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ADAPTATION OF OSCILLATORY SYSTEMS IN NETWORKS — A LEARNING SIGNAL APPROACH

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We consider a network of coupled periodic stable signals (PSS) interacting through the gradient of a coupling potential. Each PSS has its own set of parameters Ω_k, characterizing the time scale of the signal and its shape. The Ω_k are allowed to modify their values (i.e. to adapt) by introducing adaptive mechanisms on them. Together with the state variable interactions, the adaptive mechanisms drive all PSS towards a consensual oscillatory state where they all have a common, constant set of parameters Ω_c. Once reached, the consensual oscillatory state remains invariant to the interactions. This implies that if the interactions are removed, all PSS continue to deliver the consensual signal. This situation is to be contrasted with classical synchronization problems where common dynamical patterns are attained and maintained thanks to the interactions. Hence, if the interactions are removed, all PSS converge back towards their individual behavior. The resulting value Ω_c is analytically calculated. It does not depend on the network's topology. However, the conditions for convergence do depend on the connectivity of the network and on the coupling potential.

1. INTRODUCTION

Producing stable oscillatory motion is of great importance for a device delivering stable periodic signals. Due to its stability mechanism, the apparatus sends out signals that are not drastically altered even if it is placed in a noisy environment. However, structural changes within the device may occur (e.g. the stability mechanism itself may be perturbed), and these create permanent discrepancies, thus lowering the quality of the output signal. To overcome this problem, a signal can be coupled to another of its like. As an example, consider two coupled signals $r_1(t)$ and $r_2(t)$ in the setup

$$\dot{r}_k = R(r_k;\Omega_k) - \frac{\partial V}{\partial r_k}(r_1, r_2),\ k = 1,2 \tag{1}$$

with the gradient of a potential V as coupling function. Here, the set of parameters $0 \neq \|\Omega_1 - \Omega_2\| << 1$ due to a structural change. Synchronizing signals may enhance the overall quality in the sense that now, under suitable conditions, $\lim_{\tau\to\infty} r_k(t) = r_{k,V}(t)$ (for $k = 1,2$) with signals $r_{k,V}(t)$ having the same periodicity t_V.

However, synchronized signals $r_{1,V}(t)$ and $r_{2,V}(t)$ only exist at the cost of maintaining the coupling — if coupling vanishes (i.e. $V \equiv 0$), the two individual signals return, respectively, towards the signals produced by the vector fields

$R(\cdot;\Omega_k)$, $k=1,2$. Furthermore, $r_{k,V}(t)$, and consequently period t_V, is subject to any change in the coupling: if V changes, the synchronized signals, as well as their periodic behavior, are perturbed.

One way to tackle this problem is to construct systems that can synchronize and simultaneously "adapt" local characteristics (i.e. Ω_k) in order to be:

- closer to their likes (i.e. reduce the difference $\Omega_1 - \Omega_1$);
- less dependent on the coupling (i.e. find $\breve{\Omega}_k$ such that $Z(\breve{\Omega}_1, \breve{\Omega}_2) = \nabla V((\breve{r}_1(t), \breve{r}_2(t))$ is minimum over a period and $\breve{r}_k(t)$ solves Equations (1)).

An optimum solution for I and II is when there exists a consensual parameter set $\Omega_c = \Omega_k$, $k=1,2$ such that $Z(\Omega_c, \Omega_c) = 0$. In this situation, if the coupling is removed, the devices continue to deliver the same signal. Furthermore, at this consensual state, any changes in the coupling does not affect the signals since they are now independent of it.

Technically, for local parameters Ω_k to adapt, they must become time-dependent (i.e. $\Omega_k \to \Omega_k(t)$ and have their own dynamics. For n coupled signals, having each an additional phase variable controlling their time scale, the general complex networks dynamics is

$$\begin{aligned} \dot{\phi}_k &= P(\phi_k, r_k, \Omega_k) - c_k \frac{\partial V}{\partial \phi_k}(\phi, r), \\ \dot{r}_k &R\underbrace{(\phi_k, r_k, \Omega_k)}_{\text{local dynamics}} - \underbrace{c_k \frac{\partial V}{\partial r_k}(\phi, r)}_{\text{coupling dynamics}}, \quad k = 1, \ldots, n, \\ \dot{\Omega}_k &= \underbrace{A_k(\phi, r)}_{\text{adaptive mechanisms}} \end{aligned} \tag{2}$$

with $\phi = (\phi_1, \ldots \phi_n)$, $r = (r_1, \ldots, r_n)$ and coupling strengths $c_k > 0$. The local dynamics belong to the class of PR systems (i.e phase-radius systems): P and R govern the dynamics of the local oscillator's phase and radius, respectively. Adapting parameters in complex systems has long been a busy field of research [1–10]. Whereas in other contributions adaptation occurs in the coupling strength [5] or directly in the connections [2], Equations (2) describe adaptation in the local systems. As mentioned in [6], for local systems' parameter adaptation, there exist two types: flow parameters controlling the frequency or time scale on an attractor, and geometric parameters determining the shape of the attracting set. Frequency or time scale controlling parameters have, in general, a high propensity for adaptation and have been well studied in [3, 10, 1, 7]. However, not much has been accomplished for shaping local attractors, which, by nature, is a more delicate task — as stated in [8, 9].

In this paper, we present new adaptive mechanisms for modifying the local system's attractor. Whereas in [8, 9] the adaptive mechanisms implicitly depend on the parameter set Ω_k via a functional, ours solely depend on the state variables ϕ_k and r_k. Note that adaptive mechanisms should only depend on the state variables since, in practice, these are the only information available. In [8, 9], one needs to calculate or numerically compute an integral beforehand to know the sign of the function for the adaptive mechanism. Our approach is systematic for all parameters.

This contribution is organized as follows: We present individually the components of our network's dynamical system in Section 2. In Section 3 we discuss the resulting dynamics and present two related alternatives to our system. Numerical simulations are reported in Section 4, and we conclude in Section 5.

2. NETWORKS OF PERIODIC STABLE SIGNALS WITH ADAPTIVE MECHANISMS

Consider a n — vertex connected and undirected network with positive adjacency entries. To each node corresponds a local dynamical system defined in Section 2.1. While the network topology (i.e. adjacency matrix) of the underlying network indicates if the k^{th} local system is connected to the f^{th} (and vice versa), it is the coupling dynamics discussed in Section 2.2 that describes how the neighboring local dynamics interact. Described in Section 2.3, supplementary interactions directly acting on the local systems' parameters will play the role of adaptive mechanisms. Let us now individually present each three dynamical components.

2.1. Local Dynamics

The local systems belong to the class of PR systems. We here focus on Periodic Stable Signals (PSS), which we define as

$$P(\phi_k, r_k; \Omega_k) = w_k$$

$$R(\phi_k, r_k; \Omega_k) = \underbrace{-(r_k - F_k(\phi_k))}_{\text{dissipative dynamics}} + \underbrace{F'(\phi_k) w_k}_{\text{oscillatiry dynamics}}, \quad k = 1, \ldots, n, \tag{3}$$

with $F_k(\phi_k) = u_{k,0} + \sum_{m=1}^{q} u_{k,m} \cos(m\phi_k) + v_{k,m} \sin(m\phi_k)$. The set of parameters is $\Omega_k = \{w_k, u_{k,0}, u_{k,1}, v_{k,1}, \ldots, u_{k,q}, v_{k,q}\}$. Parameter w_k controls the time scale of the phase, which here oscillates uniformly (i.e. $\phi_k(t) = w_k t + \phi_k$).

The rest of the parameters determine the shape of the stable periodic signal produced by a PSS. Stable here means that if the system endures a perturbation, it will converge back to its oscillatory motion and continue to deliver the signal with its original shape given by the compact set $\mathbb{K}_k = \{(\phi, r) \in \mathbb{S}^1 \times \mathbb{R} | r - F_k(\phi) = 0\}$. The convergence towards $\mathbb{K}_k$ is discussed in Appendix A. It is the dissipative dynamics that is responsible for driving the orbits towards $\mathbb{K}_k$. This term is the gradient (with respect to the variable r) of the potential $\frac{1}{2}(r - F_k(\phi))^2$. It is seen as an energy controller that takes in and/or gives out energy (depending on the system's state) until it reaches its equilibrium state $\mathbb{K}_k$. On $\mathbb{K}_k$, the PSS's dynamics is governed by the oscillatory dynamics and so $\dot{r}_k(t) - F_k'(\phi_k(t))\dot{\phi}_k(t) = 0$, which is consistence with Equation (3) when the dissipative dynamics is zero.

When $F_k(\phi_k) = u_{k,0}$, the PSS is a limit cycle oscillator with constant angular velocity and a circle of radius $u_{k,0}$ as an attractor. As sketched in Fig. 1, PSS may form more complicated and interesting attractors.

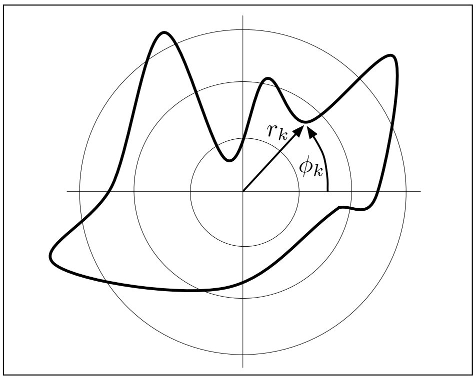

Fig. 1. Sketch of an attractor for a PSS. The dynamics evolves at a constant angular velocity $\phi_k(t) = w_k t$ on the black thick curve

2.2. Coupling Dynamics

The coupling dynamics is here given by the gradient of a positive semi-definite coupling potential $V(\phi, r) \geq 0$ (see Section 1.1.2 in [6] for a precise definition). On V, we have the following assumptions

$$\phi = y\mathbf{1} \text{ and } r = z\mathbf{1},\ y, z \in \mathbb{R} \Rightarrow V(\phi, r) = 0$$

with $\phi = (\phi_1, \ldots, \phi_n)$, $r = (r_1, \ldots, r_n)$ and $\mathbf{1} = (1, \ldots, 1)$. Bellow, we present two examples.

Example 1. Laplacian Potential

Define V as

$$V(\phi, r) = \frac{1}{2} r | L_{\cos r} \frac{1}{2} \sum_{k=1}^{n} r_k \sum_{j=1}^{n} l_{k,j} r_j \cos(\phi_k - \phi_j)$$

with $\phi = (\phi_1, \ldots, \phi_n)$ and $r = (r_1, \ldots, r_n)$, and where the matrix $L_{\cos r}$ has entries $l_{k,j} r_j \cos(\phi_k - \phi_j)$ with L being the corresponding Laplacian matrix ($L = D - A$, where D is the diagonal matrix with $d_{k,k} = \sum_{j=1}^{n} a_{k,j}$). Matrix $L_{\cos r}$ is positive semi-definite since, by Гершгорин's circle theorem [4], all its eigenvalues are positive (i.e. nonnegative). Explicitly, the coupling dynamics for this potential is

$$c_k \frac{\partial V}{\partial \phi_k}(\phi, r) = -c_k \sum_{j=1}^{n} l_{k,j} r_k r_j \sin(\phi_k - \phi_j),\ \ k = 1, \ldots, n,$$

$$c_k \frac{\partial V}{\partial r_k}(\phi, r) = c_k \sum_{j=1}^{n} l_{k,j} r_j \cos(\phi_k - \phi_j),\ \ k = 1, \ldots, n,$$

where $c_k > 0$ are coupling strengths.

Example 2. B **Potential**

Define V as

$$V(\phi,r)=\frac{1}{2}\sum_{k,j}^{n}a_{k,j}(B_{k,j}(\phi_k-\phi_j)+B_{k,j}(r_k-r_j))\geq 0$$

with edge weights $0\leq a_{kj}=a_{j,k}$, and where functions $B_{k,j}$ satisfy $B_{k,j}(x)\geq 0$, $B_{k,j}(x)=0\Leftrightarrow x=0$, $B_{k,j}(x)=B_{k,j}(-x)$ (i.e. even function) and $0<B'_{k,j}(0)$. We here impose $B_{k,j}\equiv B_{j,k}$. For the functions $B_{k,j}$, one may take one of the cases

$B_{k,j}(x)=\left(\frac{1}{2}\right)(x^2)$	Diffusion	$B_{k,j}(x)=\cosh(x)-1$
$B_{k,j}(x)=1-\cos(x)$	Kuramoto-type	$B_{k,j}(x)=\log(\cosh(x))$

.

Explicitly, the coupling dynamics for this potential is

$$c_k\frac{\partial V}{\partial\phi_k}(\phi,r)=c_k\sum_{j=1}^{n}a_{k,j}B'_{k,j}(\phi_k-\phi_j),\ \ k=1,\ldots,n,$$

$$c_k\frac{\partial V}{\partial r_k}(\phi,r)=c_k\sum_{j=1}^{n}a_{k,j}B'_{k,j}(r_k-r_j),\ \ k=1,\ldots,n$$

with coupling strengths $c_k>0$.

2.3. Adaptive Mechanisms

Here, adaptive mechanisms are additional interactions that modify the values of the local parameters. For this, the fixed and constant parameters Ω_k are now time-dependent (i.e.

$$\Omega_k=\{w_k,u_{k,0},u_{k,1},v_{k,1},\ldots,u_{k,q},v_{k,q}\}\Rightarrow$$

$$\Rightarrow(w_k(t),\mu_{k,0}(t),\mu_{k,1}(t),v_{k,1}(t),\ldots,\mu_{k,q}(t),v_{k,q}(t))=\Omega_k(t),$$

for $k=1,\ldots,n$) and each have their own dynamics depending only on state variables ϕ and r, that is, for all k, $\frac{\partial A_k}{\partial\Omega_k}=0$ with 0 a $2+2q$ dimensional vector of 0.

Time scale Adaptive Mechanisms

For adaptation on ω_k, we apply the same idea as developed in [8, 6] and so the explicit dynamics is

$$A_k^{\omega}(\phi,r)=-s_{\omega_k}\frac{\partial V}{\partial\phi_k}(\phi,r)\ \ k=1,\ldots,n,$$

where $s_{\omega_k}>0$ are susceptibility constants, technically playing the role of coupling strengths but with the following interpretation: the smaller the value of s_{ω_k}, the easier it is for the PSS to modify the value of its ω_k, and vice versa — the larger the value of s_{ω_k}, the harder it is for the PSS to modify the value of its ω_k.

Amplitude Adaptive Mechanisms

Inspired by attractor-shaping mechanisms studied in [8, 6] we propose, for the PSS's $\mu_{k,0}, \mu_{k,m}$ and $v_{k,m}$, the following new adaptive mechanisms

$$A_k^{\mu_0}(\phi, r) = -s_{\mu_{k,0}} \sum_{j=1}^{n} l_{k,j} r_j \frac{\partial F_j}{\partial \mu_{j,s}}(\phi_j) = -s_{\mu_{k,0}} \sum_{j=1}^{n} l_{k,j} r_j, \;\; k = 1, \dots, n,$$

$$A_k^{\mu_m}(\phi, r) = -s_{\mu_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \frac{\partial F_j}{\partial \mu_{j,m}}(\phi_j) = -s_{\mu_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \cos(m\phi_j), \;\; k = 1, \dots, n,$$

$$A_k^{v_m}(\phi, r) = -s_{v_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \frac{\partial F_j}{\partial v_{j,m}}(\phi_j) = -s_{v_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \sin(m\phi_j), \;\; m = 1, \dots, q,$$

where $l_{k,j}$ are the entries of L and strictly positive $s_{\mu_{k,0}}$, $s_{\mu_{k,m}}$, and $s_{v_{k,m}}$ are susceptibility constants.

3. NETWORK'S DYNAMICAL SYSTEM WITH TIME SCALE AND AMPLITUDE ADAPTATION

Combining the individual components discussed in Section 2 yields the global dynamical system

$$\dot{\phi}_k = \omega_k - c_k \frac{\partial V}{\partial \phi_k}(\phi, r), \;\; k = 1, \dots, n,$$

$$\dot{r}_k = \underbrace{-(r_k - F_k(\phi_k)) + F_k'(\phi_k)\omega_k}_{\text{local dynamics}} - \underbrace{c_k \frac{\partial V}{\partial r_k}(\phi, r)}_{\text{coupling dynamics}}, \;\; k = 1, \dots, n,$$

$$\dot{\omega}_k = \underbrace{-s_{\omega_k} \frac{\partial V}{(\partial \phi_k)(\phi, r)}}_{\text{time scale adaptive mechanisms}}, \;\; k = 1, \dots, n,$$

$$\dot{\mu}_{k,0} = -s_{\mu_{k,0}} \sum_{j=1}^{n} l_{k,j} r_j, \;\; k = 1, \dots, n,$$

$$\dot{\mu}_{k,m} = -s_{\mu_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \cos(m\phi_j), \;\; m = 1, \dots, q,$$

$$\underbrace{\dot{v}_{k,m} = -s_{v_{k,m}} \sum_{j=1}^{n} l_{k,j} r_j \sin(m\phi_j), \;\; m = 1, \dots, q.}_{\text{amplitude adaptive mechanisms}} \tag{4}$$

Equations (4) describe the dynamics of n PSS (i.e. local dynamics) coupled by the gradient of a coupling potential V (i.e. coupling dynamics) with frequency adaptation (i.e. time scale adaptive mechanisms) and attractor shaping (i.e. amplitude adaptive mechanisms). For Equations (4), we have $q+1$ constants of motion, the existence of a consensual oscillatory state and the convergence towards it.

$2q+1$ Constants of Motion

The functions

$$J_{\mu_0}(\mu_0)=\sum_{k=1}^{n}\frac{\mu_{k,0}}{s_{\mu_{k,0}}},\quad J_{\mu_m}(\mu_m)=\sum_{k=1}^{n}\frac{\mu_{k,m}}{s_{\mu_{k,m}}},$$

$$J_{v_m}(v_m)=\sum_{k=1}^{n}\frac{v_{k,m}}{s_{v_{k,m}}},\quad m=1,\ldots,q \tag{5}$$

with $\mu_0(\mu_{1,0},\ldots,\mu_{n,0})$, $\mu_m(\mu_{1,m},\ldots,\mu_{n,m})$, and $v_m(v_{1,m},\ldots,v_{n,m})$ are constants of motion. Indeed, if $\mu_0(t)$, $\mu_m(t)$, $v_m(t)$ for $m=1,\ldots,q$ are orbits of Equations (4), then

$$\frac{d[J_{\mu_0}(\mu_0(t))]}{dt}=-\sum_{k=1}^{n}\sum_{j=1}^{n}l_{k,j}r_j,\quad \frac{d[J_{\mu_m}(\mu_m(t))]}{dt}=-\sum_{k=1}^{n}\sum_{j=1}^{n}l_{k,j}r_j,\cos(m\phi_j),$$

$$\frac{d[J_{v_m}(v_m(t))]}{dt}=-\sum_{k=1}^{n}\sum_{j=1}^{n}l_{k,j}r_j,\sin(m\phi_j),\quad m=1,\ldots,q$$

by Lemma D.2 in [6]. If we further suppose that $\sum_{k=1}^{n}\left(\frac{\partial V}{\partial\phi_k}\right)(\phi,r)$ for all (ϕ,r) (and this is true for both types of coupling potentials in Example 1), then Equations (4) admit another constant of motion, namely $J_\omega(\omega)=\sum_{k=1}^{n}\frac{\omega_k}{s_{\omega_k}}$ with $\omega=(\omega_1,\cdots,\omega_n)$.

Existence of a Consensual Oscillatory State

Equations (4) admit a consensual oscillatory state. Indeed, for given common constants $\Omega_c(\omega_c,\mu_{c,0},\mu_{c,1},v_{c,1},\ldots,\mu_{c,q},v_{c,q}$,

$$(\phi_k(t),r_k(t),\Omega_k(t))=(\omega_c t,F_c(t),\Omega_c),\quad k=1,\ldots,n \tag{7}$$

is a consensual orbit of Equations (4), with here F_c taking the value Ω_c. Indeed, since points given by Equations (7) are extrema of the V, then the coupling dynamics and the adaptive time scale mechanisms are zero. Hence, $\omega_k(t)$ is a constant taking value ω_c for all k, and $(\phi_k(t),r_k(t))=(\omega_c t,F_c(t))$ solves each local dynamics and cancels all amplitude adaptive mechanisms for all k. Therefore $(\mu_{k,0}(t),(\mu_{k,1}(t),v_{k,1}(t),\ldots,\mu_{k,q}(t),v_{k,q}(t))$ are constants taking, respectively, common values $(\mu_{c,0},\mu_{c,1},v_{c,1},\ldots,\mu_{c,q},v_{c,q}$, for all k.

Convergence Towards a Consensual Oscillatory State

If perturbations are introduced in Equations (7), we say that System (4) converges towards a consensual oscillatory state if we have the following limit

$$\lim_{t\to\infty}\{\phi_k(t), r_k(t), \Omega_k(t) - (\omega_c(t), F_c(t), \Omega_c)\} = 0 \quad \forall\, k \tag{8}$$

with constant Ω_c. This limit raises two problems: determining the limit values Ω_c and finding the conditions for convergence.

Limit Values. If the constant of motion in Equation (6) exists and if Limit (8) holds, then, thanks to all the other constants of motion in Equations (5), we have

$$J_\omega(\omega(0)) = \lim_{t\to\infty} J_\omega(\omega(t)) = J_\omega(\lim_{t\to\infty}\omega(t)) = J_\omega(\omega_c 1) = \omega_c \sum_{k=1}^{n}\frac{1}{s_{\omega_k}},$$

$$J_{\mu_0}(\mu_0(0)) = \lim_{t\to\infty} J_{\mu_0}(\mu_0(t)) = J_{\mu_0}(\lim_{t\to\infty}\mu_0(t)) = J_{\mu_0}(\mu_{c,0} 1) = \mu_{c,0} \sum_{k=1}^{n}\frac{1}{s_{\mu_{k,0}}},$$

$$J_{\mu_m}(\mu_m(0)) = \lim_{t\to\infty} J_{\mu_m}(\mu_m(t)) = J_{\mu_m}(\lim_{t\to\infty}\mu_m(t)) = J_{\mu_m}(\mu_{c,m} 1) = \mu_{c,m} \sum_{k=1}^{n}\frac{1}{s_{\mu_{k,m}}},$$

$$J_{v_m}(v_m(0)) = \lim_{t\to\infty} J_{v_m}(v_m(t)) = J_{v_m}(\lim_{t\to\infty}v_m(t)) = J_{v_m}(v_{c,m} 1) = v_{c,m} \sum_{k=1}^{n}\frac{1}{s_{v_{k,m}}}$$

for $m = 1,\ldots,q$. Hence, the consensual values of Ω_c are analytically expressed as

$$\omega_c = \frac{\sum_{k=1}^{n}\frac{\omega_k(0)}{s_{\omega_k}}}{\sum_{k=1}^{n}\frac{1}{s_{\omega_k}}},\quad \mu_{c,0} = \frac{\sum_{k=1}^{n}\frac{\mu_{j,0}(0)}{s_{\mu_{k,0}}}}{\sum_{k=1}^{n}\frac{1}{s_{\mu_{k,0}}}},\quad m = 1,\ldots,q,$$

$$\mu_{c,m} = \frac{\sum_{k=1}^{n}\frac{\mu_{j,m}(0)}{s_{\mu_{k,m}}}}{\sum_{k=1}^{n}\frac{1}{s_{\mu_{k,m}}}},\quad v_{c,m} = \frac{\sum_{k=1}^{n}\frac{v_{j,m}(0)}{s_{\mu_{k,m}}}}{\sum_{k=1}^{n}\frac{1}{s_{v_{k,m}}}},\quad m = 1,\ldots,q. \tag{9}$$

Convergence Conditions. To prove the convergence in Limit (8), one can linearize Equations (4) around a consensual oscillatory state. In general, the resulting $n(4+2q)\times n(4+2q)$ Jacobian depends explicitly on time (since evaluated on a consensual oscillatory state) and therefore Floquet exponents have to be computed. Note that for certain coupling potentials V and assumptions on the coupling strengths and susceptibility constants, the Jacobian can be diagonalized in order to reduce the computation of Floquet exponents to n systems, each of size $(4+2q)\times(4+2q)$.

We emphasize that numerous numerical simulations show that Limit (8) holds — and this for different topologies, coupling potential and values of coupling strengths and susceptibility constants. For these numerical experiments, the coupling strengths were set around one and susceptibility constants around 0.1.

REMARK: ADAPTATION

Here, adaptation is to be interpreted as an asymptotic stability problem, which is directly related to the study of Limit (8). Indeed, for initially different PSS, if Limit (8) holds, then the adaptive mechanisms, with the help of the coupling dynamics, drive all local systems towards a consensual oscillatory state as defined in Equations (7). Once this state is reached, the coupling dynamics, as well as the adaptive mechanisms, may be removed — and all PSS will still continue to deliver the same signal with the same time scale (i.e. local system are no longer dependent on their environment to produce common dynamical patterns). This is because the values Ω_k have been permanently modified (i.e. $\lim_{t\to\infty}\Omega_k(t)=\Omega_c$). If the adaptive mechanisms are not switched on initially, dynamical patterns may occur (due to the coupling dynamics) — but these are maintained because of the network interactions. If the interactions are removed, all PSS converge back towards their own shape, which is determined by $\mathbb{K}_k$ and their own time scale, given by w_k.

3.1. Miscellaneous Remark: Time Scale or Amplitude Adaptation Only

We present here two alternatives of System (4). One alternative concerns amplitude r_k adaptation only (Section 3.1.1), whereas the other deals with time scale ω_k adaptation only (Section 3.1.2).

3.1.1. Amplitude Adaptation Only

Consider Equations (4) with no phases ϕ_k (and hence no time scale adaptive mechanisms), and for each local PSS, let $\phi_k(t)=t$ for all k. The system becomes

$$\dot{r}_k = \underbrace{-(r_k - F_k(t)) + \dot{F}_k(t)}_{\text{local dynamics}} - \underbrace{c_k \frac{\partial V}{\partial r_k}(r)}_{\text{coupling dynamics}} ,$$

$$\dot{\mu}_{k,0} = -s_{\mu_{k,0}} \sum_{j=1}^{n} l_{k,j} r_j, \quad k = 1,\ldots,n,$$

$$\dot{\mu}_{k,m} = -s_{\mu_{k,m}} \cos(mt) \frac{\partial V}{\partial r_k}(r), \quad k = 1,\ldots,n,$$

$$\dot{v}_{k,m} = \underbrace{-s_{v_{k,m}} \sin(mt) \frac{\partial V}{\partial r_k}(r)}_{\text{amplitude adaptive mechanisms}} , \quad m = 1,\ldots,q. \tag{10}$$

with $F_k(t) = \mu_{k,0} + \sum_{m=1}^{q} \mu_{k,m} \cos(mt) + v_{k,m} \sin(mt)$. Note that for Equations (10) we still have $q+1$ constants of motion (as given in Equations (5)), the existence of a consensual oscillatory state $(r_k(t), \Omega_k(t)) = (F_c(t), \Omega_c)$ for $k = 1, \dots, n,$ and the convergence towards it (i.e. $\lim_{t \to \infty} \{(r_k(t), \Omega_k(t)) - (F_c(t), \Omega_c)\} = 0$ for all k) as observed by numerous numerical simulations.

A priori, Equations (10) describe a system where adaptation only occurs on the shape of the local attractors. However, by adequately setting the value of one or several constants of motion in Equations (5), one can cancel the asymptotic values of the corresponding coefficients. Thus, by changing the shape of the signal, one can change its frequency.

3.1.2. Time scale Adaptation Only

We here remark that PSS (i.e. belonging to the class of PR System) can be slightly modified in order to be seen as Ortho-Gradient (OG) systems. For a precise definition and examples, see Section 1.1.1 in [6]. Briefly, OG systems are characterized by dissipative dynamics that are orthogonal to their canonical — or here, oscillatory-dynamics. Let us consider the following network of PSS that are also OG systems, and where there is only time scale adaptation

$$\begin{aligned}
\dot{\phi}_k &= \omega_k + (r_k - F(\phi_k))F'(\phi_k) - \underbrace{c_k \frac{\partial V}{\partial \phi_k}(\phi)}_{\text{coupling dynamics}}, \\
\dot{r}_k &= \underbrace{\omega_k F'(\phi_k) - (r_k - F_k(\phi_k))}_{\text{local dynamics}}, \quad k = 1, \dots, n, \\
\dot{\omega}_k &= \underbrace{-s_{\omega_k} \frac{\partial V}{\partial \phi_k}(\phi)}_{\text{time scale adaptive mechanisms}}.
\end{aligned} \tag{11}$$

As shown in Lemma 1.1 in [6], each local dynamics in Equations (11), taken individually, possesses its own attractor given by $\mathbb{K}$. Networks of OG systems with adapting angular velocities have been studied. For the particular type of coupling dynamics and time scale adaptive mechanisms (i.e. only on variables ϕ_k), one can directly apply Proposition 2.2 in [6] to show that System (11) converges towards a consensual oscillatory state with consensual value ω_c as in Equations (9). For this convergence, one needs to suppose that $\langle 1 | \nabla V(\phi) \rangle$ for all ϕ and to make a technical hypothesis on V.

4. NUMERICAL SIMULATIONS

We report two sets of numerical simulations, one with time scale and amplitude adaptation (refer to Section 4.1.) and one with amplitude adaptation only (refer to Section 4.2.).

4.1. Time Scale and Amplitude Adaptation

We consider 39 PSS as in Equations (4) with network topology as in Fig. 2,*a*). Here, each PSS is given by $F_k(\phi) = \mu_{k,0} + \sum_{m=1}^{3} \mu_{k,m} \cos(m\phi) + \nu_{k,m} \sin(m\phi)$ for

$k = 1,\ldots,39$. The coupling strengths and susceptibility constants are $c_k = 1$, $s_{\omega_k} = s_{\mu_{k,0}} = s_{\mu_{k,m}} = s_{v_{k,m}} = 0.1$ for $k = 1,\ldots,39$ and $m = 1,2,3$. A Laplacian potential, as in Example 1, is used for the coupling dynamics. The initial conditions $(\phi_k(0), \omega_k(0), \mu_{k,0}(0), \mu_{k,1}(0), v_{k,1}(0), \mu_{k,2}(0), v_{k,2}(0), \mu_{k,3}(0), v_{k,3}(0))$ are randomly uniformly drawn from $]-h,h[\times]1-h,1+h[\times]2-h,2+h[\times]5-h,5+h[\times]-3-$ $-h,-3+h[\times]-7-h,-7+h[\times]-5-h,-5+h[$ with $h = 0.225$. These initial conditions determine $F_k(0)$, and finally, the initial conditions $r_k(0)$ are randomly uniformly drawn from $]F_k(0)-h, F_k(0)+h[$.

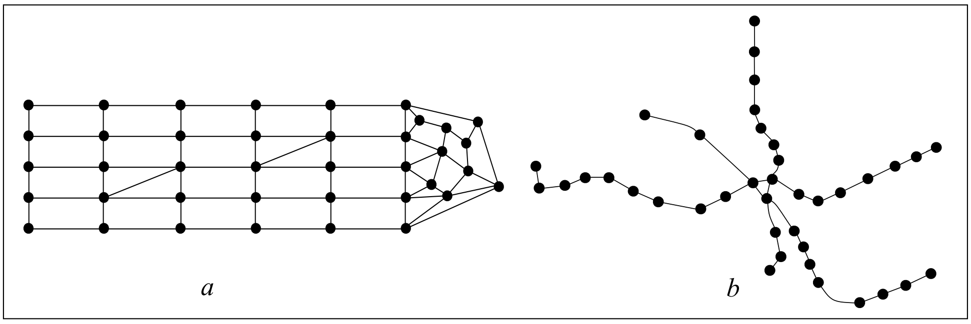

Fig. 2. Two 39-vertex Network Topologies, “Manhattan” 2,*a* and Metro of Kyiv 2,*b*

The resulting dynamics for the variables r_k, ω_k and $v_{k,2}$ is shown in Fig. 3. Note that the variables r_k converge quickly towards a common signal, whereas

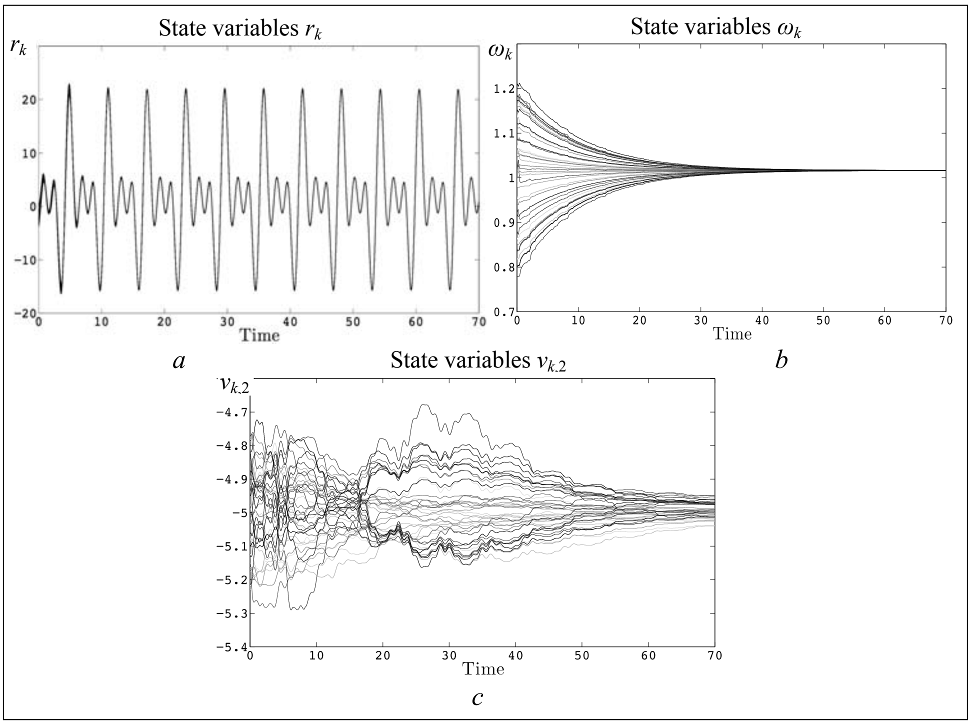

Fig. 3. Time evolution of r_k (Fig. 3,*a*), ω_k (Fig. 3,*b*) and $v_{k,2}$ (Fig. 3,*c*) for 39 PSS, interacting through the network in (Fig. 2,*a*)

the variables ω_k and $v_{k,2}$ take more time to converge towards their asymptotic values. This is due to a relatively strong coupling strength compared to the susceptibility constants. For this setup, we have always observed convergence towards a consensual oscillatory state. With the same setup, but with a network as in Fig. 2,*b*, convergence was not observed for all numerical experiments — as we report in Fig. 4,*a–c*. However, for the exact same initial conditions as in Fig. 4, if all adaptive mechanisms are switched off (i.e. all susceptibility constants are zero), the network is still able to synchronize as shown in Fig. 4,*d*.

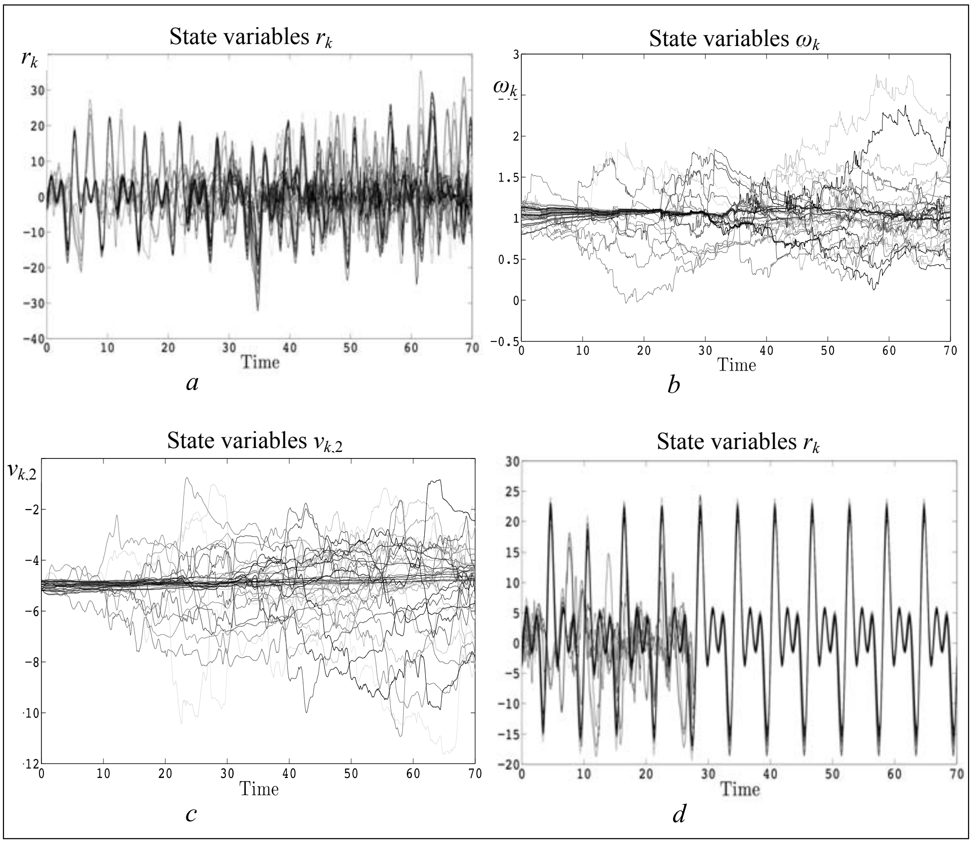

Fig. 4. Time evolution of r_k (Fig. 4,*a*), ω_k (Fig. 4,*b*) and $v_{k,2}$ (Fig. 4,*d*) for 39 PSS, interacting through the network in (Fig. 2,*b*). Time evolution of r_k for 39 PSS with all their adaptive mechanisms switched off (i.e. all susceptibility constants are zero), interacting through the network in (Fig. 2,*d*)

4.2. Amplitude Adaptation Only

Two PSS with amplitude adaptation only as in Equations (10) are considered, with here $F_k(t) = \mu_{k,0} + \sum_{m=1}^{3} \mu_{k,m} \cos(mt) + v_{k,m} \sin(mt)$ for $k = 1,2$. The coupling strengths and susceptibility constants are $c_k = 2$, $s_{\mu_{k,0}} = s_{\mu_{k,m}} = s_{v_{k,m}} = 0.5$ for $k = 1,2$. and $m = 1,2,3$. The coupling potential is $V(r) = \frac{1}{2}(r_1 - r_2)^2$.

The initial conditions $(\mu_{1,0}(0), \mu_{2,0}(0), \mu_{1,3}(0), v_{1,3}(0)\mu_{2,3}(0), v_{2,3}(0))$ are randomly uniformly drawn from $]0.8, 1.2[\times]0.8, 1.2[\times]2.8, 3.2[\times]4.8, 5.2[\times]4.8, 5.2[\times]-7.2, -6.8[$, and the others given by $(\mu_{1,1}(0), v_{1,1}(0), \mu_{1,2}(0), v_{1,2}(0)) = = (2, -2, -1, 1)$ and $(\mu_{2,1}(0), v_{2,1}(0), \mu_{2,2}(0), v_{2,2}(0)) = (-2, 2, 1, -1)$. These initial conditions determine $F_k(0)$, and finally, the initial conditions $r_k(0)$ are randomly uniformly drawn from $]F_1 k(0) - 0.2, F_1 k(0) + 0.2[$ for $k = 1, 2$.

The resulting dynamics for variables r_k and $\mu_{k,1}$ is shown in Fig. 5. For $t \in [0,15]$, the coupling dynamics and the adaptive mechanisms are switched off and so each PSS generates its individual signal. Because of the choice of the initial conditions $(\mu_{k,m}(0), v_{k,m}(0))$ $k, m = 1, 2$, the asymptotic values are $[(\mu)]_{c,m}(0), v_{c,m}(0)) = (0,0)$ for $m = 1, 2$, and so both amplitudes $r_1(t)$ and $r_2(t)$ converge towards $F_c(t) = \mu_{c,3} \cos(3t) + v_{c,3} \sin(3t)$ (i.e. Fourier series with mode $\cos(3t)$ and $\sin(3t)$ only). As a consequence, $F_c(t)$ has a higher frequency than any of the two signals before interactions are switched on. This is observed in Fig. 5,*a* where the two signals have a larger period in the interval $[0,15]$ than when they are close to $F_c(t)$.

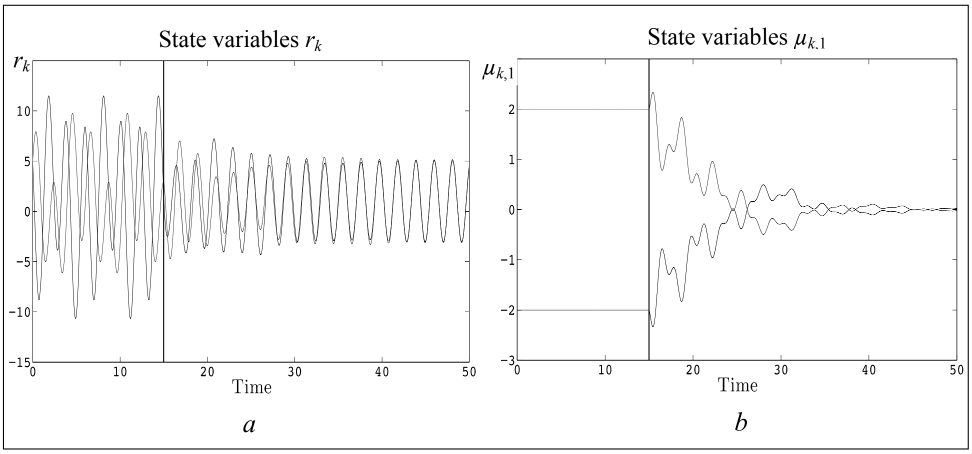

Fig. 5. Time evolution of r_k (*Fig. 5,a*) and $\mu_{k,1}$ (*Fig. 5,b*) for two PSS. Coupling dynamics and adaptive mechanisms are switched on a $t = 15$ (black solid line).

5. CONCLUSION

PSS form a suitable class of systems to investigate the interaction of multi-signal dynamics. Whereas adapting the time scale is a fairly straightforward procedure, shaping the attractor is more complicated. Nevertheless, our dynamical systems show that this can be implemented in a robust manner. The adaptive mechanisms depend solely on the state variables, and no pre-calculations or information on the curvature of the attractor is needed.

The asymptotic values from the resulting dynamics are analytically calculable. The network's topology and the nature of the coupling potential itself directly influence the conditions for attaining a consensual oscillatory state. Determining basins of attraction with respect to connectivity and coupling functions remains an open question.

Apart from investigating the resulting dynamics for directed network connections with time-dependent edges, prospective works would also include merging two adapting PSS communities — one belonging to the class of systems given by Equations (4), and the other described by Systems (10).

APPENDIX A: CONVERGENCE TOWARDS COMPACT SET $\mathbb{K}$

The convergence towards the compact set $\mathbb{K}_k = \{(\phi, r) \in \mathbb{S}^1 \times \mathbb{R} | r - F(\phi) = 0\}$ follows from Lyapunov's second method with Lyapunov function $L(\phi, r) = \frac{1}{2}(r - F(\phi))^2$. By construction, we have that $\mathbb{K} = \{(\phi, r) \in \mathbb{S}^1 \times \mathbb{R} | r - - L(\phi) = 0\}$. Computing the time derivative

$$\nabla L(\phi, r) | (\dot{\phi}, \dot{r}) = -(r - F(\phi)) F'(\phi) \dot{\phi} + (r - F(\phi)) \dot{r} =$$

$$= -(r - F(\phi)) F'(\phi) w + (r - F(\phi))(-(r - F(\phi)) + F'(\phi) w) = -(r - F(\phi))^2.$$

Hence, $\nabla L(\phi, r) | (\dot{\phi}, \dot{r})$ for all $(\phi, r) \in (\mathbb{S})^1 \times \mathbb{R} \setminus \mathbb{K}$.

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REFERENCES

1. *Acebrón J., Spigler R.* Adaptive frequency model for phase-frequency synchronization in large populations of globally coupled nonlinear oscillators // Physical Review Letters. — 1998. — № 81. — P. 2229–2232.
2. *De Lellis P., di Bernardo M., Gorochowski T.E., Russo G.* Synchronization and control of complex networks via contraction, adaptation and evolution // IEEE Circuits and Systems Magazine. — 2010. — № 10. — P. 64–82.

3. *Ermentrout B.* An adaptive model for synchrony in the firefly pteroptyx malaccae // Journal of Mathematical Biology. — 1991. — № 29. — P. 571–585.
4. *Gershgorin S.A.* Über die Abgrenzung der Eigenwerte einer Matrix // Izvestiya Akademii Nauk USSR, Otdelenie matematicheskih i estestvennih Nauk. — 1931. — № 7. — P. 749–754.
5. *Lehnert J., Hövel P., Flunkert V., Guzenko P.Y., Fradkov A.L., Schöll E.* Adaptive tuning of feedback gain in time-delayed feedback control // Chaos. — 2011. — № 21.
6. *Rodriguez J.* Networks of Self-Adaptive Dynamical Systems. PhD thesis, Ecole Polythechnique Fédérale de Lausanne, 2011. —100 p.
7. *Rodriguez J., Hongler M.-O.* Networks of Self-Adaptive Dynamical Systems. IMA Journal of Applied Mathematics (2012). doi: 10.1093/imamat/hxs057.
8. *Rodriguez J., Hongler M.-O., Blanchard Ph.* Self-Adaptive Attractor-Shaping for Oscillators Networks. In Proceedings of The Joint INDS'11 & ISTET'11 – Third International Workshop on Nonlinear Dynamics and Synchronization and Sixteenth International Symposium on Theoretical Electrical Engineering, 2011. — P. 1–6.
9. *Rodriguez J., Hongler M.-O., Blanchard Ph.* Self-Shaping Attractors for Coupled Limit Cycle Oscillators. In Selected Topics in Nonlinear Dynamics and Theoretical Electrical Engineering. — K. Kyamakya, W.A. Halang, W. Mathis, J.C. Chedjou, Z. Li (Eds.), vol. 483 of Studies in Computational Intelligence. Springer (Berlin Heidelberg), 2013. — P. 97–115.
10. *Tanaka H.-A., Lichtenberg A.J., Oishi S.* Self-synchronization of coupled oscillators with hysteretic responses // Physica D. —1997. — № 100. — P. 279–300.

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