematical tool for modeling and predicting potential vibration problems. It could help to modify engineering systems before they are manufactured.

We are going to explore vibrations of a load that is hanging from a spring.

I. Problem Statement

Let us consider the mass m, attached to a spring of a constant stiffness c (spring rigidity). Figure 2.1 shows of the spring without and with the mass attached to it.

We denote by l the deviation of the mass from equilibrium position (the string is stretched a distance l beyond its natural length L).

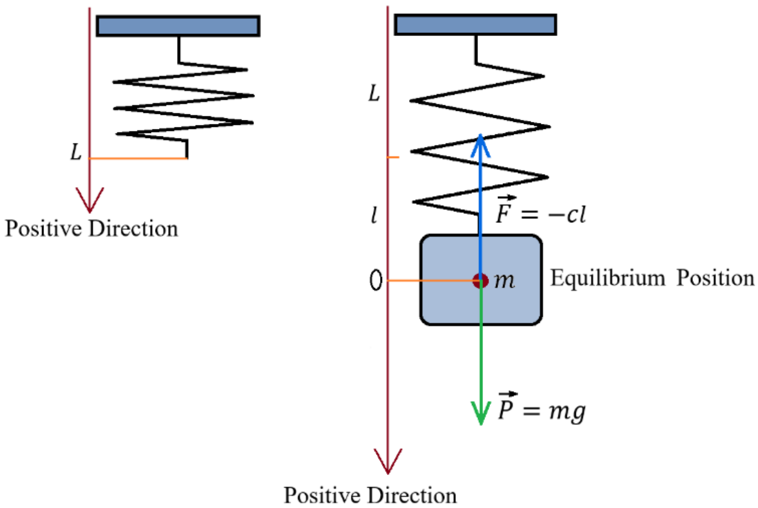

Figure 2.1

Consider the coordinate line with the origin at the equilibrium position and positive direction downward.

In the equilibrium position (Fig. 2.1), the force of the weight $\vec{P} = mg$ (g is a gravitational constant) is balanced by the restoring force of the spring, which according to Hooke's law is proportional to the length of the segment l (the mass of spring is negligible compared to m).

Thus,

$$mg = cl.$$

Let the mass be pulled down and released (Fig. 2.2). Denote by $y = y(t)$ the coordinate of the center of mass of the weigth after t seconds. The force acting on the mass is given by Newton's second law of motion

$$\vec{P} = ma,$$

where a is acceleration.

Since $a = \frac{d^2y}{dt^2}$,

$$\vec{P} = my''.$$

The force of the weight $\vec{P}$ is balanced by the restoring force of the spring

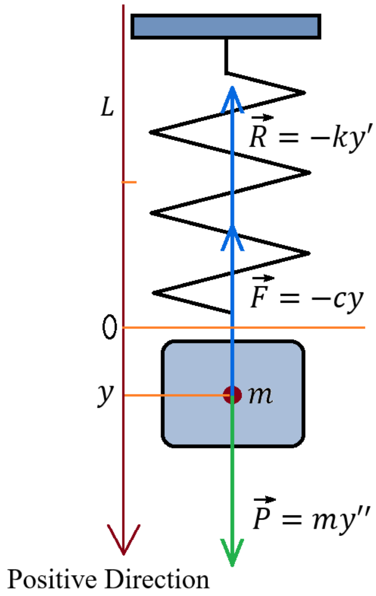

Figure 2.2

$$\vec{F} = -cy.$$

Suppose that the motion of the mass is restricted by a resistance force (the motion is **damped**) operating in a negative direction and proportional to velocity

$$\vec{R} = -ky',$$

where $k = const > 0$ (shock absorber).

If we assume that the motion is **free** (there is no alternating force or motion is applied to mechanical system) then by Newton's second law we have

$$my'' = -cy - ky',$$

or, after division by m and denoting $\frac{c}{m} = \omega^2$, $\frac{k}{m} = 2K$,

$$y'' + 2Ky' + \omega^2 y = 0.$$

If the motion is **forced** (there is some alternating force or motion is applied to mechanical system) then we have

$$y'' + 2Ky' + \omega^2 y = f(t).$$

If the mass moves in a frictionless medium (the motion is **undamped** and $\vec{R} = 0$) then $K = 0$ and the equation has a forms

$$y'' + \omega^2 y = 0$$

or

$$y'' + \omega^2 y = f(t).$$

Thus, we have a second-order linear differential equation with constant coefficients (homogeneous (free oscillations) or nonhomogeneous (forced oscillations)).

If it is known initial (when $t = 0$) displacement from the equilibrium position $y_0 = y(0)$ and initial velocity $y'_0 = y'(0)$, then we have the initial value problem (Cauchy problem) for corresponding linear differential equation.

Let us look through some examples.

II. Free, Undamped Vibrations

In this case, the differential equation becomes

$$y'' + \omega^2 y = 0.$$

Solve the characteristic equation

$$\lambda^2 + \omega^2 = 0 \implies \lambda_{1,2} = \pm\omega i,$$

and write the solution of the equation

$$y = C_1 \cos\omega t + C_2 \sin\omega t.$$

Let us make the following transformations

$$y = C_1 \cos\omega t + C_2 \sin\omega t =$$

$$= \sqrt{C_1^2 + C_2^2}\left(\frac{C_1}{\sqrt{C_1^2 + C_2^2}}\cos\omega t + \frac{C_2}{\sqrt{C_1^2 + C_2^2}}\sin\omega t\right).$$

Denote

$$\sqrt{C_1^2 + C_2^2} = A,$$

$$\frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \sin\alpha\,, \qquad \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \cos\alpha.$$

Then

$$y = A(\sin\alpha\cos\omega t + \cos\alpha\sin\omega t)$$

or

$$y = A\sin(\omega t + \alpha).$$

These oscillations are called *harmonic*. The integral curves are sine curves (Fig. 2.3).

The period of oscillations is $T = \frac{2\pi}{\omega}$.

Here

ω – *the frequency* - number of oscillations during time 2π (depends on spring rigidity and mass),

A – *the amplitude* (the greatest deviation from equilibrium),

α – *the initial phase*.

Amplitude and initial phase are expressed by C_1 and C_2, that is they are depended on initial conditions.

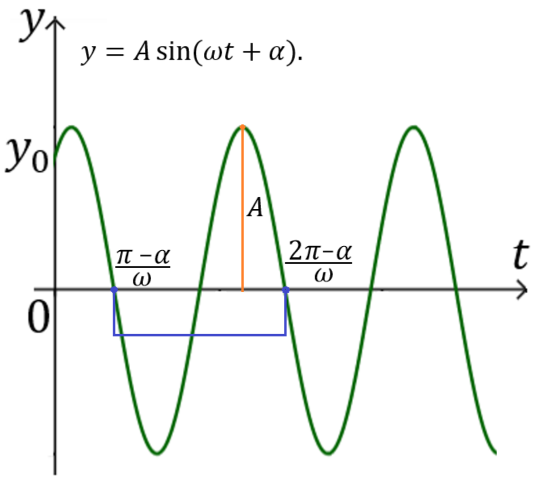

Figure 2.3

III. Free, Damped Vibrations

In this case the differential equation is

$$y'' + 2Ky' + \omega^2 y = 0.$$

Upon solving for the roots of the characteristic equation we get

$$\lambda^2 + 2K\lambda + \omega^2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -K \pm \sqrt{K^2 - \omega^2},$$

The general solution depends on K and ω.

Here we have three cases.

a) Underdamping: $K^2 < \omega^2$.

Then

$$\lambda_{1,2} = -K \pm \sqrt{K^2 - \omega^2} = -K \pm \omega_1 i$$

and

$$y = e^{-Kt}(C_1 \cos \omega_1 t + C_2 \sin \omega_1 t)$$

or

$$y = Ae^{-Kt} \sin(\omega_1 t + \alpha).$$

In this case the spring oscillates while returning to equilibrium position, that is the oscillation amplitude decreases exponentially, since $Ae^{-Kt} \to 0,\ t \to \infty$ (Fig. 2.4).

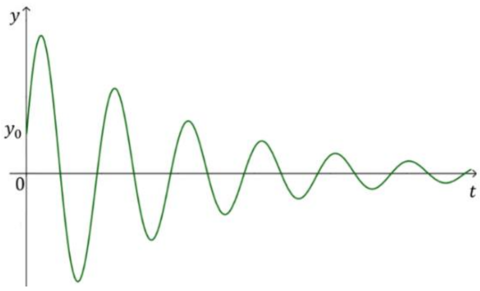

Figure 2.4

b) Overdamping: $K^2 > \omega^2$.

The roots of the characteristic equation are real and negative

$$\lambda_{1,2} = -K \pm \sqrt{K^2 - \omega^2} = -K \pm h.$$

The general solution of the differential equation has the form

$$y = C_1 e^{-(K+h)t} + C_2 e^{-(K-h)t}.$$

It follows from this formula that there are no oscillations (Fig. 2.5), the motion is aperiodical and the system returns to equilibrium relatively quickly (since $y \to 0,\ t \to \infty$).

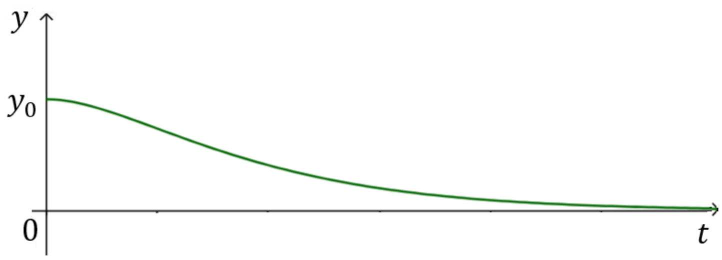

Figure 2.5

c) Critical Damping: $K^2 = \omega^2$.

In this case $\lambda_{1,2} = -K$, and

$$y = C_1 e^{-Kt} + C_2 t e^{-Kt} = (C_1 + C_2 t)e^{-Kt}.$$

The value of $y(t)$ may even increase at the beginning of the process because of the linear factor $C_1 + C_2 t$. But then $y(t)$ decreases rapidly (Fig. 2.6) due to the exponential factor e^{-Kt} and $y \to 0,\ t \to \infty$.

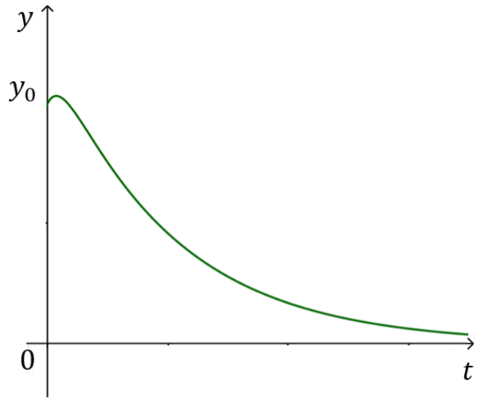

Figure 2.6

IV. Undamped, Forced Vibrations

Suppose now that an external force acts on the oscillatory system. Let us consider the case when this force varies with time according to a harmonic law with frequency q

$$f(t) = D \sin qt.$$

In the undamped case, the following differential equation is

$$y'' + \omega^2 y = H \sin qt, \quad \frac{D}{m} = H.$$

This is a linear nonhomogeneous differential equation with a right-sided function of the form (2.38). The general solution is the sum of the general solution of the homogeneous equation y_h and a particular solution of the nonhomogeneous equation y_p.

Since

$$\lambda^2 + \omega^2 = 0 \implies \lambda_{1,2} = \pm \omega i,$$

the general solution of the homogeneous equation is written as

$$y_h = A \sin(\omega t + \alpha).$$

To find the particular solution of the nonhomogeneous equation we use the method of unknown coefficients. That leads us to two different cases: when $\omega \neq q$ and when $\omega = q$.

a) if $\omega \neq q$ then

$$y_p = a \cos qt + b \sin qt.$$

Plugging into the equation and solving the system obtained, we get

$$a = 0, \qquad b = \frac{H}{\omega^2 - q^2}.$$

Hence, the general solution is

$$y = A \sin(\omega t + \alpha) + \frac{H}{\omega^2 - q^2} \sin qt.$$

The solution is a superposition of two sine waves at different frequencies. It is nice and well behaved for all time (Fig. 2.7). When ω and q are quite close to each other, the amplitude of the second wave is large and the amplitude of vibrations is also large.

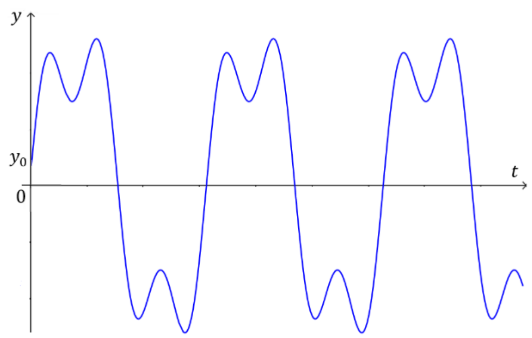

Figure 2.7

b) if $\omega = q$ then

$$y_h = t(a \cos qt + b \sin qt).$$

Plugging into the equation and solving the system obtained, we get

$$a = -\frac{H}{2\omega}, \qquad b = 0.$$

The general solution is

$$y = A \sin(\omega t + \alpha) - \frac{H}{2\omega} t \cos qt\,.$$

Here we have the solution that is a superposition of two sine waves at **equal** frequencies (the frequency of the external force coincides with the frequency of free oscillations of the system). The second term $-\frac{H}{2\omega}t\cos qt$ shows that the amplitude increases dramatically and goes to infinity as $t \to \infty$ (Fig. 2.8). This phenomenon is called **resonance**.

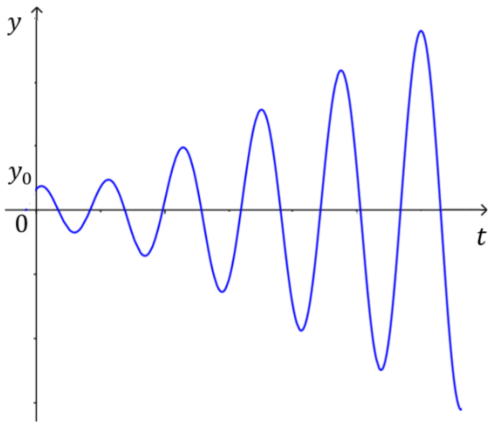

Figure 2.8

Resonance occurs often in mechanics, electronics, optics, acoustics and astrophysics.

Sometimes it is desired. For example, while projecting musical instruments or electrical resonance of turned circuits in radio that allow radio frequencies to be selectively received.

However, resonance can be destructive and dangerous phenomenon. For example, structural resonance of a suspension bridge induced by winds can lead to its catastrophic collapse (Tacoma Narrows Bridge on 7 of November 1940). Another example is a flutter of aircraft wing. Airplane wing is attached to the fuselage and has a natural structural frequency. The wind and the aerodynamic force that it generates on the wing represents the external force, which is applied with a periodic frequency. When those frequencies coincide, the system enters resonance and the amplitude of the vibration increases infinitely. If it goes on for a certain time, the wing will collapsed. You may read more about flutter in [16].

V. Forced, Damped Vibrations

Let us consider more realistic model of the forced oscillations - the damping of forced oscillations. An external force varies with time according to a harmonic law with frequency q.

In this case, the differential equation has a form

$$y'' + 2Ky' + \omega^2 y = H\sin qt.$$

Solve the characteristic equation

$$\lambda^2 + 2K\lambda + \omega^2 = 0 \implies \lambda_{1,2} = -K \pm \sqrt{K^2 - \omega^2}.$$

Consider the most common case in mechanics when $K^2 < \omega^2$, when the resistance of the medium is small.

Then $\lambda_{1,2} = -K \pm \sqrt{K^2 - \omega^2} = -K \pm \omega_1 i$ and

$$y_h = Ae^{-Kt}\sin(\omega_1 t + \alpha).$$

The particular solution has a form

$$y_p = a\cos qt + b\sin qt.$$

Plugging it into the equation, evaluating a, b and reducing the result, we obtain

$$y_p = \frac{H}{\sqrt{(\omega^2 - q^2)^2 + 4K^2q^2}}\sin(qt + \delta),$$

where $\tan\delta = \frac{-2pq}{\omega^2 - q^2}$.

Consequently, the general solution is

$$y = Ae^{-Kt}\sin(\omega t + \alpha) + \frac{H}{\sqrt{(\omega^2 - q^2)^2 + 4K^2q^2}}\sin(qt + \delta).$$

The resulting oscillation consists of two damped oscillations at different frequencies ω and q.

a) If $q \neq \omega$ (and they are very different) and $\frac{2K}{\omega}$ is quite large, then we have nonresonant case (Fig. 2.9).

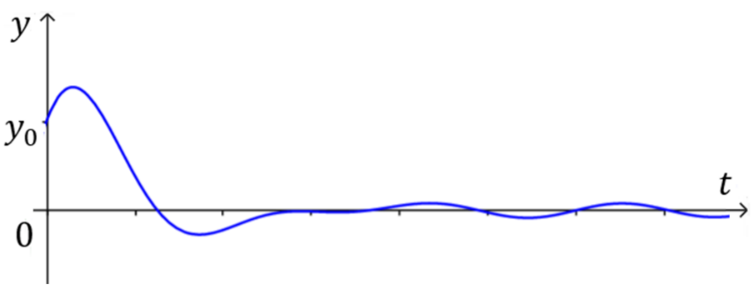

Figure 2.9

b) If $q = q^*$ and q^* is very close to ω, then, for small values of $\frac{2K}{\omega}$, coefficient $B = \frac{H}{\sqrt{(\omega^2 - q^2)^2 + 4K^2q^2}}$ acquires its maximum value. In this case, there is a dramatically increasing of the amplitude of the forced oscillations, which leads to the phenomenon of resonance (Fig. 2.10).

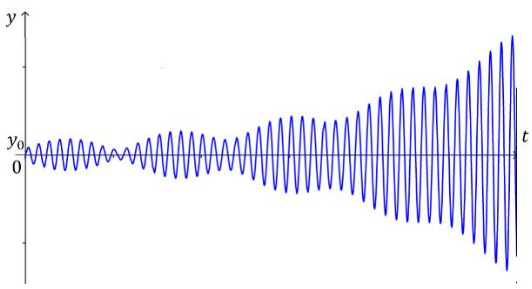

Figure 2.10.

You may read more about mechanical oscillations in [1,17,14].

3. Systems of Ordinary Differential Equations

3.1 Basic Concepts and Definitions

I. Basic Concepts

Solving some problems, we do not have single dependent variable and single differential equation. Sometimes it is required to find several functions. That leads to system of several equations and several dependent variables.

Definition. The set of differential equations each of which contains an independent variable x, the unknown functions $y_1, y_2, \dots, y_n$ and their derivatives, that is

$$F_i\left(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n', \dots, y_1^{(n)}, y_2^{(n)}, \dots, y_n^{(n)}\right) = 0 \tag{3.1}$$

where $i = 1,2, \dots, n$, is called *the system of n-order differential equations*.

Definition. *The system of first-order differential equations* that contains n unknown functions $y_1, y_2, \dots, y_n$ has a form

$$\begin{cases} F_1(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') = 0, \\ F_2(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x, y_1, y_2, \dots, y_n, y_1', y_2', \dots, y_n') = 0. \end{cases} \tag{3.2}$$

Solving system (3.2) with respect to derivatives of unknown functions, we get

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n), \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n). \end{cases} \tag{3.3}$$

The system of this form is called *normal*. Note that the number of equations is equal to the number of required functions.

Definition. *The solution* of the system (3.3) is a set of n functions $y_1, y_2, \dots, y_n$, which satisfy **each** equation of the system.

Definition. *Initial conditions* for the system (3.3) has a form:

$$y_1(x_0) = y_1^0, y_2(x_0) = y_2^0, \ \dots \ y_n(x_0) = y_n^0. \tag{3.4}$$

Definition. *Initial value problem (Cauchy problem)* for the system (3.3) consists of the system (3.3) and initial conditions (3.4). To solve Cauchy problem means to determine the functions $y_1, y_2, \dots, y_n$, which satisfy the system and given initial conditions.

Theorem.

About Existence and Uniqueness of the Solution

If in the system of differential equations (3.3)

$$\begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2, \dots, y_n), \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2, \dots, y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dy_n}{dx} = f_n(x, y_1, y_2, \dots, y_n), \end{cases}$$

each of functions $f_i(x, y_1, y_2, \dots, y_n)$ $(i = 1,2, \dots, n)$ and its partial derivative with respect to y_i are continuous in some region $D \subset R^{n+1}$ containing some point $M_0(x_0, y_1^0, y_2^0, \dots, y_n^0)$, then there is only one solution to this system $y_1 = \phi_1(x), y_2 = \phi_2(x), \dots, y_n = \phi_n(x)$ which satisfies the initial condition

$$y_1(x_0) = y_1^0, y_2(x_0) = y_2^0, \ \dots \ y_n(x_0) = y_n^0.$$

Definition. The set of n functions $y_1 = \phi_1(x, C_1, \dots, C_n)$, $y_2 = \phi_2(x, C_1, \dots, C_n)$, … , $y_n = \phi_n(x, C_1, \dots, C_n)$ is called *the general solution of the system* (3.3) if these functions satisfy each equation of the system and constants $C_1, \dots, C_n$ are uniquely determined from the condition (3.4)

$$\begin{cases} \phi_1(x_0, C_1, C_2, \dots, C_n) = y_1^0, \\ \phi_2(x_0, C_1, C_2, \dots, C_n) = y_2^0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \phi_n(x_0, C_1, C_2, \dots, C_n) = y_n^0. \end{cases}$$

Definition. The solution obtained from general solution for certain values of constants $C_1, C_2, \dots, C_n$ is called *the particular solution of the system*.

II. Some Applications of the Systems of Differential Equations

1. Biology (Simple Case of Competition of Species)

Let us consider populations of two species in terms of competition. Assume that the growth rate per individual of each population is reduced by an amount proportional to the other population. If positive functions $x(t)$ and $y(t)$ present the amount of populations then we have the system of linear differential equations of the form

$$\begin{cases} x'(t) = ax - \alpha y, \\ y'(t) = -\beta x + by, \end{cases}$$

where $a,\ \alpha,\ \beta,\ b$ are positive costants. Here $a,\ b$ represent the exponential or Malthusian growth at low densities, and $\alpha,\ \beta$ are coefficients of reduction of corresponding population.

2. Biology (Predator-prey System)

Let us consider populations of two species in terms of predator–prey interactions (for example, foxes and rabbits).

Assume that the amount of population of prey is described by positive function $x(t)$, the rate of change $x'(t)$ is proportional to its size and that species always finds enough food. Meanwhile, the positive function $y(t)$ presents the population of predators, the rate of change $y'(t)$ is also proportional to its size, but their food supply depends on the size of the prey population. To simplify the system we do not consider environmental changes, genetic adaptation etc.

Under these assumptions, we get the following system of differential equations

$$\begin{cases} x'(t) = ax - \alpha xy, \\ y'(t) = -by + \beta xy, \end{cases}$$

where $a,\ \alpha,\ \beta,\ b$ are positive costants.

Here a represents the reproductive exponential growth of prey population, parameter α shows how often predators meet prey (that leads to killing the prey). Number b is a loss rate predators (natural death or emigration), β represents the growth of the population the rate at which predators consume prey (that leads to growth of predators population).

The system obtained is called the Lotka-Volterra equations (predator-prey equations). This is the system of the first-order nonlinear differential equations. The Lotka-Volterra equations were developed in the beginning of 20th century. They describe not only the dynamics of biological systems, but are used in the theory of autocatalytic chemical reactions, economic theory etc.

3. Physics (Series RL Circuit)

Let us consider two electrical circuits (Fig. 3.1). The first one consists of a resistor R_1 an inductor L_2 connected in seriesand AC voltage source $V(t)$. The second one consists of a resistor R_2 an inductor L_2. These circuits interact due to mutual induction M.

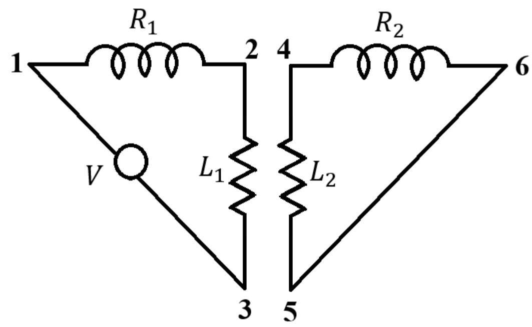

Figure 3.1

Let $I_1(t)$ and $I_2(t)$ are currents of the left and rightelectrical circuits respectively and u_{km} is a voltage drop on the segment between points k and m. According to the Kirchhoff's Law for electrical circuits (the directed sum of the voltages around a circuit is zero) we get

$$u_{12} + u_{23} + u_{31} = 0, \;\; u_{45} + u_{56} + u_{64} = 0.$$

Considering, that

$$u_{12} = R_1 I_1, \quad u_{23} = L_1 I'_1 + M I'_2, \;\; u_{31} = -V(t),$$
$$u_{45} = L_2 I'_2 + M I'_1, \qquad u_{56} = R_2 I_2, \qquad u_{64} = 0,$$

we obtain the system of linear differential equations

$$\begin{cases} L_1 I'_1 + M I'_2 + R_1 I_1 = V(t), \\ L_2 I'_2 + M I'_1 + R_2 I_2 = 0. \end{cases}$$

Review Questions

1. What is the system of n-order differential equations?

2. What is the general (particular) solution of the system of first-order differential equations?

3. What system of differential equations is called normal?

4. Formulate the Cauchy problem for the system of first-order differential equations.

5. Formulate the Existence and Uniqueness Theorem for the solution of the Cauchy problem of the system of first-order differential equations.

3.2 Systems of Linear Differential Equations

Definition. A normal system of differential equations is called *linear* if each equation of the system is linear with respect to the required function and their derivatives

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \ldots + a_{1n}(x)y_n + f_1(x), \\ y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \ldots + a_{2n}(x)y_n + f_2(x), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \ldots + a_{nn}(x)y_n + f_n(x), \end{cases} \tag{3.5}$$

where functions $a_{ij}(x)$ and $f_i(x)(i, j = 1,2,\ldots,n)$ are determined and continuous for $x \in (a, b)$.

If a_{ij} are constants then (3.5) is called *the linear system with constant coefficients*.

Definition. The linear system is called homogeneous if for all $x \in (a, b)$ functions $f_i(x)$ are identically equal zero

$$\begin{cases} y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + \ldots + a_{1n}(x)y_n, \\ y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + \ldots + a_{2n}(x)y_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n' = a_{n1}(x)y_1 + a_{n2}(x)y_2 + \ldots + a_{nn}(x)y_n. \end{cases} \tag{3.6}$$

Definition. System of functions $y_{1k}(x), y_{2k}(x), \ldots, y_{nk}(x)$ $(k = 1,2,\ldots,n)$ is called fundamental if Wronskian

$$W(y_{ik}) = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

is not equal zero at least in one point from interval (a, b).

Theorem.

General solution $\bar{y}_k$ of the homogeneous linear system (3.6) in the interval (a, b) is a linear combination

$$\bar{y}_k = C_1 y_{1k} + C_2 y_{2k} + \ldots + C_n y_{nk} = \sum_{i=1}^{n} C_i y_{ik}, \quad (k = 1,2,\ldots,n) \tag{3.7}$$

of the n particular linear independent solutions $y_{1k}, y_{2k}, \ldots, y_{nk}$ (fundamental system of solutions).

Theorem.

General solution y_k of the homogeneous linear system (3.5) in interval (a, b) is a sum of general solution $\bar{y}_k$ of the corresponding homogeneous linear system (3.6) and any particular solution y_k^* of the system (3.5):

$$y_k = \bar{y}_k + y_k^* = \sum_{i=1}^{n} C_i y_{ik} + y_k^*, \quad (k = 1,2,\ldots,n). \tag{3.8}$$

You may read more about linear systems in [1,11,13,17].

Review Questions

1. What system of differential equations is called linear?

2. What system of differential equations is called homogeneous (nonhomogeneous)?

3. Which of the following systems is homogeneous (nonhomogeneous)?

1) $\begin{cases} x'(t) = 3x + 2y, \\ y'(t) = 2x - y, \end{cases}$ 2) $\begin{cases} x'(t) = y, \\ y'(t) = -x, \end{cases}$

3) $\begin{cases} x'(t) = 2x - y + 3, \\ y'(t) = x + 3y - 1, \end{cases}$ 4) $\begin{cases} x'(t) = 4x - 5y + t, \\ y'(t) = -x + y. \end{cases}$

4. What form has the linear system of differential equations with constant coefficients?

5. What is called the fundamental system of solutions of a linear homogeneous system of differential equations?

6. Formulate the theorem about the structure of the general solution of a linear homogeneous system of differential equations.

7. Formulate the theorem about the structure of the general solution of a linear nonhomogeneous system of differential equations.

3.3 Systems of Linear Differential Equations with Constant Coefficients

There are several methods of solving homogeneous systems of equations with constant coefficients. The following methods are the most commonly used:

- elimination method;
- method of eigenvalues and eigenvectors.

I. Elimination Method

Let us consider first the elimination method. The main idea of this method is the reduction of n equations to a single linear equation of the n-th order. This method is useful for simple systems, especially for systems of the second order.

Consider the homogeneous system of two first-order differential equations with constant coefficients:

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + a_2 y, \\ \dfrac{dy}{dt} = b_1 x + b_2 y, \end{cases} \quad \text{or} \quad \begin{cases} x'_t = a_1 x + a_2 y, \\ y'_t = b_1 x + b_2 y, \end{cases} \tag{3.9}$$

where $x = x(t)$, $y = y(t)$ are differentiable functions of independent variable t, coefficients a_i and b_i are constants (in general case, a_i and b_i are functions of t).

1. Differentiate the first equation of the system with respect to t:

$$x''_{tt} = a_1 x'_t + a_2 y'_t.$$

2. Plug the expression for y'_t (from the second equation) into the equation obtained:

$$x''_{tt} = a_1 x'_t + a_2 (b_1 x + b_2 y).$$

3. Find the expression for y from the first equation

$$y = \frac{1}{a_2}(x'_t - a_1 x)$$

and put it into the equation obtained

$$x''_{tt} = a_1 x'_t + a_2 \left(b_1 x + b_2 \frac{1}{a_2}(x'_t - a_1 x) \right).$$

Simplify the expression

$$x''_{tt} = a_1 x'_t + a_2 b_1 x + b_2 x'_t - b_2 a_1 x,$$

$$x''_{tt} = (a_1 + b_2) x'_t + (a_2 b_1 - b_2 a_1) x,$$

$$x''_{tt} - (a_1 + b_2) x'_t - (a_2 b_1 - b_2 a_1) x = 0,$$

or

$$x''_{tt} + A x'_t + B x = 0,$$

where $A = -(a_1 + b_2)$ and $B = -(a_2 b_1 - b_2 a_1)$.

4. The equation obtained is the second-order linear differential equation with constant coefficients. Solve the equation and find the general solution of the equation with respect to $x(t)$:

$$x(t) = \phi(t, C_1, C_2).$$

5. Plug function $x(t)$ and its derivative $x'(t)$ into

$$y = \frac{1}{a_2}(x'_t - a_1 x)$$

and find the general solution with respect to $y(t)$

$$y(t) = \psi(t, C_1, C_2).$$

6. The set of functions

$$x(t) = \phi(t, C_1, C_2), \quad y(t) = \psi(t, C_1, C_2)$$

give the general solution of given linear system.

Example 1.

Find the general solution of linear system

$$\begin{cases} x'_t = 5x + 4y, \\ y'_t = 4x + 5y. \end{cases}$$

Here we have the homogeneous system with respect to unknown functions $x(t)$ and $y(t)$. Let us apply the elimination method.

Differentiate the first equation of the system with respect to t:

$$x''_{tt} = 5x'_t + 4y'_t.$$

Plug the expression for y'_t (from the second equation $y'_t = 4x + 5y$) into the equation obtained:

$$x''_{tt} = 5x'_t + 4(4x + 5y),$$

$$x''_{tt} = 5x'_t + 16x + 20y.$$

Find the expression for y from the first equation

$$y = \frac{1}{4}(x'_t - 5x)$$

and put it into the equation obtained

$$x''_{tt} = 5x'_t + 16x + 20 \cdot \frac{1}{4}(x'_t - 5x).$$

Thus,

$$x''_{tt} - 10x'_t + 9x = 0.$$

Here we have the second order linear homogeneous differential equation of the form. Write the corresponding characteristic equation

$$\lambda^2 - 10\lambda + 9 = 0$$

and solve it

$$\left[\begin{array}{l}\lambda_1 = 1, \\ \lambda_2 = 9.\end{array}\right.$$

Since all roots are real and distinct, then the partial solutions are

$$x_1(t) = e^x \text{ and } x_2(t) = e^{9x}.$$

Thus, the general solution of the original equation is

$$x(t) = C_1 e^x + C_2 e^{9x}.$$

Let us find the derivative of $x(t)$

$$x'(t) = C_1 e^x + 9C_2 e^{9x}.$$

Put $x(t)$ and $x'(t)$ into $y = \frac{1}{4}(x'_t - 5x)$ and find

$$y(t) = \frac{1}{4}\left(C_1 e^x + 9C_2 e^{9x} - 5(C_1 e^x + C_2 e^{9x})\right) =$$

$$= \frac{1}{4}(C_1 e^x + 9C_2 e^{9x} - 5C_1 e^x - 5C_2 e^{9x}) = -C_1 e^x + C_2 e^{9x}.$$

Then the general solution of given linear system is

$$x(t) = C_1 e^x + C_2 e^{9x}, \quad y(t) = -C_1 e^x + C_2 e^{9x}.$$

It could be written in the vector form

$$\begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = C_1 \begin{Bmatrix} e^x \\ -e^x \end{Bmatrix} + C_2 \begin{Bmatrix} e^{9x} \\ e^{9x} \end{Bmatrix},$$

where $\begin{Bmatrix} x_1(t) \\ y_1(t) \end{Bmatrix} = \begin{Bmatrix} e^x \\ -e^x \end{Bmatrix}$ and $\begin{Bmatrix} x_2(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} e^{9x} \\ e^{9x} \end{Bmatrix}$ are two linear independent solutions formed the fundamental system of solutions.

Example 2.

Integrate the system

$$\begin{cases} \dfrac{dy}{dx} + 2y - 4z = 0, \\ \dfrac{dz}{dx} + y - 3z = 3x, \end{cases}$$

with the initial conditions

$$y(0) = 0, \;\; z(0) = 0.$$

First, we have to solve the system.

The given system is nonhomogeneous with respect to unknown function $y(x)$ and $z(x)$. Let us apply the elimination method.

Differentiate the first equation with respect to x:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4\frac{dz}{dx} = 0.$$

Put the expression for $\frac{dz}{dx}$ from the second equation of system into the equation obtained

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4(3x - y + 3z) = 0,$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 12x + 4y - 12z = 0.$$

Substituting $z = \frac{1}{4}\left(\frac{dy}{dx} + 2y\right)$ we get

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 12x + 4y - 12 \cdot \frac{1}{4}\left(\frac{dy}{dx} + 2y\right) = 0.$$

Thus, we obtain second order nonhomogeneous linear differential equation with constant coefficients with respect to $y(x)$:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12x + 3.$$

Let us find the general solution for the corresponding homogeneous differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

The characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

has two distinct real roots $\lambda_1 = 2$ and $\lambda_2 = -1$.

Hence, the general solution for the homogeneous equation is

$$y_h = C_1 e^{2x} + C_2 e^{-x}.$$

Since the right-sided function has the form of polynomial $P_n(x) = 12x + 3, (n = 1)$, then $\alpha = 0$ and $\beta = 0$, that is $\alpha + \beta i = 0 \neq \lambda_{1,2}$ and $r = 0$.

Hence,

$$y_p = Ax + B.$$

To determine $A,\ B$ put y_p into the original equation

$$-A - 2(Ax + B) = 12x + 3,$$

$$-2Ax - A - 2B = 12x + 3.$$

Comparing the coefficients with the like powers of x in the right and left sides, we obtain

$$\begin{array}{c|l} x & -2A = 12, \\ x^0 & -A - 2B = 3. \end{array}$$

This gives us $A = -6,\ B = \frac{3}{2}$.

Consequently

$$y_p = -6x + \frac{3}{2}.$$

Finally, the general solution is

$$y(x) = y_h + y_p = C_1 e^{2x} + C_2 e^{-x} - 6x + \frac{3}{2}.$$

Now, let us find $z(x)$.

Plug $y(x)$ and its derivative $\frac{dy}{dx} = 2C_1 e^{2x} - C_2 e^{-x} - 6$ into $z = \frac{1}{4}\left(\frac{dy}{dx} + 2y\right)$:

$$z(x) = \frac{1}{4}\left(2C_1 e^{2x} - C_2 e^{-x} - 6 + 2\left(C_1 e^{2x} + C_2 e^{-x} - 6x + \frac{3}{2}\right)\right) =$$

$$= \frac{1}{4}(4C_1 e^{2x} + C_2 e^{-x} - 12x - 3).$$

Hence,

$$z(x) = C_1 e^{2x} + \frac{1}{4} C_2 e^{-x} - 3x - \frac{3}{4}.$$

Finally, the general solution of the given system is

$$y(x) = C_1 e^{2x} + C_2 e^{-x} - 6x + \frac{3}{2},$$

$$z(x) = C_1 e^{2x} + \frac{1}{4} C_2 e^{-x} - 3x - \frac{3}{4}.$$

Let us choose the costants C_1 and C_2 so that the initial conditions are satisfied

$$y(0) = 0, \qquad z(0) = 0.$$

Putting $y(x)$ and $z(x)$ into the conditions, we get

$$\begin{cases} 0 = C_1 + C_2 + \frac{3}{2}, \\ 0 = C_1 + \frac{1}{4} C_2 - \frac{3}{4}, \end{cases}$$

whence

$$\begin{cases} C_1 = \frac{3}{2}, \\ C_2 = -3. \end{cases}$$

Thus, the solution that satisfies the given initial conditions has the form

$$y(x) = \frac{3}{2} e^{2x} - 3e^{-x} - 6x + \frac{3}{2},$$

$$z(x) = \frac{3}{2} e^{2x} - \frac{3}{4} C_2 e^{-x} - 3x - \frac{3}{4}.$$

Note.

1. The elimination method can be applied to both homogeneous and nonhomogeneous systems. In the first case, the system is reduced to a linear homogeneous equation, and in the second - usually to a linear nonhomogeneous equation. Moreover, if all the coefficients a_{ij} of the linear system are constants, then the equations obtained are with constant coefficients.

2. The elimination method can be applied to normal systems with more equations and unknown functions, but its realization is complicated. Therefore, there are other methods for solving such systems.

II. Method of eigenvalues and eigenvectors

Let us consider the method for the linear homogeneous differential equations with constant coefficients with respect to three unknown functions $x(t), y(t), z(t)$:

$$\begin{cases} x'(t) = a_{11}x + a_{12}y + a_{13}z, \\ y'(t) = a_{21}x + a_{22}y + a_{23}z, \\ z'(t) = a_{31}x + a_{32}y + a_{33}z, \end{cases} \tag{3.10}$$

where coefficients a_{ij} $(i, j = 1,2,3)$ are real numbers.

This system could be written in the matrix form (matrix differential equation)

$$X'(t) = A \cdot X(t),$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a matrix of coefficients of the system, $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ is a column of unknown functions, $X'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix}$ is a column of derivatives of unknown functions.

Let us seek the particular solution of the system (3.10) in the following form

$$x(t) = \alpha e^{\lambda t}, \quad y(t) = \beta e^{\lambda t}, \quad z(t) = \gamma e^{\lambda t}, \tag{3.11}$$

where $\alpha, \beta, \gamma, \lambda$ are constants that is required to determine in such a way that the functions (3.11) should satisfy the system (3.10).

Plugging these functions into (3.10), we get

$$\begin{cases} \alpha\lambda e^{\lambda t} = a_{11}\alpha e^{\lambda t} + a_{12}\beta e^{\lambda t} + a_{13}\gamma e^{\lambda t}, \\ \beta\lambda e^{\lambda t} = a_{21}\alpha e^{\lambda t} + a_{22}\beta e^{\lambda t} + a_{23}\gamma e^{\lambda t}, \\ \gamma\lambda e^{\lambda t} = a_{31}\alpha e^{\lambda t} + a_{32}\beta e^{\lambda t} + a_{33}\gamma e^{\lambda t}, \end{cases}$$

and, canceling out the factor $e^{\lambda t} \neq 0$, obtain

$$\begin{cases} \alpha\lambda = a_{11}\alpha + a_{12}\beta + a_{13}\gamma, \\ \beta\lambda = a_{21}\alpha + a_{22}\beta + a_{23}\gamma, \\ \gamma\lambda = a_{31}\alpha + a_{32}\beta + a_{33}\gamma, \end{cases}$$

or, in matrix form,

$$A \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Definition. Vector $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ that satisfies this equation is called *an eigenvector* of the matrix A, and the number λ is called *an eigenvalue*.

Collecting coefficients of α, β, γ, we get the system

$$\begin{cases}(a_{11}-\lambda)\alpha + a_{12}\beta + a_{13}\gamma = 0,\\ a_{21}\alpha + (a_{22}-\lambda)\beta + a_{23}\gamma = 0,\\ a_{31}\alpha + a_{32}\beta + (a_{33}-\lambda)\gamma = 0.\end{cases} \qquad (3.12)$$

This is a homogeneous system of linear algebraic equations in α, β, γ. This system has a non-trivial solution, if and only if the determinant of the system is zero:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0 \qquad (3.13)$$

Definition. Equation (3.13) is called *the auxiliary or characteristic equation* of the system (3.10) (of a matrix A).

It could be written in the form

$$|\,A - \lambda E\,| = 0, \qquad (3.14)$$

where A is a matrix of coefficients of the system (3.10), $E = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ is a unit matrix.

Solving determinant (3.13), we get the algebraic equation of third degree for λ. It has three roots $\lambda_1, \lambda_2, \lambda_3$ (*eigenvalue or characteristic numbers of A*). Substituting each eigenvalue λ_i into the system (3.12) and solving it, we find corresponding the eigenvector $\{\alpha_i, \beta_i, \gamma_i\}$ (this system of equations will have an infinite set of solutions, thus, eigenvectors can be determined only to within a constant factor).

Let us consider the possible cases.

Case 1. The roots of the characteristic equation are real and distinct:

$$\lambda_1,\ \lambda_2,\ \lambda_3 \in R.$$

For each root λ_i $(i = 1, 2, 3)$ write the system (3.12) and determine the coefficients α_i, β_i, γ_i. One of them is arbitrary and usally we choose it equal to unity. As a result, we have for each **eigenvalue** λ_i corresponding **eigenvector** $\{\alpha_i, \beta_i, \gamma_i\}$.

Thus, we obtain:

1. for root λ_1 the particular solution of the system (3.10) is

$$x_1(t) = \alpha_1 e^{\lambda_1 t}, \qquad y_1(t) = \beta_1 e^{\lambda_1 t}, \qquad z_1(t) = \gamma_1 e^{\lambda_1 t};$$

2. for root λ_2 the particular solution of the system (3.10) is

$$x_2(t) = \alpha_2 e^{\lambda_2 t}, \qquad y_2(t) = \beta_2 e^{\lambda_2 t}, \qquad z_2(t) = \gamma_2 e^{\lambda_2 t};$$

3. for root λ_3 the particular solution of the system (3.10) is

$$x_3(t) = \alpha_3 e^{\lambda_3 t}, \qquad y_3(t) = \beta_3 e^{\lambda_3 t}, \qquad z_3(t) = \gamma_3 e^{\lambda_3 t}.$$

It can be shown that these functions form the fundamental system of particular solutions of the system. The general solution of the system (3.10) is written in the form

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + C_3 x_3(t),$$
$$y(t) = C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t),$$
$$z(t) = C_1 z_1(t) + C_2 z_2(t) + C_3 z_3(t).$$

Hence,

$$\begin{aligned} x(t) &= C_1\alpha_1 e^{\lambda_1 t} + C_2\alpha_2 e^{\lambda_2 t} + C_3\alpha_3 e^{\lambda_3 t}, \\ y(t) &= C_1\beta_1 e^{\lambda_1 t} + C_2\beta_2 e^{\lambda_2 t} + C_3\beta_3 e^{\lambda_3 t}, \\ z(t) &= C_1\gamma_1 e^{\lambda_1 t} + C_2\gamma_2 e^{\lambda_2 t} + C_3\gamma_3 e^{\lambda_3 t}. \end{aligned} \tag{3.15}$$

or

$$\begin{Bmatrix} x(t) \\ y(t) \\ z(t) \end{Bmatrix} = C_1 e^{\lambda_1 t} \begin{Bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{Bmatrix} + C_2 e^{\lambda_2 t} \begin{Bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{Bmatrix} + C_3 e^{\lambda_3 t} \begin{Bmatrix} \alpha_3 \\ \beta_3 \\ \gamma_3 \end{Bmatrix}.$$

Here C_1, C_2, C_3 are arbitrary constants. One can find values of the constants such that the solution will satisfy the given initial conditions.

Example 3.

Integrate the system

$$\begin{cases} x'(t) = 6x - 12y - z, \\ y'(t) = x - 3y - z, \\ z'(t) = -4x + 12y + 3z. \end{cases}$$

This is linear homogeneous system with respect to three unknown functions $x(t)$, $y(t)$ and $z(t)$.

Let us apply the matrix method.

Form the characteristic equation of that system

$$\begin{vmatrix} 6-\lambda & -12 & -1 \\ 1 & -3-\lambda & -1 \\ -4 & 12 & 3-\lambda \end{vmatrix} = 0.$$

Solving the determinant, we get

$$(6-\lambda)(\lambda^2-9)-48-12+4(3+\lambda)+12(6-\lambda)+12(3-\lambda)=0,$$

and, finally,

$$\lambda^3-6\lambda^2+11\lambda-6=0.$$

The equation obtained has three real and distinct roots $\lambda_1=1, \lambda_2=2, \lambda_3=3$. These numbers are eigenvalues of matrix $A=\begin{pmatrix} 6 & -12 & -1 \\ 1 & -3 & -1 \\ -4 & 12 & 3 \end{pmatrix}$. Now, let us find corresponding eigenvectors $\{\alpha_i, \beta_i, \gamma_i\}$ and particular solutions of given system

$$x_i(t)=\alpha_i e^{\lambda_i t}, \qquad y_i(t)=\beta_i e^{\lambda_i t}, \qquad z_i(t)=\lambda_i e^{\lambda_i t}, (i=1{,}2{,}3).$$

Form the system (3.12) for each root λ_i and determine $\{\alpha_i, \beta_i, \gamma_i\}$.

1. For $\lambda_1=1$ we have

$$\begin{cases} (6-1)\alpha_1-12\beta_1-\gamma_1=0, \\ \alpha_1+(-3-1)\beta_1-\gamma_1=0, \\ -4\alpha_1+12\beta_1+(3-1)\gamma_1=0; \end{cases}$$

$$\begin{cases} 5\alpha_1-12\beta_1-\gamma_1=0, \\ \alpha_1-4\beta_1-\gamma_1=0, \\ -4\alpha_1+12\beta_1+2\gamma_1=0. \end{cases}$$

Adding the first equation to the third one, we obtain

$$\begin{cases} \alpha_1+\gamma_1=0, \\ \alpha_1-4\beta_1-\gamma_1=0. \end{cases}$$

Choose α_1 as a free variable. Let $\alpha_1=1$, then from the first equation we have $\gamma_1=-1$, and from the second equation we get $\beta_1=\frac{1}{2}$. Thus, the eigenvector corresponding λ_1 is

$$\{\alpha_1, \beta_1, \gamma_1\}=\left\{1, \frac{1}{2}, -1\right\},$$

and particular solution of the system is

$$x_1(t)=\alpha_1 e^{\lambda_1 t}=e^t, \qquad y_1(t)=\beta_1 e^{\lambda_1 t}=\frac{1}{2}e^t, \qquad z_1(t)=\gamma_1 e^{\lambda_1 t}=-e^t.$$

2. For $\lambda_2 = 2$ we get

$$\begin{cases}(6-2)\alpha_2 - 12\beta_2 - \gamma_2 = 0,\\ \alpha_2 + (-3-2)\beta_2 - \gamma_2 = 0,\\ -4\alpha_2 + 12\beta_2 + (3-2)\gamma_2 = 0;\end{cases}$$

$$\begin{cases}4\alpha_2 - 12\beta_2 - \gamma_2 = 0,\\ \alpha_2 - 5\beta_2 - \gamma_2 = 0,\\ -4\alpha_2 + 12\beta_2 + \gamma_2 = 0.\end{cases}$$

In this system, the first and third equations are the same. We consider two independent equations

$$\begin{cases}4\alpha_2 - 12\beta_2 - \gamma_2 = 0,\\ \alpha_2 - 5\beta_2 - \gamma_2 = 0.\end{cases}$$

Add them by pre-multiplying the second equation by (-1). This yields:

$$\begin{cases}3\alpha_2 - 7\beta_2 = 0,\\ \gamma_2 = \alpha_2 - 5\beta_2.\end{cases}$$

Take α_2 as a free variable. Let $\alpha_2 = 1$, then $\beta_2 = \frac{3}{7}, \gamma_2 = -\frac{8}{7}$.

Consequently, we get the eigenvector

$$\{\alpha_2, \beta_2, \gamma_2\} = \left\{1, \frac{3}{7}, -\frac{8}{7}\right\},$$

and the particular solution is

$$x_2(t) = \alpha_2 e^{\lambda_2 t} = e^{2t}, \qquad y_2(t) = \beta_2 e^{\lambda_2 t} = \frac{3}{7}e^{2t}, \qquad z_2(t) = \gamma_2 e^{\lambda_2 t} = -\frac{8}{7}e^{2t}.$$

3. For the root $\lambda_3 = 3$ we have

$$\begin{cases}(6-3)\alpha_3 - 12\beta_3 - \gamma_3 = 0,\\ \alpha_3 + (-3-3)\beta_3 - \gamma_3 = 0,\\ -4\alpha_3 + 12\beta_3 + (3-3)\gamma_3 = 0;\end{cases}$$

$$\begin{cases}3\alpha_3 - 12\beta_3 - \gamma_3 = 0,\\ \alpha_3 - 6\beta_3 - \gamma_3 = 0,\\ -4\alpha_3 + 12\beta_3 = 0.\end{cases}$$

Adding the first equation to the third one, we get

$$\begin{cases}-\alpha_2 - \gamma_2 = 0,\\ \alpha_2 - 6\beta_2 - \gamma_2 = 0;\end{cases}$$

$$\begin{cases}\alpha_2 = -\gamma_2,\\ 6\beta_2 = \alpha_2 - \gamma_2.\end{cases}$$

Put $\alpha_3 = 1$, then $\gamma_3 = -1$ and $\beta_3 = \frac{1}{3}$.

Thus, the eigenvector is

$$\{\alpha_3, \beta_3, \gamma_3\} = \left\{1, \frac{1}{3}, -1\right\}$$

and the particular solution is

$$x_3(t) = \alpha_3 e^{\lambda_3 t} = e^{3t}, \qquad y_3(t) = \beta_3 e^{\lambda_3 t} = \frac{1}{3}e^{3t}, \qquad z_3(t) = \gamma_3 e^{\lambda_3 t} = -e^{3t}.$$

Particular solutions form the fundamental system of solutions. According to formula (3.15) the general solution of the system is written as

$$x(t) = C_1 e^t + C_2 e^{2t} + C_3 e^{3t},$$

$$y(t) = \frac{1}{2}C_1 e^t + \frac{3}{7}C_2 e^{2t} + \frac{1}{3}C_3 e^{3t},$$

$$z(t) = -C_1 e^t - \frac{8}{7}C_2 e^{2t} - C_3 e^{3t}.$$

Case 2. The roots of the characteristic equation are distinct, but include two complex conjugate roots:

$$\lambda_1 = a + ib, \ \lambda_2 = a - ib, \ \lambda_3 \in R.$$

In this case, the form of particular solutions is determined similarly to the Case I. The solutions corresponding the pair $\lambda_{1,2} = a \pm ib$ have the form

$$C_{1,2} e^{(a \pm ib)t}.$$

However, since the original system of equations does not include complex numbers and functions, the solutions should be real. Using Euler's formulas, we separate real and imaginary parts of particular solutions

$$C_1 e^{(a+ib)t} = C_1 e^{at}\cos bt + C_1 i e^{at}\sin bt,$$

$$C_2 e^{(a-ib)t} = C_2 e^{at}\cos bt - C_2 i e^{at}\sin bt.$$

Obviously, both solutions are a linear combination of two linearly independent functions real part $e^{at}\cos bt$ and imaginary part $e^{at}\sin bt$. Thus, we obtain two real linearly independent particular solutions.

On practice it enough to find the particular solution only for $\lambda_1 = a + ib$, since the root $\lambda_2 = a - ib$ does not give us new linearly independent real solutions.

Then the general solution of the system (3.10) can be written as a linear combination of all linearly independent partial solutions with arbitrary constants.

Example 4.

Solve the initial value problem for the system

$$\begin{cases} x'(t) = x + y, & x(0) = 7, \\ y'(t) = -x + y - z, & y(0) = 2, \\ z'(t) = 3y + z, & z(0) = 1. \end{cases}$$

Let us solve the system by matrix method.

Form the characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & -1 \\ 0 & 3 & 1-\lambda \end{vmatrix} = 0,$$

and solve it

$$(1-\lambda)^3 + 3(1-\lambda) + (1-\lambda) = 0,$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0.$$

There are three roots (one real and pair of complex conjugate)

$$\lambda_1 = 1,\ \lambda_2 = 1 + 2i,\ \lambda_3 = 1 - 2i.$$

For each eigenvalue $\lambda_i$we solve the system (3.12).

1. For $\lambda_1 = 1$

$$\begin{cases} 0 \cdot \alpha_1 + \beta_1 + 0 \cdot \gamma_1 = 0, \\ -\alpha_1 + 0 \cdot \beta_1 - \gamma_1 = 0, \\ 0 \cdot \alpha_1 + 3\beta_1 + 0 \cdot \gamma_1 = 0; \end{cases}$$

$$\begin{cases} \beta_1 = 0, \\ -\alpha_1 - \gamma_1 = 0. \end{cases}$$

Plugging $\alpha_1 = 1$, we obtain $\gamma_1 = -1$.

Thus, the corresponding eigenvector is

$$\{\alpha_1, \beta_1, \gamma_1\} = \{1,0,-1\},$$

and the particular solution is written as following

$$x_1(t) = \alpha_1 e^{\lambda_1 t} = e^t, \qquad y_1(t) = \beta_1 e^{\lambda_1 t} = 0, \qquad z_1(t) = \gamma_1 e^{\lambda_1 t} = -e^t.$$

2. For $\lambda_2 = 1 + 2i$ we get

$$\begin{cases} -2i\alpha_2 + \beta_2 = 0, \\ -\alpha_2 - 2i\beta_2 - \gamma_2 = 0, \\ 3\beta_2 - 2i\gamma_2 = 0. \end{cases}$$

Put $\alpha_2 = 1$ and obtain from the system: $\beta_2 = 2i, \gamma_2 = 3$.

Hence, the eigenvector is

$$\{\alpha_2, \beta_2, \gamma_2\} = \{1, 2i, 3\},$$

and the particular solution is

$$x_2(t) = \alpha_2 e^{\lambda_2 t} = e^{(1+2i)t}, y_2(t) = \beta_2 e^{\lambda_2 t} = 2ie^{(1+2i)t}, z_2(t) = \gamma_2 e^{\lambda_2 t} = 3e^{(1+2i)t}.$$

Let us find real (Re) and imaginary (Im) parts of the obtained solutions:

$$x_2(t) = e^{(1+2i)t} = e^t(\cos 2t + i \sin 2t),$$

$$Re\, x_2\,(t) = e^t \cos 2t\,, \qquad Im\, x_2\,(t) = e^t \sin 2t\,;$$

$$y_2(t) = 2ie^{(1+2i)t} = e^t(2i \cos 2t - 2 \sin 2t),$$

$$Re\, y_2\,(t) = -2e^t \sin 2t\,, \qquad Im\, y_2\,(t) = 2e^t \cos 2t\,;$$

$$z_2(t) = 3e^{(1+2i)t} = e^t(3 \cos 2t + 3i \sin 2t),$$

$$Re\, z_2\,(t) = 3e^t \cos 2t\,, \qquad Im\, z_2\,(t) = 3e^t \sin 2t.$$

3. Since for $\lambda_3 = 1 - 2i$ particular solutions consist of the same linear independent parts, we do not solve the system (3.12) for that eigenvalue.

Consequently, the particular solutions obtained in steps 1. and 2. form the fundamental system of solution.

Thus the general solution of the given system is

$$x(t) = C_1 e^t + C_2 e^t \cos 2t + C_3 e^t \sin 2t\,,$$

$$y(t) = C_1 \cdot 0 - 2C_2 e^t \sin 2t + 2C_3 e^t \cos 2t\,,$$

$$z(t) = -C_1 e^t + 3C_2 e^t \cos 2t + 3C_3 e^t \sin 2t.$$

Now, let us choose the constants C_1, C_2, C_3 so that the initial conditions $x(0) = 7$, $y(0) = 2,\ z(0) = 1$ are satisfied.

$$\begin{cases} 7 = C_1 + C_2 + 0, \\ 2 = 0 - 0 + 3C_3, \\ 1 = -C_1 + 3C_2 + 0, \end{cases}$$

$$C_1 = 5, C_2 = 2, C_3 = 1.$$

As a result, we obtain the solution of the initial value problem

$$x(t) = 5e^t + 2e^t \cos 2t + e^t \sin 2t\,,$$

$$y(t) = -4e^t \sin 2t + 2e^t \cos 2t\,,$$

$$z(t) = -5e^t + 6e^t \cos 2t + 3e^t \sin 2t.$$

Case 3. The roots of the characteristic equation are real and λ is a root of multiplicity m $(m = 2, 3)$.

The solution of the system corresponding to this eigenvalue is defined as follows

- if $m = 2$, then

$$x(t) = (At + B)e^{\lambda t},$$
$$y(t) = (Ct + D)e^{\lambda t},$$
$$z(t) = (Et + F)e^{\lambda t};$$

- if $m = 3$, then

$$x(t) = (At^2 + Bt + C)e^{\lambda t},$$
$$y(t) = (Dt^2 + Et + F)e^{\lambda t},$$
$$z(t) = (Gt^2 + Ht + N)e^{\lambda t}.$$

Solution depends on m arbitrary constants: $A, B, C, \dots, N$. These constants could be determined by using the method of undetermined coefficients.

We have to express the coefficients through arbitrary m of them, then, putalternately one of them equal to one and the rest constants equal to zero. As a result, we obtain m linearly independent partial solutions of the system (3.10).

Example 5.

Solve the system

$$\begin{cases} x'(t) = x - y + z, \\ y'(t) = x + y - z, \\ z'(t) = -y + 2z. \end{cases}$$

Let us form the characteristic equation of that system

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0,$$

$$(1-\lambda)^2(2-\lambda) - 1 - (1-\lambda) + (2-\lambda) = 0,$$

$$(1 - 2\lambda + \lambda^2)(2-\lambda) = 0.$$

The solutions of that equation are

$$\lambda_1 = 2,$$
$$\lambda_2 = \lambda_3 = 1.$$

1. For the root $\lambda_1 = 2$ system (3.12) has a form

$$\begin{cases} -\alpha_1 - \beta_1 + \gamma_1 = 0, \\ \alpha_1 - \beta_1 - \gamma_1 = 0, \\ -\beta_1 = 0, \end{cases}$$

or

$$\begin{cases} \beta_1 = 0, \\ \alpha_1 - \gamma_1 = 0. \end{cases}$$

Plugging $\alpha_1 = 1$, we get $\gamma_1 = 1$.

Thus, we obtain one of the particular solutions of the system

$$x_1(t) = \alpha_1 e^{\lambda_1 t} = e^{2t}, y_1(t) = \beta_1 e^{\lambda_1 t} = 0, z_1(t) = \gamma_1 e^{\lambda_1 t} = e^{2t}.$$

2. For the solution of multiplicity $m = 2$: $\lambda = \lambda_2 = \lambda_3 = 1$, the corresponding solution has a form:

$$x_{2,3}(t) = (At + B)e^{\lambda t}, y_{2,3}(t) = (Ct + D)e^{\lambda t}, z_{2,3}(t) = (Et + F)e^{\lambda t},$$

where A, B, C, D, E, F are undetermined coefficients.

Finding derivatives of these functions and putting them into the original system, we obtain

$$\begin{cases} A \cdot e^t + (At + B)e^t = (At + B)e^t - (Ct + D)e^t + (Et + F)e^t, \\ C \cdot e^t + (Ct + D)e^t = (At + B)e^t + (Ct + D)e^t - (Et + F)e^t, \\ E \cdot e^t + (Et + F)e^t = -(Ct + D)e^t + 2(Et + F)e^t. \end{cases}$$

After reducing by $e^t \neq 0$ and collecting similar addends, we get

$$\begin{cases} (C - E)t + A + D - F = 0, \\ (A - E)t + B - C - F = 0, \\ (C - E)t + D + E - F = 0. \end{cases}$$

Comparing the coefficients with the same powers of t in the right and left sides in each equation, we get a system for the coefficients A, B, C, D, E, F:

$$\begin{cases} C - E = 0, \\ A - E = 0, \\ A + D - F = 0, \\ B - C - F = 0, \\ D + E - F = 0. \end{cases}$$

Since the solution depends on only two different arbitrary constants, let us express, for example, C, D, E, F by A and B. From the second equation we have: $E = A$. Then, from the first equation, we get $C = A$. Forth equation gives us $F = B - C$, that is $F = B - A$. From the third equation we get $D = F - A$, or $D = B - A - A = B - 2A$.

Thus,

$$E = A,$$

$$C = A,$$

$$F = B - A,$$

$$D = B - 2A,$$

where A and B are arbitrary real numbers.

Now we find two different solution:

- let $A = 1, B = 0$, then $C = 1, D = -2, E = 1, F = -1$.
- if $A = 0, B = 1$, then $C = 0, D = 1, E = 0, F = 1$.

Thus we obtain two linearly independent particular solutions corresponding to the root $\lambda = 1$:

$$x_2(t) = te^t, \qquad y_2(t) = (t-2)e^t, \qquad z_2(t) = (t-1)e^t,$$

$$x_3(t) = e^t, \qquad y_3(t) = e^t, \qquad z_3(t) = e^t.$$

Finally, the general solution of the original system is

$$x(t) = C_1e^{2t} + C_2te^t + C_3e^t,$$

$$y(t) = C_2(t-2)e^t + C_3e^t,$$

$$z(t) = C_1e^{2t} + C_2(t-1)e^t + C_3e^t.$$

Review Questions

1. What methods are the most commonly used for solving linear homogeneous systems of differential equations?

2. What is the idea of the elimination method?

3. Is it possible to use the elimination method for solving linear homogeneous (nonhomogeneous) systems of differential equations?

4. What are advantages and disadvantages of the elimination method?

5. Is it possible to use the matrix method (method of eigenvalues and eigenvectors) for solving linear homogeneous (nonhomogeneous) systems of differential equations?

6. What is the idea of the matrix method for solving linear homogeneous systems of differential equations?

7. What form have the particular solutions of linear homogeneous systems of differential equations?

8. What is the characteristic equation of the linear homogeneous systems of differential equations with constant coefficients?

9. What are eigenvalues of a matrix of coefficients of the system? What are eigenvectors?

10. How many arbitrary constants does the general solution of the following systems contain?

1) $\begin{cases} x'(t) - 4x - y = 0, \\ y'(t) - 5x + 3y = 0; \end{cases}$ 2) $\begin{cases} x'(t) = x + 4y, \\ y'(t) = 4x + 3y - z, \\ z'(t) = 2x + z. \end{cases}$

Exercises 3.3

1-10. Solve the systems by elimination method

1. $\begin{cases} x'(t) = -y, \\ y'(t) = -4x; \end{cases}$ **2.** $\begin{cases} x'(t) = y, \\ y'(t) = -x; \end{cases}$

3. $\begin{cases} x'(t) = -y + e^{3t}, \\ y'(t) = -x + 2e^{3t}; \end{cases}$ **4.** $\begin{cases} x'(t) = 2y - 5x + e^{t}, \\ y'(t) = x - 6y + e^{-2t}; \end{cases}$

5. $\begin{cases} x'(t) = y + t, \\ y'(t) = x + e^{t}, \end{cases}$ $x(0) = 1, \quad y(0) = 0;$

6. $\begin{cases} x'(t) = 2x + y + \cos t, \\ y'(t) = -x + 2\sin t, \end{cases}$ $x(0) = 0, \quad y(0) = 0;$

7. $\begin{cases} 4x'(t) - y'(t) + 3x = \sin t, \\ x'(t) + y = \cos t; \end{cases}$ **8.** $\begin{cases} x''(t) = x, \\ y''(t) = y; \end{cases}$

9. $\begin{cases} x'(t) = x - z, \\ y'(t) = x, \\ z'(t) = x - y; \end{cases}$

10. $\begin{cases} x'(t) = y + z, \\ y'(t) = x + z, \\ z'(t) = x + y, \end{cases}$ $x(0) = -1, \quad y(0) = 1, \quad z(0) = 0.$

11-17. Find general solutions of the systems by matrix method

11. $\begin{cases} x'(t) = x - y, \\ y'(t) = -4x + y; \end{cases}$

12. $\begin{cases} x'(t) = 3x - y, \\ y'(t) = 4x - y; \end{cases}$

13. $\begin{cases} x'(t) = x - 3y, \\ y'(t) = 3x + y; \end{cases}$

14. $\begin{cases} x'(t) = x - 2y - z, \\ y'(t) = -x + y + z, \\ z'(t) = x - z; \end{cases}$

15. $\begin{cases} x'(t) = x - y + z, \\ y'(t) = x + y - z, \\ z'(t) = 2x - y; \end{cases}$

16. $\begin{cases} x'(t) = 3x - y + z, \\ y'(t) = -x + 5y - z, \\ z'(t) = x - y + 3z; \end{cases}$

17. $\begin{cases} x'(t) = x - z, \\ y'(t) = x, \\ z'(t) = x - y. \end{cases}$

18-24. Find particular solutions of the systems by matrix method

18. $\begin{cases} x'(t) = 2x + y, \\ y'(t) = 3x + 4y, \end{cases}$ $x(0) = 1, \quad y(0) = 3;$

19. $\begin{cases} x'(t) = 3x + 2y, \\ y'(t) = x + 2y, \end{cases}$ $x(0) = 2, \quad y(0) = -\frac{1}{2};$

20. $\begin{cases} x'(t) = -3x + 2y, \\ y'(t) = -2x + y, \end{cases}$ $x(0) = 1, \quad y(0) = 0;$

21. $\begin{cases} x'(t) = x - 5y, \\ y'(t) = x - y, \end{cases}$ $x(0) = 3, \quad y(0) = \frac{1}{5};$

22. $\begin{cases} x'(t) = x + y, \\ y'(t) = -2x + 3y, \end{cases}$ $x(0) = 2, \quad y(0) = 1;$

23. $\begin{cases} x'(t) = -x + y + z, \\ y'(t) = x - y + z, \\ z'(t) = x + y + z, \end{cases}$ $x(0) = 1, \quad y(0) = 0, \quad z(0) = 0;$

24. $\begin{cases} x'(t) = y + z, \\ y'(t) = x + z, \\ z'(t) = x + y, \end{cases}$ $x(0) = -1, \quad y(0) = 1, \quad z(0) = 0.$

Answers.

4. Equilibrium Solutions and Their Classification

4.1 Elements of the Stability Theory

Since differential equations are widely used in various practical problems, it should be remembered that usually the initial conditions are obtained by some kind of measurement. Unfortunately, any measurements cannot be accurate. This puts us before the question of the influence of small changes in the initial conditions on the solution of the differential equation.

If small changes in the initial conditions can significantly change the solution, then such a solution cannot describe the phenomenon under study. Thus, we must understand under what conditions small changes in the initial conditions occur small changes in the solution.

This question is one of basic in so-called qualitative theory of differential equations. It helps us understand what happens with the solution (and phenomenon) as time increases.

I. Basic Definitions and Concepts

Let us consider the first order differential equation

$$y' = f(t, y). \tag{4.1}$$

Function $\varphi(t), y(t), t \geq t_0$, are the particular solutions of the equation (4.1) under the initial conditions $y(t_0) = \varphi(t_0)$ and $y(t_0) = y(t_0)$ respectively.

Definition. Solution $\varphi(t)$ is called *stable in the sense of Lyapunov* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that if

$$|y(t_0) - \varphi(t_0)| < \delta(\varepsilon),$$

then

$$|y(t) - \varphi(t)| < \varepsilon$$

for all values $t \geq t_0$ (Fig. 4.1).

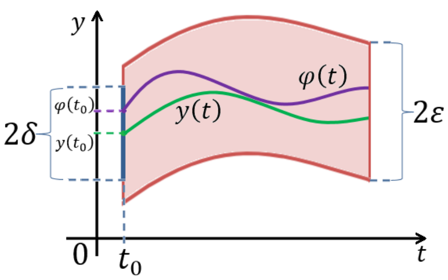

Figure 4.1

Stability in the sense of Lyapunov means that solution $\varphi(t)$ starting "close enough" to $y(t)$ remains "close enough" forever.

Definition. Solution $\varphi(t)$ is called *asymptotically stable* if

- it is stable in the sense of Lyapunov;
- there exists $\delta_0 > 0$, such that if

$$|y(t_0) - \varphi(t_0)| < \delta_0,$$

then

$$\lim_{t\to\infty}(y(t) - \varphi(t)) = 0.$$

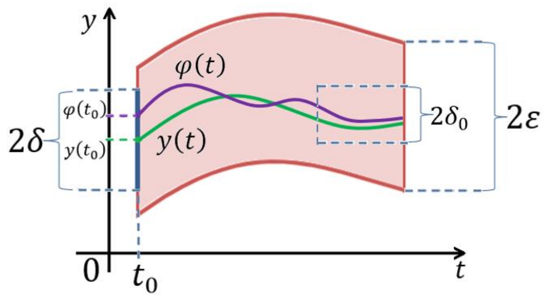

Figure 4.2

In this case, solution $\varphi(t)$ starting "close enough" not only remains "close enough" to $y(t)$, but also gradually converge to $y(t)$ as t increases (Fig. 4.2).

Definition. If the solution $\varphi(t)$ is not stable, it is called *unstable*.

To determine the unstability it is enough to find $\varepsilon_0 > 0$ that for any $\delta > 0$ there exist at least one solution $y(t)$ such that

$$|y(t_0) - \varphi(t_0)| < \delta$$

and for $t_1 > t_0$

$$|y(t_1) - \varphi(t_1)| = \varepsilon_0.$$

Example 1.

Investigate for stability the solution of initial value problem

$$y' = -y, \qquad y(0) = y_0.$$

Let us integrate the differential equation. Using separation of variables, we obtain

$$y(t) = Ce^{-t},$$

where C is an arbitrary constant.

From the initial condition $y(0) = y_0$ we get the particular solution

$$\varphi(t) = y_0 e^{-t}.$$

Consider the initial value problem

$$y' = -y, \qquad y(0) = \widetilde{y_0},$$

Thus,

$$y(t) = \widetilde{y_0} e^{-t}.$$

Let $\varepsilon > 0$ be known. According to the definition of stability in the sense of Lyapunov, we have to find the corresponding number $\delta(\varepsilon) > 0$, such that if

$$|\widetilde{y_0} - y_0| < \delta(\varepsilon),$$

then

$$|y(t) - \varphi(t)| < \varepsilon, \qquad t \geq t_0.$$

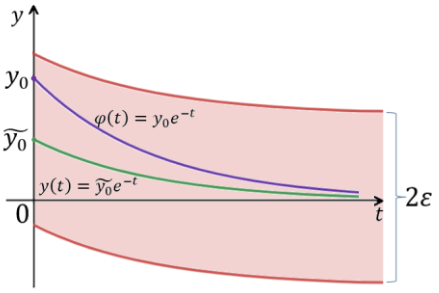

Figure 4.3

Since

$$|y(t) - \varphi(t)| = |\widetilde{y_0} e^{-t} - y_0 e^{-t}| = |\widetilde{y_0} - y_0| e^{-t},$$

it easy to see that the last inequality holds true for any $\delta \leq \varepsilon$.

Thus, the solution $\varphi(t)$ is stable (Fig. 4.3).

More over, since

$$\lim_{t\to\infty}(y(t) - \varphi(t)) = \lim_{t\to\infty}(\widetilde{y_0} - y_0)\, e^{-t} = 0,$$

the solution $\varphi(t)$ is asymptotically stable.

Example 2.

Investigate for stability the solution of initial value problem

$$y' = \sin^2 y, \qquad y(0) = 0.$$

Let us integrate the differential equation. Using separation of variables, we obtain

$$y(t) = \operatorname{arccot}(C - t),$$

where C is an arbitrary constant.

From the initial condition $y(0) = 0$ we get the particular solution

$$\varphi(t) = 0.$$

Consider the initial value problem

$$y' = \sin^2 y, \qquad y(0) = \widetilde{y_0} \quad \Longrightarrow \quad y(t) = \operatorname{arccot}(\cot \widetilde{y_0} - t).$$

Since

$$\lim_{t\to\infty}\big(y(t) - \varphi(t)\big) = $$
$$= \lim_{t\to\infty} \operatorname{arccot}(\cot \widetilde{y_0} - t) = \pi,$$

we get that however small $\widetilde{y_0} > 0$ is, there exists $t_1 > 0$ such that $y(t_1) > 1$.

Thus, the solution $\varphi(t) = 0$ is unstable (Fig. 4.4).

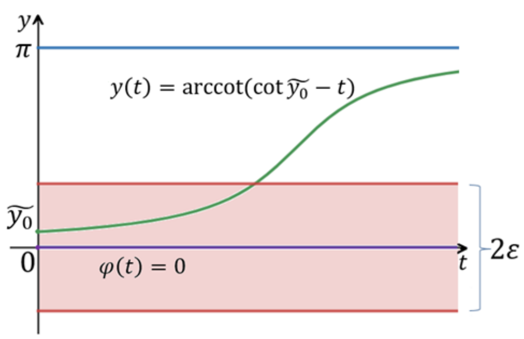

Figure 4.4

The investigation the solution $y = \varphi(t)$ for stability is reduced to the investigation for stability the solution $x(t) \equiv 0$ (trivial solution) of some auxiliary differential equation.

The auxiliary differential equation is obtained by substituting

$$x(t) = y(t) - \varphi(t)$$

into the original equation (16.1)

$$y' = f(t, y).$$

As a result, we get

$$x' = f\big(t, x(t) + \varphi(t)\big) - \varphi'(t)$$

and, since $\varphi'(t) = f\big(t, \varphi(t)\big)$,

$$x' = f\big(t, x(t) + \varphi(t)\big) - f\big(t, \varphi(t)\big) = F(t, x). \tag{4.2}$$

It is obvious, that equation (4.2) has trivial solution

$$x(t) \equiv 0$$

and $F(t, 0) \equiv 0$ for any t.

Theorem.

The solution $\varphi(t)$ of differential equation (4.1) is stable in the sense of Lyapunov (asymptotically stable) if and only if the trivial solution $x(t) \equiv 0$ of auxiliary differential equation (4.2) is stable in the sense of Lyapunov (asymptotically stable).

Similarly, we could define the concept of stability for the solution of the system of differential equations

$$\begin{cases} \dfrac{dy_1}{dt} = f_1(t, y_1, y_2, \dots, y_n), \\ \dfrac{dy_2}{dt} = f_2(t, y_1, y_2, \dots, y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dy_n}{dt} = f_n(t, y_1, y_2, \dots, y_n), \end{cases} \tag{4.3}$$

Let functions $\varphi_1(t), \varphi_2(t)$, …, $\varphi_n(t)$ be partial solution of (4.3) which satisfies the initial conditions

$$y_1(t_0) = \varphi_1(t_0), y_2(t_0) = \varphi_2(t_0), \ \dots \ y_n(t_0) = \varphi_n(t_0),$$

and $y_1(t), y_2(t), \ldots, y_n(t)$ be partial solution of (4.3) which satisfies the changed initial conditions

$$y_1(t_0) = y_1(t_0), y_2(t_0) = y_2(t_0), \ \ldots \ y_n(t_0) = y_n(t_0),$$

Definition. Solution $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ is called *stable in the sense of Lyapunov* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that if

$$|y_k(t_0) - \varphi_k(t_0)| < \delta(\varepsilon), k = 1,2, \ldots n,$$

then

$$|y_k(t) - \varphi_k(t)| < \varepsilon, k = 1,2, \ldots n,$$

for all values $t \geq t_0$.

Definition. Solution $\varphi_1(t), \varphi_2(t), \ldots \varphi_n(t)$ is called *asymptotically stable* if

- it is stable in the sense of Lyapunov;
- there exists $\delta_0 > 0$, such that if

$$|y_k(t_0) - \varphi_k(t_0)| < \delta_0, k = 1,2, \ldots n,$$

then

$$\lim_{t \to \infty} (y_k(t) - \varphi_k(t)) = 0, k = 1,2, \ldots n.$$

Similarly, the investigation the solution $\varphi_1(t), \varphi_2(t), \ldots \varphi_n(t)$ for stability is reduced to the investigation for stability the solution $y_k(t) \equiv 0, k = 1,2, \ldots n$ (trivial solution) of some auxiliary system of differential equations obtained by substituting

$$x_k(t) = y_k(t) - \varphi_k(t), k = 1,2, \ldots n.$$

In this case, system (4.3) could be written in the form

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(t, x_1, x_2, \ldots, x_n), \\ \dfrac{dx_2}{dt} = F_2(t, x_1, x_2, \ldots, x_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dx_n}{dt} = F_n(t, x_1, x_2, \ldots, x_n), \end{cases}$$

and $F_k(t, 0,0, \ldots, 0) \equiv 0, k = 1,2, \ldots n$, for any t.

Thus, it is enough to investigate for stability the trivial solution

$$x_k(t) \equiv 0, k = 1,2, \ldots n.$$

Definition of stability could be written as follows.

Definition. Trivial solution $x_k(t) \equiv 0, k = 1,2, \dots n,$ is called *stable* if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that if

$$\sum_{k=1}^{n} x_k^2(t_0) < \delta^2(\varepsilon),$$

then

$$\sum_{k=1}^{n} x_k^2(t) < \varepsilon$$

for all values $t \geq t_0$.

Example 3.

Investigate for stability the trivial solution of initial value problem

$$\begin{cases} \dfrac{dx_1}{dt} = -x_2, & x_1(t_0) = x_{10}, \\ \dfrac{dx_2}{dt} = x_1, & x_2(t_0) = x_{20}. \end{cases}$$

Let us integrate the initial value problem for this system (for example by elimination method).

We obtain

$$x_1(t) = x_{10} \cos t - x_{20} \sin t,$$
$$x_2(t) = x_{10} \sin t + x_{20} \cos t.$$

It easy to see, that

$$x_1^2 + x_2^2 = x_{10}^2 + x_{20}^2.$$

Thus, for any $\varepsilon > 0$ we may choose $\delta < \varepsilon$, and

$$x_1^2 + x_2^2 < \varepsilon^2$$

for all values $t \geq t_0$.

Hence, the trivial solution is stable.

You may read more about stability theory in [1,9,14,17].

Review Questions

1. What is the idea of stability theory?
2. Formulate the definition of stability in the sense of Lyapunov of the solution of differential equations (system of differential equations).
3. What does the stability in the sense of Lyapunov mean?
4. Formulate the definition of asymptotical stability of the solution of differential equations (system of differential equations).
5. What does the asymptotical stability mean?
6. What solutios do we called unstable? What does it mean?
7. How could we simplify the process of investigation for stability?

Exercises 4.1

1-5. Investigate for stability the solution of initial value problem

1. $y' = 1 + t - y,\ y(0) = 0;$

2. $y' = t(y - 1),\ y(1) = 2;$

3. $y' = t(y - 1),\ y(1) = 0;$

4. $y' = y(y^2 - 1),\ y(0) = 0;$

5. $y' = y(y^2 - 1),\ y(0) = -1.$

6-7. Investigate for stability the solution of initial value problem depending on the parameter $\boldsymbol{a}$.

6. $y' = \frac{ay}{t},\ y(1) = 0;$

7. $y' = ay,\ y(t_0) = y_0.$

Answers.

4.2 Equilibrium Solutions of the Autonomous Differential Equation

Definition. The first-order differential equation of the form

$$\frac{dy}{dt} = f(y) \tag{4.4}$$

is called *an autonomous differential equation* (the right-sided function does not depend on t).

Since any autonomous differential equation is separable, the general solution is

$$\int \frac{dy}{f(y)} = \int dt$$

or

$$\int \frac{dy}{f(y)} = t + C. \tag{4.5}$$

Unfortunately, integration of function $\frac{1}{f(y)}$ could be complicated and lead to the cumbersome expressions. This, in turn, leads to difficulties in the qualitative analysis of solutions, in particular, investigation for stability.

It is convenient to consider the independent variable t as indicating time and solution of the equation as describing some motion. Then function $f(y)$ is the rate of change of some function $y(t)$ expressed implicitly (without variable t).

Definition. Any constant solution of differential equation $(y(t) = C)$ is called *an equilibrium solution*.

Since the derivative of a constant function is zero, we may find equilibrium solutions of the equation (4.4) by solving the equation

$$f(y) = 0.$$

Note.

1. According to Cauchy theorem about existence and uniqueness of the solution of differential equation, if function $f(y)$ and $\frac{\partial f}{\partial y}$ are continuous in some region, then the equilibrium solution $y(t) = C$ is unique and no other solution can intersect it.

2. If differential equation (4.4) describes a certain process, then the equilibrium solutions show cases where the process does not change (balance level), since the rate of change is zero.

3. You may read more about autonomous differential equation in [1,11].

Let us remind that function $f(y)$ defines a direction field on the plane. This direction field could show us some properties of the solutions. The equilibrium solutions, when $\frac{dy}{dt} = 0$, give us horizontal dashes of a direction field.

In addition, we consider two more situations that appear between each pair of equilibrium solutions (or $\pm\infty$):

- when $\frac{dy}{dt} = f(y) > 0$, the dashes have positive slopes and the solutions increase (from the previous equilibrium solution (or $-\infty$) to the next equilibrium solution (or $+\infty$));
- when $\frac{dy}{dt} = f(y) < 0$, the dashes have negative slopes and the solutions decrease.

Applying this investigation we can classify the equilibrium solutions by the behavior of other solutions near them.

Definition. The equilibrium solution $y(t) = C$ is called *stable (asymptotically)* if all nearby solutions approach to C as t increases (the straight line $y = C$ attracts nearby integral curves to itself) (Fig. 4.5).

In this case $f(y) > 0$ for $y \in (C - \delta, C)$ and $f(y) < 0$ for $y \in (C, C + \delta)$ with sufficiently small $\delta > 0$.

Definition. The equilibrium solution $y(t) = C$ is called *unstable* if **none** of nearby solutions approaches to C as t increases (the straight line $y = C$ repel nearby integral curves from itself) (Fig. 4.5).

In this case $f(y) < 0$ for $y \in (C - \delta, C)$ and $f(y) > 0$ for $y \in (C, C + \delta)$ with sufficiently small $\delta > 0$.

Definition. The equilibrium solution $y(t) = C$ is called *semistable* if, when t increases, nearby solutions approach to C from one side but do not approach to C from the other side (Fig. 4.5).

In this case $f(y)$ holds the sign when moving through the point $y = C$.

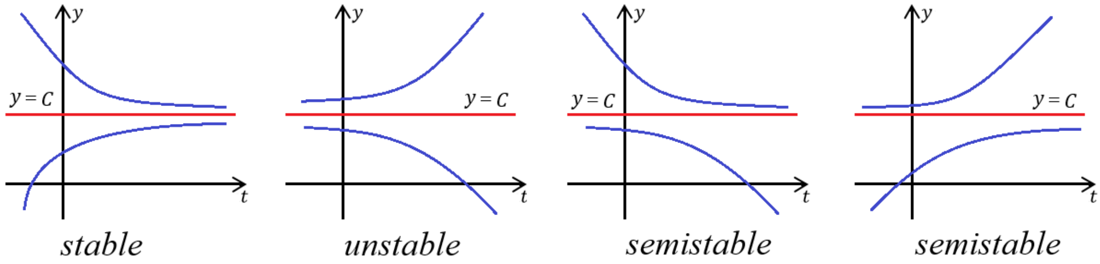

Figure 4.5

Example 1.

Investigate for stability the equilibrium solutions of the equation

$$\frac{dy}{dt} = (y-2)(y-1).$$

First, let us solve the equation

$$(y-2)(y-1) = 0.$$

There are two equilibrium solutions

$$y = 2 \;\text{ and }\; y = 1.$$

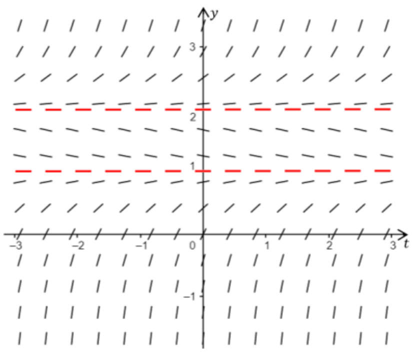

Figure 4.6

Second, let us consider the direction field for the equation (Fig. 4.6). Notice that here we have two sets of horizontal directions, that correspond to equilibrium solutions.

Thus,

y	$(-\infty, 1)$	$y = 1$	$(1,2)$	$y = 2$	$(2, +\infty)$
$\frac{dy}{dt} = (y-2)(y-1)$	positive increase	equilibrium solution	negative decrease	equilibrium solution	positive increase

Since moving through the point $y = 1$ derivative changes the sign from $''+''$ to $''-''$, the equilibrium solution $y = 1$ is stable.

Since moving through the point $y = 2$ derivative changes the sign from $''-''$ to $''+''$, the equilibrium solution $y = 2$ is unstable.

According to the results obtained, we could make the sketch of the solutions of given equation (Fig. 4.7).

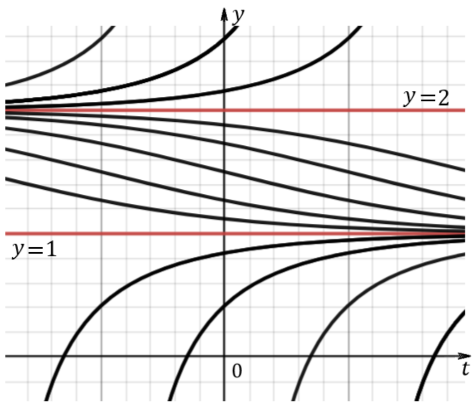

Figure 4.7

The straight line $y = 1$ attracts nearby solutions to itself. The straight line $y = 2$ repel nearby integral curves from itself.

Example 2.

Investigate for stability the equilibrium solution of the differential equation

$$\frac{dy}{dt} = (y-1)^2.$$

Solving the equation

$$(y-1)^2 = 0,$$

we obtain the equilibrium solution is

$$y = 1.$$

Since derivative holds the sign $'' + ''$ when moving through the point $y = 1$, this equilibrium solution is semistable.

The sketch of the integral curves is on the Figure 4.8.

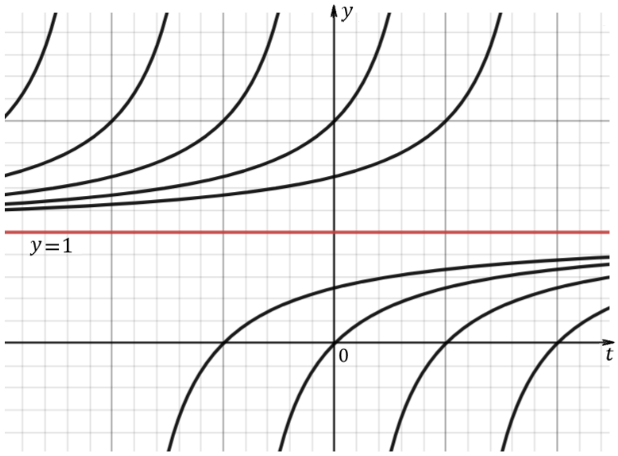

Figure 4.8

Review Questions

1. What is the autonomous differential equation?
2. What solutions of differential equation are called equilibrium?
3. What does the stability of equilibrium solution mean? Formulate the conditions.
4. What equilibrium solution is called unstable? What does it mean? Formulate the conditions.
5. What equilibrium solution do we called semistable? What does it mean?
7. Formulate the process of investigation the equilibrium solution for stability.

Exercises 4.2

1-5. Investigate for stability the equilibrium solutions of the autonomous differential equation

1. $y' = -\frac{1}{3}(y+1);$ **2.** $y' = 3(y-3);$ **3.** $y' = y^3 - 2y^2;$

4. $y' = \frac{1}{2}y(y-2)^2(y-4);$ **5.** $y' = \sin y.$

Answers.

4.3 Equilibrium Solutions of the Linear System of Differential Equations with Constant Coefficients

Let us consider the linear system of the form

$$\begin{cases} x'(t) = a_{11}x + a_{12}y, \\ y'(t) = a_{21}x + a_{22}y, \end{cases} \tag{4.6}$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are real numbers.

It is easy to see that $x(t) = 0,\ y(t) = 0$ is a solution of given system.

Definition. Solution $x(t) = 0,\ y(t) = 0$ is called an *equilibrium solution* (*equilibrium point*) of the system.

It is unique if the determinant of the matrix of coefficients $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is nonzero. In opposite case the system has an infinite number of equilibrium solutions. Further we consider only the case $detA \neq 0$.

We could investigate the behavior of other solutions of the system (4.6) near the equilibrium solution. Do the other solutions to the system move towards this solution or do they move away from this solution as $t \to \infty$? Is equilibrium solution stable or unstable? To answer these questions we need only the eigenvalues of matrix A.

Let us apply the matrix method.

Form the characteristic equation of that system

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0. \tag{4.7}$$

Solving the determinant, we get the quadratic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) = 0.$$

It has two roots λ_1, λ_2 (*eigenvalues or characteristic numbers of* A).

Let us investigate the behavior of solutions of the system (4.6) near the equilibrium point with respect to *eigenvalues of* A.

More over we could plot their graphs, since the solution of the system we could consider as parametric equations $\{x(t), y(t)\}$ in the plane.

Such curves are called *trajectories* (*orbits*). The XOY plane is called *the phase plane* and the set of trajectories form *the phase portrait* of solutions.

I. The roots of the characteristic equation are real and distinct

Substituting each eigenvalue λ_i into the system

$$\begin{cases}(a_{11}-\lambda_i)\alpha_i + a_{12}\beta_i = 0,\\ a_{21}\alpha_i + (a_{22}-\lambda_i)\beta_i = 0,\end{cases} \qquad (4.8)$$

and solving it, we find corresponding eigenvectors $\{\alpha_i, \beta_i\}, i = 1, 2.$

The general solution of the system (4.6) is written in the form

$$\begin{cases}x(t) = C_1\alpha_1 e^{\lambda_1 t} + C_2\alpha_2 e^{\lambda_2 t},\\ y(t) = C_1\beta_1 e^{\lambda_1 t} + C_2\beta_2 e^{\lambda_2 t},\end{cases} \qquad (4.9)$$

where $C_1,\ C_2$ are arbitrary constants.

Let us consider the possible cases.

Case 1. $\lambda_1 \neq \lambda_2, \lambda_1 < \lambda_2 < 0$

According to the form of solution (4.9), the equilibrium point $x = 0, y = 0$ is asymptotically stable. For any values of C_1 and C_2 every solution tends exponentially to the equilibrium point as $t \to \infty$, since $\frac{e^{\lambda_1 t}}{e^{\lambda_2 t}} \to 0, t \to \infty.$

If $C_2 \neq 0$ then $\frac{y}{x} \to \frac{\beta_2}{\alpha_2}, t \to \infty$ and corresponding curves have the the same tangent line in the origin of coordinates. If $C_2 = 0$ then straight line $y = \frac{\beta_1}{\alpha_1}x$ is one of trajectories. The phase portrait is shown in Figure 4.9 (schematically). Arrows show the direction of moving along the curves as t increases. In this case the equilibrium point is called ***the stable node***.

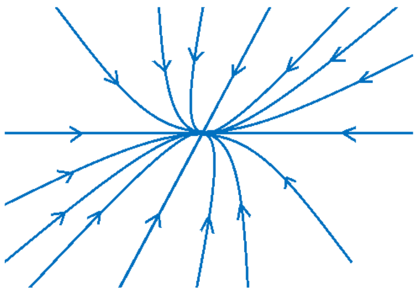

Figure 4.9

Case 2. $\lambda_1 \neq \lambda_2, \lambda_1 > 0, \lambda_2 > 0$

The shape of trajectories is the same as in Case1. but the direction of moving along the curves changes, since every solution tends outwards the equilibrium point as $t \to \infty$ (Fig. 4.10). In this case the equilibrium point is called ***the unstable node***.

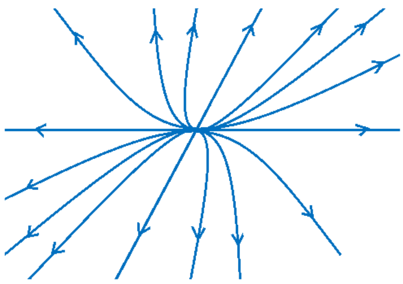

Figure 4.10

Case 3. $\lambda_1 \neq \lambda_2, \lambda_1 > 0, \lambda_2 < 0$

If $C_1 \neq 0, C_2 \neq 0$ it is easy to see from the solution (4.9) that the point moves along the trajectories outwards the equilibrium point $x = 0, y = 0$ as $t \to \infty$ and as $t \to -\infty$.

If $C_1 = 0, \alpha_2 \neq 0$ then the solution is

$$\begin{cases} x(t) = C_2\alpha_2 e^{\lambda_2 t}, \\ y(t) = C_2\beta_2 e^{\lambda_2 t}. \end{cases}$$

Hence, it is the straight line $y = \frac{\beta_2}{\alpha_2}x$. The point moves along that line towards the equilibrium point as $t \to \infty$.

If $C_2 = 0, \alpha_1 \neq 0$ then the solution is

$$\begin{cases} x(t) = C_1\alpha_1 e^{\lambda_1 t}, \\ y(t) = C_1\beta_1 e^{\lambda_1 t}. \end{cases}$$

Thus, the point moves along straight line $y = \frac{\beta_1}{\alpha_1}x$ outwards the origin of coordinates as $t \to \infty$.

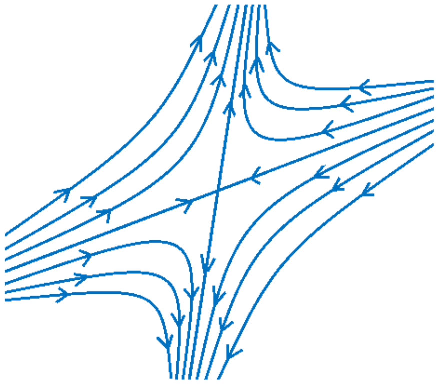

Figure 4.11

In this case the equilibrium point is called ***the saddle*** (Fig. 4.11). It is unstable.

The straight lines $y = \frac{\beta_2}{\alpha_2}x$ and $y = \frac{\beta_1}{\alpha_1}x$ are called *separatrices*.

Case 4. $\lambda_1 = 0, \lambda_2 < 0$

Let us rewrite the solution (4.9) in the form

$$\begin{cases} x(t) = C_1\alpha_1 + C_2\alpha_2 e^{\lambda_2 t}, \\ y(t) = C_1\beta_1 + C_2\beta_2 e^{\lambda_2 t}. \end{cases}$$

If $C_2 = 0, \alpha_1 \neq 0$ then the solution is

$$\begin{cases} x(t) = C_1\alpha_1, \\ y(t) = C_1\beta_1, \end{cases} \Rightarrow y = \frac{\beta_1}{\alpha_1}x.$$

In this case the whole line $y = \frac{\beta_1}{\alpha_1}x$ consists of the equilibrium points.

If $C_2 \neq 0, \alpha_2 \neq 0$ then trajectories are rays parallel to the straight line $y = \frac{\beta_2}{\alpha_2}x$ (Fig. 4.12).

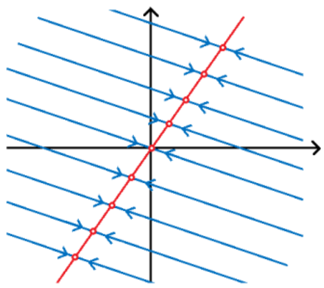

Figure 4.12

The point moves along that rays towards the equilibrium points as $t \to \infty$. Line $y = \frac{\beta_1}{\alpha_1}x$ is called ***the line of stable fixed points***.

Case 5. $\lambda_1 = 0, \lambda_2 > 0$

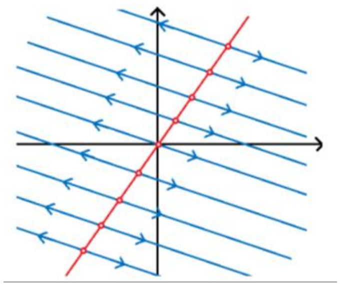

Figure 4.13

The shape of trajectories is the same as in Case 4. but the direction of moving along the rays is outwards the equilibrium points as $t \to \infty$ (Fig. 4.13). Line $y = \frac{\beta_1}{\alpha_1} x$ is called ***the line of unstable fixed points***.

Example 1.

Investigate the equilibrium point $x = 0,\ y = 0$ of the linear system

$$\begin{cases} x'(t) = -x, \\ y'(t) = 2x - 2y. \end{cases}$$

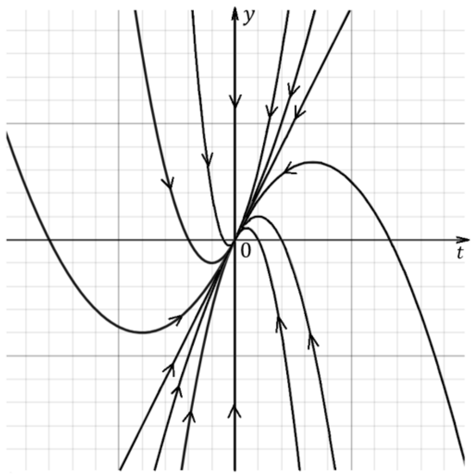

Figure 4.14

Let us find the eigenvalues of matrix A.

Form the characteristic equation of that system

$$\begin{vmatrix} -1-\lambda & 0 \\ -2 & -2-\lambda \end{vmatrix} = 0.$$

Thus, we get the quadratic equation

$$(\lambda + 1)(\lambda + 2) = 0,$$

and eigenvalues of A are $\lambda_1 = -2,\ \lambda_2 = -1$.

Both eigenvalues are negative, thus, the equilibrium point is **the stable node** (Case 1.).

Using (4.8), we could find eigenvectors $\{\alpha_1, \beta_1\} = \{0,1\},\ \{\alpha_2, \beta_2\} = \{1,2\}$.

Thus, the general solution of the system is

$$\begin{cases} x(t) = C_1 e^{-t}, \\ y(t) = 2C_1 e^{-t} + C_2 e^{-2t}, \end{cases}$$

where $C_1,\ C_2$ are arbitrary constants. The phase portrait is shown in Figure 4.14.

Example 2.

Investigate the equilibrium point $x = 0,\ y = 0$ of the linear system

$$\begin{cases} x'(t) = x, \\ y'(t) = -2y. \end{cases}$$

First, we write down the characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -2-\lambda \end{vmatrix} = 0.$$

Solving determinant, we obtain the quadratic equation

$$(1 - \lambda)(-2 - \lambda) = 0.$$

Thus, eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = -2$.

Since eigenvalues has different signs, we obtain that the equilibrium point is **the saddle (unstable)** (Case 3.).

Finding eigenvectors

$\{\alpha_1, \beta_1\} = \{1,0\},\ \{\alpha_2, \beta_2\} = \{0,-1\}$,

we get the general solution of the system

$$\begin{cases} x(t) = C_1 e^t, \\ y(t) = -C_2 e^{-2t}, \end{cases}$$

where $C_1,\ C_2$ are arbitrary constants.

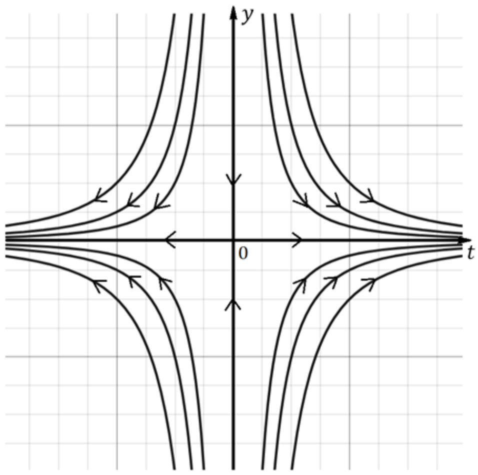

Figure 4.15

The phase portrait is shown in Figure 4.15. In this case, the separatrices coincide with the coordinate axes.

II. The roots of the characteristic equation are complex conjugate roots:

$$\boldsymbol{\lambda_1 = a + ib, \quad \lambda_2 = a - ib.}$$

General solution have the form

$$\begin{cases} x(t) = e^{at}(\alpha_1 \cos bt + \beta_1 \sin bt), \\ y(t) = e^{at}(\alpha_2 \cos bt + \beta_2 \sin bt), \end{cases} \tag{4.10}$$

where α_2, β_2 are linear combinations of α_1, β_1.

Let us consider the possible cases.

Case 1. $\lambda_{1,2} = \pm ib$

Here we have

$$\begin{cases} x(t) = \alpha_1 \cos bt + \beta_1 \sin bt, \\ y(t) = \alpha_2 \cos bt + \beta_2 \sin bt. \end{cases}$$

Since solutions represented by periodic functions of t, trajectories are closed curves (ellipses or circles centered the equilibrium point $x = 0, y = 0$) (Fig. 4.16).

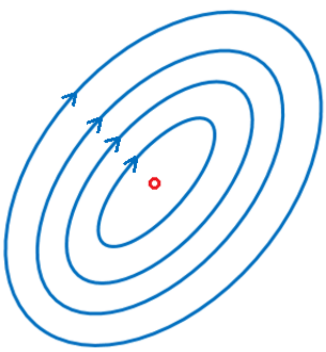
Figure 4.16

Obviously, that points of curves near the origin of coordinates stay close to $x = 0, y = 0$ for any $t > t_0$. Thus, this point is stable. But it is not asymptotically stable for $t \to \infty$.

In this case, equilibrium point is called ***the center***.

The direction of rotation is determined by the sign of the coefficient a_{21} ($a_{21} > 0$ − counterclockwise, $a_{21} < 0$ − clockwise).

Case 2. $\boldsymbol{\lambda_{1,2} = a \pm ib,\ a < 0, b \neq 0}$

From (4.10) we get that the first factor $e^{at}, a < 0$ tends to zero as $t \to \infty$, but the second factor is a T-periodic function. Thus, for $t = t_0 + kT, k = 1, 2, ...,$ trajectories do not close, unlike the center (Case 1), but approach the origin of coordinates (Fig. 4.17). The phase trajectories are spirals. If $a_{21} > 0$ they are twisted counterclockwise, if $a_{21} < 0$ − clockwise.

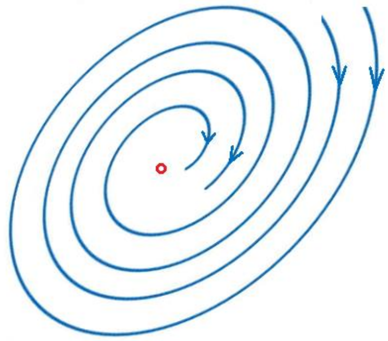

Figure 4.17

The equilibrium point is called ***the stable focus***.

Case 3. $\boldsymbol{\lambda_{1,2} = a \pm ib,\ a > 0, b \neq 0}$

The shape of trajectories is the same as in previous case but motion along the curves occurs in opposite direction as $t \to \infty$ (Fig. 4.18). If $a_{21} > 0$ they are untwisted counterclockwise, if $a_{21} < 0$ − clockwise.

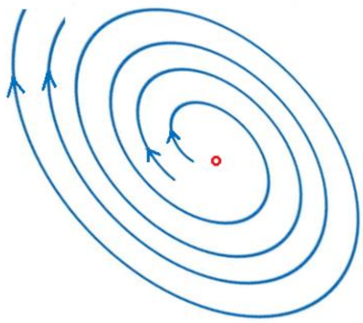

Figure 4.18

In this case the equilibrium point is called ***the unstable focus***.

Example 3.

Investigate the equilibrium point $x = 0,\ y = 0$ of the linear system

$$\begin{cases} x'(t) = x + y, \\ y'(t) = -x + y. \end{cases}$$

Write down the characteristic equation of the system

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0.$$

Hence, we obtain the quadratic equation

$$(1-\lambda)^2 + 1 = 0.$$

Eigenvalues of A are complex conjugate

$$\lambda_{1,2} = 1 \pm i.$$

Thus, since $a = 1 > 0, b \neq 0$, the equilibrium point is **the unstable focus** (Case 3).

General solution could be written in the form

$$\begin{cases} x(t) = e^t(C_1 \cos t + C_2 \sin t), \\ y(t) = e^t(C_2 \cos t - C_1 \sin t), \end{cases}$$

where C_1, C_2 are arbitrary constants.

The phase portrait is shown in Figure 4.19.

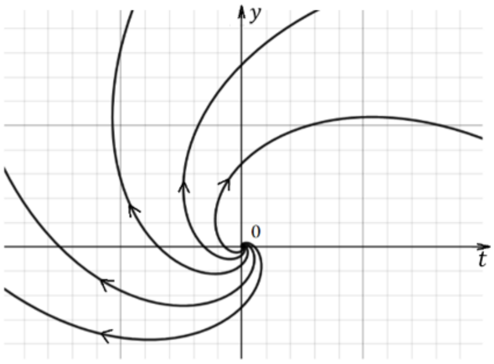

Figure 4.19

Example 4.

Investigate the equilibrium point $x = 0$, $y = 0$ of the linear system

$$\begin{cases} x'(t) = y, \\ y'(t) = -x. \end{cases}$$

Since the characteristic equation of the system is

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

we obtain complex conjugate eigenvalues of A

$$\lambda_{1,2} = \pm i.$$

Here $a = 0$, $b \neq 0$, and the equilibrium point is **the center (stable)** (Case 1).

General solution is

$$\begin{cases} x(t) = C_1 \cos t + C_2 \sin t, \\ y(t) = C_2 \cos t - C_1 \sin t, \end{cases}$$

where C_1, C_2 are arbitrary constants.

The phase portrait is shown in Figure 4.20.

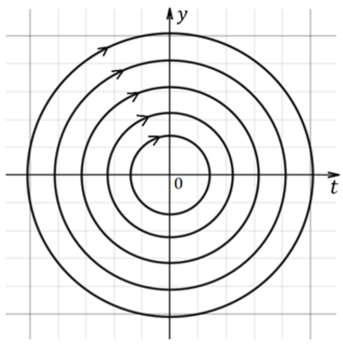

Figure 4.20

III. The roots of the characteristic equation are real and equal

The general solution of the system corresponding to this eigenvalues has the form

$$\begin{cases} x(t) = (\alpha_1 + \beta_1 t)e^{\lambda t}, \\ y(t) = (\alpha_2 + \beta_2 t)e^{\lambda t}, \end{cases} \tag{4.11}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are arbitrary constants.

Case 1. $\lambda_1 = \lambda_2 < 0$

If $\beta_1 \neq 0,\ \beta_2 \neq 0$ then the phase portrait is shown schematically in Figure 4.21.

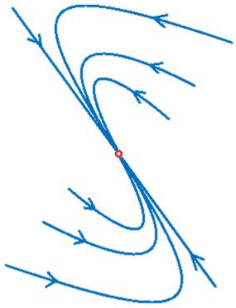

Figure 4.21

The equilibrium point is called ***the stable singular node***.

If $\beta_1 = \beta_2 = 0$ then the general solution is

$$\begin{cases} x(t) = \alpha_1 e^{\lambda_1 t}, \\ y(t) = \alpha_2 e^{\lambda_1 t}, \end{cases}$$

where α_1, α_2 are arbitrary constants.

The trajectories are straight lines $y = \frac{\alpha_2}{\alpha_1} x$ (Fig. 4.22). In this case, the equilibrium point is called ***the stable dicritical node***.

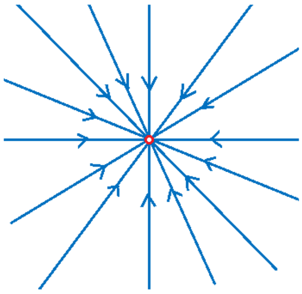

Figure 4.22

Both types of equilibrium points are asymptotically stable, since exponential function $e^{\lambda_1 t}$, $\lambda_1 < 0$, decreases faster than other polynomial factor and $\lim\limits_{t\to\infty} te^{\lambda_1 t} = 0$.

Case 2. $\lambda_1 = \lambda_2 > 0$

The shape of trajectories is the same as in previous case but motion along the curves is outwards the origin of coordinates. Both types of equilibrium points are unstable.

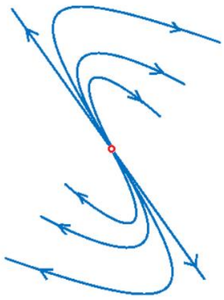

Figure 4.23

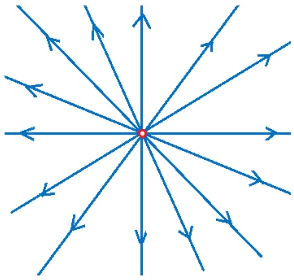

Figure 4.24

Figure 4.23 represent us ***the unstable singular node***, Figure 4.24 – ***the unstable dicritical node***.

Example 5.

Investigate the equilibrium point $x = 0,\ y = 0$ of the linear system

$$\begin{cases} x'(t) = 5x - y, \\ y'(t) = x + 3y. \end{cases}$$

Let us solve the characteristic equation of the system

$$\begin{vmatrix} 5-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0.$$

Eigenvalues of A are

$$\lambda_{1,2} = 4.$$

General solution is

$$\begin{cases} x(t) = (C_1 t + C_2)e^{4t}, \\ y(t) = (C_1 t + C_2 - C_1)e^{4t}, \end{cases}$$

where C_1, C_2 are arbitrary constants.

Since $\lambda_{1,2} = 4 > 0$ and $C_1 \not\equiv 0$, the equilibrium point is **the unstable singular node** (Case 2).

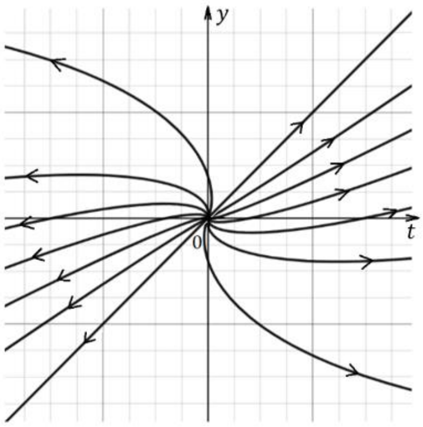

Figure 4.25

The phase portrait is shown in Figure 4.25.

You may find more examples in [9,17].

Review Questions

1. What is equilibrium solution of the linear homogeneous systems of differential equations with constant coefficients?

2. How many equilibrium solutions does the linear homogeneous systems of differential equations with constant coefficients have?

3. What is an idea of investigation of the equilibrium solution of the linear homogeneous systems of differential equations with constant coefficients?

4. What is a phase portrait?

5. Under what condition is an equilibrium solution a stable node (an unstable node)? What is the phase portrait in this case?

6. Under what condition is an equilibrium solution a saddle? What is the phase portrait in this case?

7. Under what condition is an equilibrium solution a center? What is the phase portrait in this case?

8. Under what condition is an equilibrium solution a stable focus (an unstable focus)? What is the phase portrait in this case?

9. Under what condition is an equilibrium solution a stable singular node (an unstable singular node)? What is the phase portrait in this case?

10. Under what condition is an equilibrium solution a stable dicritical node (an unstable dicritical node)? What is the phase portrait in this case?

Exercises 4.3

1-10. Investigate the equilibrium point $x = 0,\ y = 0$ of the linear system

1. $\begin{cases} x'(t) = -x - 2y, \\ y'(t) = 5x - 12y; \end{cases}$

2. $\begin{cases} x'(t) = -3x + y, \\ y'(t) = -5x + y; \end{cases}$

3. $\begin{cases} x'(t) = x - 2y, \\ y'(t) = 2x - 3y; \end{cases}$

4. $\begin{cases} x'(t) = 2x, \\ y'(t) = x + y; \end{cases}$

5. $\begin{cases} x'(t) = x - 2y, \\ y'(t) = x - y; \end{cases}$

6. $\begin{cases} x'(t) = 2y, \\ y'(t) = 2x + 3y; \end{cases}$

7. $\begin{cases} x'(t) = -5x - 5y, \\ y'(t) = 2x + 2y; \end{cases}$

8. $\begin{cases} x'(t) = 12x - 5y, \\ y'(t) = 5x + 12y. \end{cases}$

9-10. Investigate equilibrium points of the system depending on the parameter a.

9. $\begin{cases} x'(t) = ax + y, \\ y'(t) = -x + ay; \end{cases}$

10. $\begin{cases} x'(t) = ax + y, \\ y'(t) = x + ay. \end{cases}$

Answers.

5. Integral Equations

Along with differential equations, so-called integral equations are often used. They can be applied for solving and analyzing problems in geo- and astrophysics, radiative transfer, oscillation, heat flow, diffusion, electrostatic potential theory, ect.

I. Basic Definitions and Concepts.

Definition. Equation containing the unknown function under an integral sign is called *an integral equation*.

If it involves the unknown function linearly then it is called *linear integral equation*.

For example,

$y(t) - \int_0^1 tzy(z)dz = 2t,\ t \in [0,2],$ is a linear integral equation;

$y(t) = \int_0^t \frac{(t-1)(z+2)}{y^2(z)} dz$ is a nonlinear integral equation.

Definition. Function $y = \phi(x)$ which satisfies the integral equation (when put into the equation, converts it into an identity) is called *the solution of a integral equation*.

For example, it is easy to see (by direct substitution), that function $y(t) = 3t$ is the solution of the first equation

$$y(t) - \int_0^1 tzy(z)dz = 2t, \qquad t \in [0,2].$$

Let us look through the classification of the linear integral equations.

Definition. Integral equation is called *Fredholm integral equation* if the limits of the integral are both fixed numbers.

Integral equation is called *Volterra integral equation* if one of limits is variable.

Definition. Integral equation is called *an equation of the first kind* if the unknown function is placed only in the integral.

It is called *an equation of the second kind* if the unknown function is placed both inside and outside the integral.

Fredholm integral equations are more complicated then Volterra equations, and integral equations of the first kind are more complicated then equations the second of kind.

II. Volterra equation

Definition. The integral equation of the form

$$\int_a^t K(t,z)y(z)dz = f(t) \tag{5.1}$$

is called *Volterra integral equation of the first kind*.

Definition. The integral equation of the form

$$y(t) - \int_a^t K(t,z)y(z)dz = f(t) \tag{5.2}$$

is called *Volterra integral equation of the second kind*.

Definition. Function $K(t,z)$, defined on $\{(t,z): t \in [a,b], z \in [a,x]\}$, is called *a kernel function*.

Definition. Function $f(t)$, defined on $[a,b]$, is called *a right-sided function*.

If $f(t) \equiv 0$ then an integral equation is called *homogeneous*, otherwise, it is *nonhomogeneous*.

Without limiting the generality we may define $a = 0$.

Thus, futher we are going to consider:

- Volterra integral equation of the first kind

$$\int_0^t K(t,z)y(z)dz = f(t) \tag{5.3}$$

- Volterra integral equation of the second kind

$$y(t) - \int_0^t K(t,z)y(z)dz = f(t) \tag{5.4}$$

Since Volterra integral equation of the second kind are seems to be easier to solve, it is convenient to transform Volterra equation of the first kind into Volterra integral equation of the second kind.

Consider the equation (5.3), where $K(t,z)$, $\frac{\partial K(t,z)}{\partial t}$, $f(t)$ and $f'(t)$ are continuos on $\{(t,z): t \in [0,b], z \in [0,x]\}$.

Let us differentiate (5.3) with respect to t.

Hence,

$$K(t,t)y(t) + \int_0^t \frac{\partial K(t,z)}{\partial t} y(z)dz = f'(t).$$

If $K(t,t) \neq 0,\ t \in [0,b]$, and $f'(0) = 0$, then, dividing the equality by $K(t,t)$, we obtain Volterra integral equation of the second kind

$$y(t) + \frac{1}{K(t,t)} \int_0^t \frac{\partial K(t,z)}{\partial t} y(z)dz = \frac{f'(t)}{K(t,t)}.$$

III. Method of Successive Approximations for Volterra Integral Equation

Consider Volterra integral equation of the second kind

$$y(t) - \int_0^t K(t,z)y(z)dz = f(t) \tag{5.4}$$

Let us constract the sequence of fuctions $\{y_0(t), y_1(t), \dots, y_n(t), \dots\}$:

$$y_0(t) = f(t),$$

$$y_1(t) = f(t) + \int_0^t K(t,z)y_0(z)dz,$$

$$y_2(t) = f(t) + \int_0^t K(t,z)y_1(z)dz,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_n(t) = f(t) + \int_0^t K(t,z)y_{n-1}(z)dz.$$

The solution of a given integral equation is the limit as $n \to \infty$ of the sequence obtained

$$y(t) = \lim_{n\to\infty} y_n(t).$$

Example 1.

Find the solution of integral equation

$$y(t) - \int_0^t (t-z)y(z)dz = t.$$

Let

$$y_0(t) = f(t) = t.$$

Evaluate $y_1(t)$

$$y_1(t) = t + \int_0^t (t-z)y_0(z)dz = t + \int_0^t (t-z)zdz = t - \left(t\frac{z^2}{2} - \frac{z^3}{3}\right)\Bigg|_0^t = t - \frac{t^3}{2\cdot 3}.$$

Then we find $y_2(t)$

$$y_2(t) = t + \int_0^t (t-z)\left(z - \frac{z^3}{2\cdot 3}\right)dz = t - \frac{t^3}{2\cdot 3} + \frac{t^5}{2\cdot 3\cdot 4\cdot 5}.$$

Continuing the process, we obtain

$$y_n(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{k=0}^{n} \frac{(-1)^k t^{2k+1}}{(2k+1)!}.$$

Finally,

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = \sin t.$$

Thus, the solution of given integral equation is

$$y(t) = \sin t.$$

IV. Fredholm equation

Definition. The integral equation of the form

$$\lambda \int_a^b K(t,z)y(z)dz = f(t) \tag{5.5}$$

is called *Fredholm integral equation of the first kind*.

Definition. The integral equation of the form

$$y(t) - \lambda \int_a^b K(t,z)y(z)dz = f(t) \quad (5.6)$$

is called *Fredholm integral equation of the second kind*.

Number λ is some numerical parameter. It can be both real and complex.

Definition. Function $K(t,z)$, defined on $\{(t,z): t \in [a,b], z \in [a,b]\}$, is called *a kernel function*.

Definition. Function $f(t)$, defined on $[a,b]$, is called *a right-sided function*.

If $f(t) \equiv 0$ then an integral equation is called *homogeneous*, otherwise, it is *nonhomogeneous*.

Obviously that homogeneous equation of the second kind

$$y(t) - \lambda \int_a^b K(t,z)y(z)dz = 0 \quad (5.7)$$

always has solution $y(t) = 0$ (trivial solution).

Definition. Values of λ that corresponds to nontrivial solutions of (5.7) are called *eigenvalues of the equation* (of the kernel $K(t,z)$), and every corresponding non-trivial solution is called *eigenfunction*.

Theorem.

Integral equation (5.7) has at least one real eigenvalue.

Theorem.

If kernel $K(t,z)$ is square integrable on the rectangle $\{(t,z): t \in [a,b], z \in [a,b]\}$, a, b – real numbers

$$\int_a^b \int_a^b |K(t,z)|^2 dtdz < \infty,$$

then each eigenvalue corresponds to a finite number of eigenfunctions.

Note.

The number of these eigenfunctions is called *range* of corresponding eigenvalue.

Theorem.

The Fredholm Alternative

There are only two possibilities

1. *Nonhomogeneous linear integral Fredholm equation of the second kind (5.6)*

$$y(t) - \lambda \int_a^b K(t,z)y(z)dz = f(t)$$

has unique solution, whatever its right-sided function $f(t)$.

2. *The corresponding homogeneous equation (5.7)*

$$y(t) - \lambda \int_a^b K(t,z)y(z)dz = 0$$

has nontrivial solutions.

Theorem.

The necessary and sufficient condition for the existence of a solution $y(t)$ of the nonhomogeneous equation (5.6) is

$$\int_a^b \varphi(z)y(z)dz = 0, \qquad (5.8)$$

where $\varphi(z)$ is a solution of the transposed equation

$$\varphi(t) - \bar{\lambda} \int_a^b \overline{K(z,t)}\varphi(z)dz = 0.$$

Note.

If the condition (5.8) holds, then the equation (5.6) has an infinite number of solutions.

The Fredholm Alternative is very convenient to use in practice. Instead of proving existense of the solution of (5.6), it is sometimes easier to prove that the homogeneous equation (5.7) has **only trivial solution**. Then the equation (5.6) also has solutions.

V. Method of Successive Approximations for Fredholm Integral Equation

Consider Fredholm integral equation of the second kind

$$y(t) - \lambda \int_a^b K(t,z)y(z)dz = f(t) \tag{5.6}$$

If $|\lambda| < \frac{1}{M}$, where $M > 0$ and

$$M^2 = \int_a^b \int_a^b |K(t,z)|^2 dtdz,$$

then there exists the solution of the equation (5.6) and it may be found by method of successive approximations.

Let us constract the sequence of fuctions $\{y_0(t), y_1(t), \dots, y_n(t), \dots\}$:

$$y_0(t) = f(t),$$

$$y_1(t) = f(t) + \lambda \int_a^b K(t,z)y_0(z)dz,$$

$$y_2(t) = f(t) + \lambda \int_a^b K(t,z)y_1(z)dz,$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$y_n(t) = f(t) + \lambda \int_a^b K(t,z)y_{n-1}(z)dz.$$

The solution of a given integral equation is the limit as $n \to \infty$ of the sequence obtained

$$y(t) = \lim_{n\to\infty} y_n(t).$$

Example 2.

Find the solution of integral equation

$$y(t) - \frac{1}{2\pi}\int_0^\pi y(z)dz = \sin t - \frac{1}{\pi}.$$

First, let us check the condition of existence.

Here the kernel function $K(t,z) = 1$ and

$$M^2 = \int_0^\pi \int_0^\pi |K(t,z)|^2 dtdz = \int_0^\pi \int_0^\pi dtdz = \pi^2 \implies M = \pi.$$

Since $\lambda = \frac{1}{2\pi} < \frac{1}{M} = \frac{1}{\pi}$, the solution of a given equation exists.

Let

$$y_0(t) = \sin t - \frac{1}{\pi}.$$

Next, evaluate $y_1(t)$

$$y_1(t) = \sin t - \frac{1}{\pi} + \frac{1}{2\pi}\int_0^{\pi}\left(\sin z - \frac{1}{\pi}\right)dz = = \sin t - \frac{1}{\pi} + \frac{1}{2\pi}\left(-\cos z - \frac{z}{\pi}\right)\Big|_0^{\pi} =$$

$$= \sin t - \frac{1}{\pi} + \frac{1}{2\pi}.$$

Then

$$y_2(t) = \sin t - \frac{1}{\pi} + \frac{1}{2\pi}\int_0^{\pi}\left(\sin z - \frac{1}{\pi} + \frac{1}{2\pi}\right)dz = \sin t - \frac{1}{\pi} + \frac{3}{4\pi},$$

$$y_3(t) = \sin t - \frac{1}{\pi} + \frac{1}{2\pi}\int_0^{\pi}\left(\sin z - \frac{1}{\pi} + \frac{3}{4\pi}\right)dz = \sin t - \frac{1}{\pi} + \frac{7}{8\pi},$$

$$y_4(t) = \sin t - \frac{1}{\pi} + \frac{1}{2\pi}\int_0^{\pi}\left(\sin z - \frac{1}{\pi} + \frac{7}{8\pi}\right)dz = \sin t - \frac{1}{\pi} + \frac{15}{16\pi},$$

As a result, we obtain

$$y_n(t) = \sin t - \frac{1}{\pi} + \frac{2^n - 1}{2^n\pi}.$$

The solution could be found from the limit

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty}\left(\sin t - \frac{1}{\pi} + \frac{2^n - 1}{2^n\pi}\right) = \sin t.$$

Thus,

$$y(t) = \sin t.$$

You may find more information about integral equations in [10].

Review Questions

1. What is integral equation?
2. What is the integral equation of the first kind? of the second kind?
3. What difference between Volterra and Fredholm integral equations?
4. What is kernel function in integral equation?

5. What is eigenvalue and eigenfunction for integral equation?

6. What is the condition of existence of solution of Fredholm equations?

7. Formulate the Fredholm Alternative. Why this theorem is important?

8. What is the method of successive approximations for integral equations?

Exercises 5.1

1-8. Solve the integral equations by method of successive approximations

1. $y(t) - \int_0^t (t-z)y(z)dz = 1;$

2. $y(t) - \int_0^t y(z)dz = 1;$

3. $y(t) - \int_0^t tzy(z)dz = 1 - t^2;$

4. $y(t) - \int_0^t y(z)dz = 2t + t^2;$

5. $y(t) + \frac{1}{\pi}\int_0^\pi \cos^2 z\, y(z)dz = 1;$

6. $y(t) + \int_0^{\frac{\pi}{2}} \sin t\, y(z)dz = 2\sin t;$

7. $y(t) - 4\int_0^1 (t^2 - t)zy(z)dz = t;$

8. $y(t) + \frac{1}{5}\int_0^t ze^t y(z)dz = \frac{6}{5}e^t.$

Answers.

ANSWERS

Answers to exercises 1.2

1. $\frac{1}{(y-1)}+\frac{1}{2(x-1)^2}=C$; **2.** $\ln(4+x^2)+\sqrt{9-y^2}=C$; **3.** $\sin x\cos y=C$; **4.** $\ln^2 x-\cot^2 y=C$; **5.** $\ln|xy|+\frac{y-x}{xy}=C$; **6.** $(x-1)^2+y^2=C$; **7.** $y-\sqrt{x}+C=\ln|y|$; **8.** $(e^y+1)e^x=C$; **9.** $y=\frac{C\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$; **10.** $y=e^{C\arcsin x}$; **11.** $y=\sqrt{\ln^3|1-x^2|}$; **12.** $\arctan(x/2)=\ln^2 y+\pi/4$; **13.** $\tan y=1-x+\tan x$; **14.** $y=e^x(x-2)$; **15.** $\ln(\sqrt{x}+1)=-\sqrt{1-y}$.

[Return to exercises]

Answers to exercises 1.3

1. $y=2x(C+\ln|x|)$; **2.** $x+(x+3y)^2=C$; **3.** $\arctan\frac{y}{x}+\ln C\sqrt{x^2+y^2}=0$; **4.** $x=(y-x)\ln C(y-x)$; **5.** $\sqrt{x}+\sqrt{y}\ln C y=0$; **6.** $e^{-\frac{y}{x}}+\ln C x=0$; **7.** $Cx=e^{\cos\frac{y}{x}}$; **8.** $y^2=Cxe^{-\frac{y}{x}}$; **9.** $\ln x=\frac{y}{x}\left(\ln\frac{y}{x}-1\right)+C$; **10.** $\ln|Cx|=-\cos\frac{y}{x}$; **11.** $x^2+xy-y^2-x+3y=C$; **12.** $x^2+2xy-y^2-4x+8y=C$; **13.** $y=x-2x^3$; **14.** $2-\ln|x|=\frac{2}{5}\sqrt{\frac{y}{x}}$; **15.** $x=3e^{\tan\frac{y}{x}}$; **16.** $x-2=\ln\frac{y}{x}$; **17.** $y=x\cdot\arcsin x$; **18.** $3x+2y-4+2\ln|x+y-1|=0$.

[Return to exercises]

Answers to exercises 1.4

1. $y=x(C+3x)$; **2.** $y=C/x^4-x^2/6)$; **3.** $y=(C+x)/x^2$; **4.** $y=Ce^{7x}-2e^{3x}$; **5.** $y=C\sqrt{x^2+1}+x^2+1$; **6.** $y=e^{2\sqrt{x}}(C+x)$; **7.** $y=\frac{1}{\cos x}\left(C+\frac{x}{2}+\frac{1}{4}\sin 2x\right)$; **8.** $y=\frac{C+5x}{\ln x}$; **1.9.** $y=Ce^{-\tan x}+\tan x-1$; **10.** $y=\sqrt{x^2+4}\left(C+\frac{1}{2}\arctan\frac{x}{2}\right)$; **11.** $x=\frac{C+y^4}{y^5}$; **12.** $x=e^{-1/y}(C+y)$; **13.** $y=\frac{1}{2}x^2e^{-3x}$; **14.** $y=\frac{\arcsin x}{\sqrt{1-x^2}}$; **15.** $y=e^{-e^x}+e^x-1$; **16.** $y=x^3+x^2+x+1$; **17.** $y=x(2-\cos x)$; **18.** $y=\ln x\cdot\ln|\ln x|$.

[Return to exercises]

Answers to exercises 1.5

1. $y=\left(x-2+Ce^{-\frac{x}{2}}\right)^2$; **2.** $y^{-\frac{1}{3}}=Cx^{\frac{2}{3}}-\frac{3}{7}x^3$; **3.** $y=\frac{x-1}{C-x}$; **4.** $y^{\frac{1}{2}}-\tan x=\frac{\ln\cos x+C}{x}$; **5.** $y^{-4}=x^3(e^x+C)$; **6.** $y=e^{-x}\left(\frac{1}{2}e^x+1\right)^2$; **7.** $y=\frac{\sec x}{x^3+1}$; **8.** $x=\frac{1}{y(y+C)}$; **9.** $y=\frac{\sec^2 x}{\tan x-x+C}$; **10.** $x^2+y^2=e^{-y}$.

[Return to exercises]

Answers to exercises 1.6

1. $y+\frac{x^2}{2}-\cos y+x\ln|y|=C$; **2.** $2xy-3x+y^2=C$; **3.** $2y^2+x^2\cos 2y+x^2=0$; **4.** $x^3e^y-y+1=0$; **5.** $xe^{-y}+y=C$; **6.** $y^3x-2y^2x^2+3x^4-3=0$; **7.** $\mu=\frac{1}{x^2}$, $y^2x+\frac{1}{x}=C$; **8.** $\mu=\frac{1}{x^4}$, $\frac{y^2-x^2}{x^3}=C$; **9.** $\mu=e^{-x}$, $e^{-x}\cos y-y-1=0$; **10.** $\mu=\frac{1}{\sin y}$, $x^3+\frac{x}{\sin y}=C$. **11.** $\mu=e^{-2y}$, $x^2+(2y-1)e^{2y}=0$, **12.** $\mu=\frac{1}{y}$, $y\sin x+x\ln y=C$.

[Return to exercises]

Answers to exercises 2.2

1. $y=(-1/4)\sin 2x+C_1x+C_2$; **2.** $y=-64e^{-x/4}+C_1x^2+C_2x+C_3$; **3.** $y=(1/2)x^2\ln x-(3/4)x^2+C_1x+C_2$; **4.** $y=-2\ln|x|+C_1x+C_2$; **5.** $y=C_1x^2+C_2$; **6.** $y=C_1(x-e^{-x})+C_2$; **7.** $y=$

$(1/2)\ln^2|x| + C_1\ln|x| + C_2$; **8.** $y = C_2 + C_1 \sin x - x - (1/2)\sin 2x$; **9.** $y = \frac{2}{3C_1}\sqrt{(C_1x-1)^3} + C_2$; **10.** $y = (C_1x + C_2)^2$; **11.** $y = (1/3)\ln|3y+4| = C_1x + C_2$; **12.** $x = (1/C_1)\ln|y/(y+C_1)| + C_2$; **13.** $(C_1x + C_2)^2 = C_1y^2 - 8$; **14.** $x = (1/2)\sqrt{y^2 + C_1} + C_2$; **15.** $y = 1/3 - (1/9)\sin 3x$; **16.** $y = x\ln|x| + x - 1$; **17.** $y = x^4/8 - x^3/6 + x^2/2 - x$; **18.** $x = y + \ln|y| - 1$; **19.** $y = x^3/3 + x - 1$; **20.** $y = 1 + \sin x$.

Return to exercises

Answers to exercises 2.3

1. yes; **2.** no; **3.** no; **4.** yes; **5.** $y = C_1\frac{\sin x}{x} - C_2\frac{\cos x}{x}$; **6.** $y = C_1x + C_2x\ln x$; **7.** $y = C_1\sin x + C_2\left(1 - \sin x \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|\right)$; **8.** $y = C_1x - C_2(x^2 - 1)$;

Return to exercises

Answers to exercises 2.4

1. $y = C_1e^{-3x} + C_2e^{4x}$; **2.** $y = C_1e^{-x} + C_2e^{-6x}$; **3.** $y = e^{-2x}(C_1 + C_2x)$; **4.** $y = e^{2x}(C_1\cos 3x + C_2\sin 3x)$; **5.** $y = C_1 + C_2e^{-8x}$; **6.** $y = C_1\cos 5x + C_2\sin 5x$; **7.** $y = C_1e^{-x} + C_2e^{x} + C_3e^{2x}$; **8.** $y = C_1 + C_2e^{-5x} + C_3e^{3x}$; **9.** $y = C_1 + e^{4x}(C_2 + C_3x)$; **10.** $y = e^{-x}(C_1 + C_2x) + C_3e^{3x}$; **11.** $y = e^{x}(C_1 + C_2x + C_3x^2)$; **12.** $y = C_1 + C_2\cos 8x + C_3\sin 8x$; **13.** $y = C_1 + C_2e^{x} + C_3\cos x + C_4\sin x$; **14.** $y = \cos x\,(C_1 + C_2x) + \sin x\,(C_3 + C_4x)$; **15.** $y = C_1 + C_2x + C_3x^2 + C_4e^{-2x} + C_5e^{2x}$; **16.** $y = C_1 + C_2e^{-3x} + C_3e^{3x} + C_4\cos 3x + C_5\sin 3x$; **17.** $y = 4e^{-3x} - 3e^{-2x}$; **18.** $y = xe^{5x}$; **19.** $y = -\frac{1}{3}e^{x}\cos 3x$; **20.** $y = 2\sin\frac{x}{3}$; **21.** $y = \frac{1}{3}(5 - 2e^{-3x})$.

Return to exercises

Answers to exercises 2.5

1. $y = C_1\cos x + C_2\sin x + \cos x\ln|\cos x|$; **2.** $y = C_1\cos x + C_2\sin x + \sin x\ln|\tan(x/2)|$; **3.** $y = C_1e^{x} + C_2e^{-x} + \frac{1}{2}((e^{x} + e^{-x})\ln(e^{x} + 1) - (xe^{x} + 1))$; **4.** $y = C_1e^{x} + C_2xe^{x} + xe^{x}\ln|x|$; **5.** $y = C_1\cos 3x + C_2\sin 3x - \frac{1}{9}\cos 3x + \frac{1}{9}\sin 3x\ln|\sin 3x|$; **6.** $y = C_1e^{-x} + C_2xe^{-x} + xe^{-x}\ln|x|$; **7.** $y = C_1\cos 2x + C_2\sin 2x - \cos 2x\ln|\sin x| - \left(x + \frac{1}{2}\cot x\right)\sin 2x$; **8.** $y = C_1\cos x + C_2\sin x + \frac{1}{2\cos x}$; **9.** $y = 2\sin x + \cos x \cdot \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$; **10.** $y = C_1 + C_2\cos x + C_3\sin x + \ln\left|\frac{1+\sin x}{\cos x}\right| - x\cos x + \sin x \cdot \ln|\cos x|$.

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Answers to exercises 2.6

1. $y = e^{-x}(C_1\cos 2x + C_2\sin 2x) + (Ax^2 + Bx + C)e^{-2x}$; **2.** $y = C_1e^{-3x} + C_2e^{-x} + (Ax^2 + Bx)e^{-3x}$; **3.** $y = C_1 + C_2e^{8x} + Ax^4 + Bx^3 + Cx^2 + Dx$; **4.** $y = (C_1 + C_2x)e^{5x} + (Ax^3 + Bx^2 + Cx)e^{5x}$; **5.** $y = e^{x}(C_1\cos 3x + C_2\sin 3x) + e^{x}\big((Ax + B)\cos 2x + (Cx + D)\sin 2x\big)$; **6.** $y = C_1\cos 4x + C_2\sin 4x + (Ax^3 + Bx^2 + Cx)\cos 4x + (Dx^3 + Ex^2 + Fx)\sin 4x$; **7.** $y = e^{3x}(C_1\cos 2x + C_2\sin 2x) + e^{3x}(Ax\cos 2x + Bx\sin 2x)$; **8.** $y = C_1 + C_2e^{-10x} + Ax^3 + Bx^2 + Cx + e^{-10x}\big((Dx + E)\cos x + (Fx + G)\sin x\big)$; **9.** $y = C_1e^{-6x} + C_2e^{6x} + (Ax^2 + Bx)e^{-6x} + C\cos 6x + D\sin 6x$; **10.** $y = e^{-3x}(C_1\cos 4x + C_2\sin 4x) + e^{-3x}(Ax\cos 4x + Bx\sin 4x) + (Cx + D)\cos 4x + (Ex + F)\sin 4x$; **11.** $y = C_1e^{2x} + C_2e^{-5x} - x^2 - x$; **12.** $y = C_1e^{-x} + C_2e^{5x} + (x - 1)e^{-4x}$; **13.** $y = C_1e^{x} + C_2e^{3x} + 5xe^{3x}$; **14.** $y = C_1 + C_2e^{-4x} + (x^2/4 + x/8)e^{-4x}$; **15.** $y = C_1\cos 4x + C_2\sin 4x + (2x + 1)e^{-x}$; **16.** $y = e^{-2x}(C_1 + C_2x) + (1/2)x^3e^{-2x}$; **17.** $y = C_1 + C_2e^{-5x} + \sin 5x - \cos 5x$; **18.** $y = e^{2x}(C_1\cos x + C_2\sin x) + \cos x + (1/2)\sin x$; **19.** $y = C_1\cos 2x + C_2\sin 2x + (3/2)x\cos 2x + (5/2)x\sin 2x$; **20.** $y = C_1e^{-2x} + C_2e^{2x} - e^{2x}(2\cos 2x + \sin 2x)/20$; **21.** $y = e^{x}(C_1\cos x + C_2\sin x) - (1/2)xe^{x}\cos x$; **22.** $y = C_1 + C_2e^{3x} + (1/3)xe^{3x} - 2x^2 + x$; **23.** $y = -\cos 2x + \sin 2x + (3x/4 + 1)e^{-2x}$; **24.** $y = (1/3)e^{-4x} - (1/3)e^{2x} +$

$(x^2+3x)e^{2x}$; **25.** $y = e^x(-6\cos 3x + \sin 3x) + 12\cos 3x + 2\sin 3x$; **26.** $y = -3\cos x + \pi \sin x + x(4\cos x - 3\sin x)$; **27.** $y = e^{2x}(\cos 3x - \sin 3x + (1/6)x\sin 3x)$.

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Answers to exercises 3.3

1. $x(t) = C_1e^{-2t} + C_2e^{2t}, y(t) = 2C_1e^{-2t} - 2C_2e^{2t}$; **2.** $x(t) = C_1\cos t + C_2\sin t, y(t) = -C_1\sin t + C_2\cos t$; **3.** $x(t) = C_1e^t + C_2e^{-t} + \frac{1}{8}e^{3t}, y(t) = -C_1e^t + C_2e^{-t} + \frac{5}{8}e^{3t}$; **4.** $x(t) = C_1e^{-4t} + C_2e^{-7t} + \frac{7}{40}e^t + \frac{1}{5}e^{-2t}, y(t) = \frac{1}{2}C_1e^{-4t} + C_2e^{-7t} + \frac{1}{40}e^t + \frac{3}{10}e^{-2t}$; **5.** $x(t) = \frac{3}{4}e^t + \frac{5}{4}e^{-t} + \frac{1}{2}te^t - 1$, $y(t) = \frac{5}{4}e^t - \frac{5}{4}e^{-t} + \frac{1}{2}te^t - t$; **6.** $x(t) = \left(-\frac{1}{2} + \frac{3}{2}t\right)e^t - \frac{1}{2}\cos t, y(t) = \left(2 - \frac{3}{2}t\right)e^t - 2\cos t - \frac{1}{2}\sin t$; **7.** $x(t) = C_1e^{-t} + C_2e^{-3t}, y(t) = C_1e^{-t} + 3C_2e^{-3t} + \cos t$; **8.** $x(t) = C_1e^t + C_2e^{-t} + C_3\sin t + C_4\cos t, y(t) = C_1e^t + C_2e^{-t} - C_4\sin t - C_3\cos t$; **9.** $x(t) = C_1e^t + C_2\cos t + C_3\sin t, y(t) = C_1e^t - C_2\sin t + C_3\cos t, z(t) = (C_2 - C_3)\cos t + (C_2 + C_3)\sin t$; **10.** $x(t) = -e^{-t}, y(t) = e^{-t}, z(t) = 0$; **11.** $x(t) = C_1e^{-t} + C_2e^{3t}, y(t) = 2C_1e^{-t} - 2C_2e^{3t}$; **12.** $x(t) = C_1e^t + C_2te^t, y(t) = (2C_1 - C_2)e^t + 2C_2te^t$; **13.** $x(t) = C_1e^t\cos 3t + C_2\sin 3t, y(t) = C_1e^t\sin 3t - C_2e^t\cos 3t$; **14.** $x(t) = C_1 + 3C_2e^{2t}, y(t) = -2C_2e^{2t} + C_3e^{-t}, z(t) = C_1 + C_2e^{2t} - 2C_3e^{-t}$; **15.** $x(t) = C_1e^t + C_2e^{2t} + C_3e^{-t}, y(t) = C_1e^t - 3C_2e^{-t}, z(t) = C_1e^t + C_2e^{2t} - 5C_3e^{-t}$; **16.** $x(t) = C_1e^{2t} + C_2e^{3t} + C_3e^{6t}, y(t) = C_2e^{3t} + 2C_3e^{6t}, z(t) = -C_1e^{2t} + C_2e^{3t} - C_3e^{6t}$; **17.** $x(t) = C_1e^t + C_2\cos t + C_3\sin t, y(t) = C_1e^t - C_2\sin t + C_3\cos t, z(t) = (C_2 - C_3)\cos t + (C_2 + C_3)\sin t$; **18.** $x(t) = e^{5t}, y(t) = 3e^{5t}$; **19.** $x(t) = e^t + e^{4t}, y(t) = -e^t + \frac{1}{2}e^{4t}$; **20.** $x(t) = -e^{-t}, y(t) = \frac{1}{2}te^{-t}$; **21.** $x(t) = 3\cos 2t + \sin 2t, y(t) = \frac{1}{5}\cos 2t + \frac{7}{5}\sin 2t$; **22.** $x(t) = 2e^{2t}\cos t - e^{2t}\sin t, y(t) = e^{2t}\cos t - 3e^{2t}\sin t$; **23.** $x(t) = \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t} + \frac{1}{2}e^{-2t}, y(t) = \frac{1}{3}e^{-t} + \frac{1}{6}e^{2t} - \frac{1}{2}e^{-2t}, z(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}$; **24.** $x(t) = -e^{-t}, y(t) = e^{-t}, z(t) = 0$.

Return to exercises

Answers to exercises 4.1

1. asymptotically stable; **2.** unstable; **3.** unstable; **4.** asymptotically stable; **5.** unstable; **6.** $a \geq 0$, asymptotically stable; $a < 0$, unstable; **7.** $a < 0$, asymptotically stable; $a > 0$, unstable.

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Answers to exercises 4.2

1. $y = -1$, stable; **2.** $y = 3$, unstable; **3.** $y = 0$, semistable; $y = 2$, unstable; **4.** $y = 0$, stable; $y = 2$, semistable; $y = 4$, unstable; **5.** $y = \pm 2k\pi, k \in Z$, unstable; $y = \pm(2k+1)\pi, k \in Z$, stable.

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Answers to exercises 4.3

1. stable node; **2.** stable focus; **3.** stable singular node; **4.** unstable node; **5.** center; **6.** saddle; **7.** line of stable fixed points; **8.** unstable focus; **9.** $a = 0$, center; $a < 0$, stable focus; $a > 0$, unstable focus; **10.** $a < -1$, stable node; $a = -1$, any point on $y = x$, stable; $-1 < a < 1$, saddle; $a = 1$, any point on $y = -x$, unstable; $a < 1$, unstable node.

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Answers to exercises 5.1

1. $y = \cos t$; **2.** $y = e^t$; **3.** $y = 1$; **4.** $y = 2t$; **5.** $y = \frac{2}{3}$; **6.** $y = \sin t$; **7.** $y = t^2$; **8.** $y = e^t$.

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